Climbing the Temperley-Lieb Tower

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Why are Temperley–Lieb algebras important?

A common planar calculus

Quantum $\mathfrak{sl}_2$

$$\mathrm{TL}_n \cong \mathrm{End}_{U_q(\mathfrak{sl}_2)}(V^{\otimes n})$$

tensor powers and centralizers

Link invariants

$$\Big\langle \quad \Big\rangle \longmapsto J_L(q)$$

Jones polynomial, Markov traces

Subfactors

Jones tower

Fusion

conformal field theory

topological quantum computation

Statistical mechanics

Potts model, loop models, 6-vertex model

Slides: alistairsavage.ca/talks

Paper: A graphical calculus for induction and restriction on Temperley–Lieb modules (arXiv:2606.23340)

The motivating question

The tower of Temperley–Lieb algebras

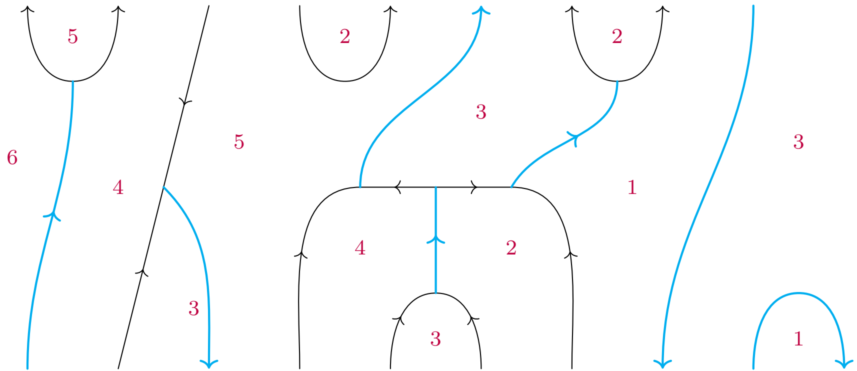
$$\mathrm{TL}_0 \hookrightarrow \mathrm{TL}_1 \hookrightarrow \mathrm{TL}_2 \hookrightarrow \mathrm{TL}_3 \hookrightarrow \cdots$$

gives rise to induction and restriction functors

$$\begin{tikzcd}
\mathrm{TL}_0\text{-mod} \arrow[r, bend left, "\mathrm{Ind}"] & \mathrm{TL}_1\text{-mod} \arrow[l, bend left, "\mathrm{Res}"] \arrow[r, bend left, "\mathrm{Ind}"] & \mathrm{TL}_2\text{-mod} \arrow[l, bend left, "\mathrm{Res}"] \arrow[r, bend left, "\mathrm{Ind}"] & \mathrm{TL}_3\text{-mod} \arrow[l, bend left, "\mathrm{Res}"] \arrow[r, bend left, "\mathrm{Ind}"] & \cdots \arrow[l, bend left, "\mathrm{Res}"]
\end{tikzcd}$$

Can we describe every composite of induction and restriction—and every natural transformation between them?

String diagrams



	region labels	string labels/decorations	diagrams
monoidal category	none	objects	morphisms
2-category	objects	1-morphisms	2-morphisms

General setup: towers from monoidal categories

A tower of algebras often arises from a monoidal category $\mathcal{C}$ and an object $X \in \mathcal{C}$: one considers

$$\mathrm{End}_{\mathcal{C}}(X^{\otimes n}), \qquad n \in \mathbb{N},$$

with inclusions

$$1_X \otimes -\colon \mathrm{End}_{\mathcal{C}}(X^{\otimes n}) \hookrightarrow \mathrm{End}_{\mathcal{C}}\left(X^{\otimes (n+1)}\right).$$

Examples

- group algebras of symmetric groups
- Hecke algebras of type A
- Brauer algebras
- Birman–Murakami–Wenzl (BMW) algebras
- partition algebras

Symmetric groups

Consider the tower of symmetric groups:

$$S_0 = S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots .$$

The Grothendieck group of the category

$$\bigoplus_{n \in \mathbb{N}} S_n\text{-mod}$$

has a product and coproduct coming from induction and restriction:

$$\sum_{m,n \in \mathbb{N}} \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}, \qquad \sum_{m,n \in \mathbb{N}} \mathrm{Res}_{S_m \times S_n}^{S_{m+n}},$$

and

$$K_0 \left(\bigoplus_{n \in \mathbb{N}} S_n\text{-mod} \right) \cong \text{Hopf algebra of symmetric functions.}$$

Heisenberg categorification

Khovanov defined a monoidal Heisenberg category $\mathcal{H}eis$ with two generating objects $\uparrow$, $\downarrow$, together with generating morphisms

satisfying certain relations. This category

- acts naturally on $\bigoplus_{n\in\mathbb{N}} S_n$-mod: $\uparrow \leftrightsquigarrow \mathrm{Ind}$, $\downarrow \leftrightsquigarrow \mathrm{Res}$,
- categorifies the infinite-rank Heisenberg algebra and its Fock space representation.

Generalizations and applications

- quantum analogues,
- more general version acting on modules over cyclotomic (degenerate) affine Hecke algebras,
- connections to categorification of quantum groups,
- geometry of Hilbert schemes,
- W-algebras and elliptic Hall algebras.

Previous work: Temperley–Lieb algebras

Harper–Samuelson (2025) developed an analogue of the Heisenberg category for the tower of Temperley–Lieb algebras.

Again, there were two generating objects

$$\uparrow \leftrightsquigarrow \mathrm{Ind}, \qquad \downarrow \leftrightsquigarrow \mathrm{Res},$$

and morphisms corresponded to natural transformations.

Challenges

- Relations are much more complicated than those of the Heisenberg category.
- Descriptions of bases for morphisms spaces are conjectural.
- Fullness and faithfulness of functors to representation-theory side are conjectural.
- Description of Grothendieck ring (decategorification) is conjectural.

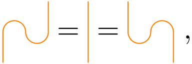
Goal: Revisit this work and try to resolve the outstanding issues.

The Temperley–Lieb category

Define $\mathcal{TL}$ to be the $\mathbb{C}(q)$-linear strict monoidal category generated by the object $\mathtt{T}$ and morphisms

$$\cap \colon \mathtt{T} \otimes \mathtt{T} \to \mathbb{1}, \qquad \cup \colon \mathbb{1} \to \mathtt{T} \otimes \mathtt{T},$$

subject to the relations



$$\bigcirc = [2] 1_{\mathbb{1}}, \qquad \text{where } [n] := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The Temperley–Lieb algebra on n strands

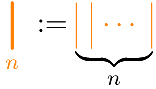
$$\mathrm{TL}_n := \mathcal{TL}(\mathtt{T}^{\otimes n}, \mathtt{T}^{\otimes n}) := \mathrm{Hom}_{\mathcal{TL}}(\mathtt{T}^{\otimes n}, \mathtt{T}^{\otimes n}).$$

In the Temperley–Lieb category, diagrams can have different numbers of endpoints at the top and bottom:

$$\in \mathcal{TL}(\mathtt{T}^{\otimes 5}, \mathtt{T}^{\otimes 7}).$$

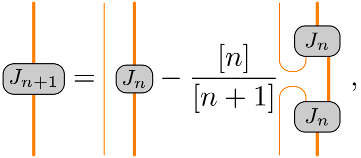
Jones–Wenzl projectors

We use thick strands to denote multiple strands:



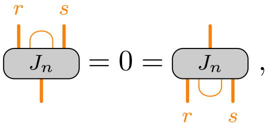
Define the Jones–Wenzl projectors recursively by:

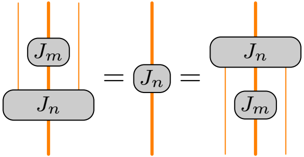
$$J_0 = 1_{\mathbb{1}}, \qquad J_1 = 1_1,$$



$$n \geq 1$$

Key properties:





Simple modules

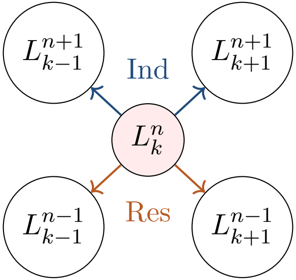
The simple TL_n-modules are

$$L_k^n = \left\{ \begin{array}{c} n \\ \boxed{f} \\ \boxed{J_k} \end{array} : f \in \mathcal{TL}(\mathsf{T}^{\otimes k}, \mathsf{T}^{\otimes n}) \right\}, \qquad 0 \le k \le n, \quad k \equiv n \mod 2,$$

with TL_n-action given by composing on the top. We define $L_k^n = 0$ if $k < 0$ or $k > n$.

We have a natural inclusion of algebras

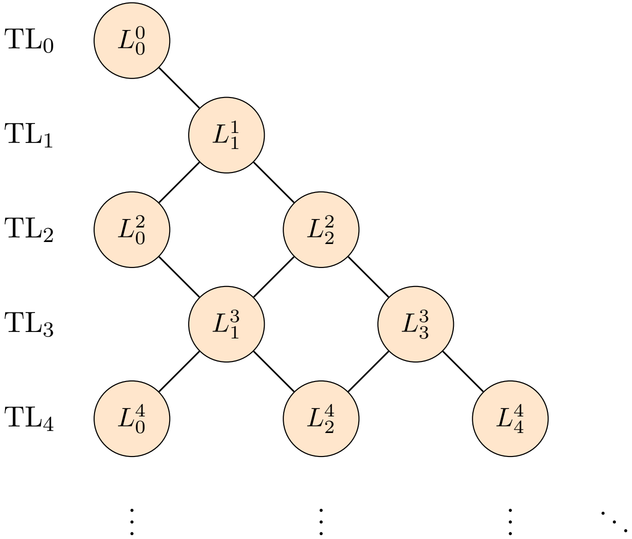
$$\mathrm{TL}_m \hookrightarrow \mathrm{TL}_{m+n}, \qquad \boxed{f} \mapsto \underset{n}{\big|}\ \boxed{f}$$



and isomorphisms

$$\mathrm{Ind}_n^{n+1}(L_k^n) \cong L_{k+1}^{n+1} \oplus L_{k-1}^{n+1}, \qquad \mathrm{Res}_n^{n+1}(L_k^{n+1}) \cong L_{k+1}^n \oplus L_{k-1}^n.$$

Branching graph for the Temperley–Lieb algebra



The diagrammatic 2-category: objects and 1-morphisms

We define a $\mathbb{C}(q)$-linear 2-category $\mathfrak{D}$ as follows.

Objects: X_n, $n \in \mathbb{N}$.

1-morphisms:

$$1_n = \text{identity 1-morphism of } X_n.$$

The 1-morphisms are generated by

$$\mathtt{S}_+1_n \colon X_n \to X_{n+1}, \qquad \mathtt{S}_-1_{n+1} \colon X_{n+1} \to X_n,$$
$$\mathtt{D}_+1_n \colon X_n \to X_{n+2}, \qquad \mathtt{D}_-1_{n+2} \colon X_{n+2} \to X_n.$$

We draw the identity 1-morphisms of the generating objects as follows:

$$1_{\mathtt{S}_+1_n} = \ n{+}1 \uparrow n \ , \qquad 1_{\mathtt{S}_-1_{n+1}} = \ n \downarrow n{+}1 \ ,$$
$$1_{\mathtt{D}_+1_n} = \ n{+}2 \Uparrow n \ , \qquad 1_{\mathtt{D}_-1_{n+2}} = \ n \Downarrow n{+}2 \ .$$

Here, and in what follows, we omit labels for regions when they can be deduced.

The diagrammatic 2-category: 2-morphisms

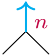
The 2-morphisms are generated by

$: \mathtt{S}_+\mathtt{S}_-1_{n+1} \to 1_{n+1},$

$: \mathtt{S}_-\mathtt{S}_+1_n \to 1_n,$

$: \mathtt{D}_+\mathtt{D}_-1_{n+2} \to 1_{n+2},$

$: 1_n \to \mathtt{D}_-\mathtt{D}_+1_n,$



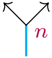
$: \mathtt{S}_+\mathtt{S}_+1_n \to \mathtt{D}_+1_n,$

$: 1_n \to \mathtt{S}_-\mathtt{S}_+1_n,$

$: 1_{n+1} \to \mathtt{S}_+\mathtt{S}_-1_{n+1},$

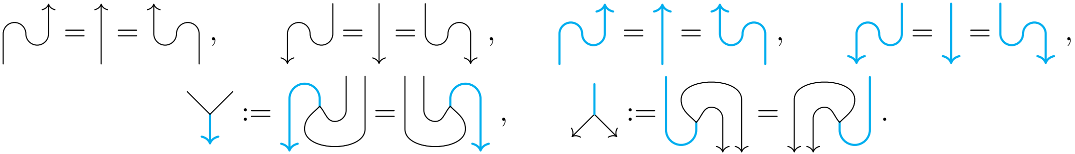
$: \mathtt{D}_-\mathtt{D}_+1_n \to 1_n,$

$: 1_{n+2} \to \mathtt{D}_+\mathtt{D}_-1_{n+2},$



$: \mathtt{D}_+1_n \to \mathtt{S}_+\mathtt{S}_+1_n.$

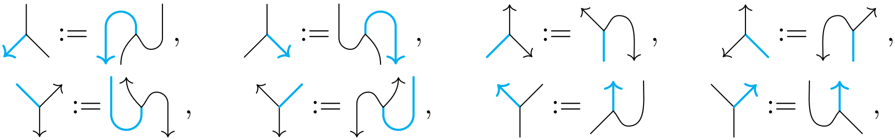
The first relations imply the 2-category is pivotal (2-morphisms are invariant under isotopy):



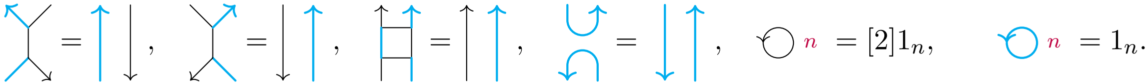
(We omit region labels in equations that hold for all valid region labelings.)

The diagrammatic 2-category: relations

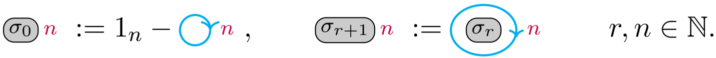
Defining



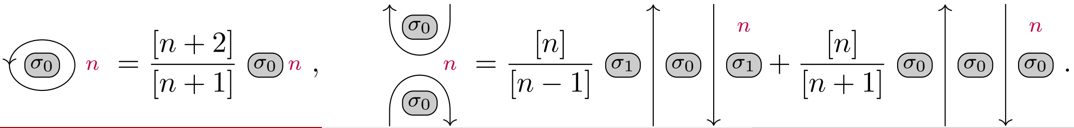
we also impose the following relations:



We recursively define



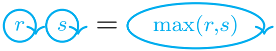
The final relations (which depend on the region!) we impose are:



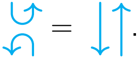
Bubbles: an example of diagrammatic reasoning

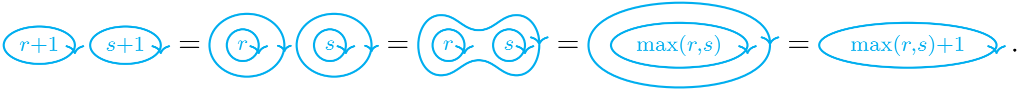
For $r \in \mathbb{Z}$ we recursively define

Lemma:



for all $r, s \in \mathbb{N}$.

Proof: Recall that  The lemma is trivial for $r = 0$ or $s = 0$. If it holds for some $r, s \in \mathbb{N}$,



Since $\max(r, s) + 1 = \max(r + 1, s + 1)$, the result follows by induction on $\min(r, s)$.

Important idempotents

We defined

$$\boxed{\sigma_0}\; n := 1_n - \bigcirc\; n \,, \qquad \boxed{\sigma_{r+1}}\; n := \boxed{\sigma_r}\; n \,.$$

It follows that

$$\boxed{\sigma_r}\; n = r\; n - r+1\; n \,.$$

Proposition

$$\sum_{r=0}^{\lfloor n/2 \rfloor} \boxed{\sigma_r}\; n = 1_n,$$

$$\boxed{\sigma_r}\;\boxed{\sigma_s} = \delta_{r,s}\;\boxed{\sigma_r}\,,$$

$$\uparrow \boxed{\sigma_r} = \boxed{\sigma_{r+1}} \uparrow \,,$$

$$\boxed{\sigma_r} \uparrow = \boxed{\sigma_r} \uparrow \left(\boxed{\sigma_r} + \boxed{\sigma_{r-1}} \right), \qquad \uparrow \boxed{\sigma_r} = \left(\boxed{\sigma_r} + \boxed{\sigma_{r+1}} \right) \uparrow \boxed{\sigma_r}\,.$$

The incarnation functor

Let $\mathfrak{B}$ be the $\mathbb{C}(q)$-linear 2-category of bimodules over the Temperley–Lieb algebras.

- Objects: X_n, $n \in \mathbb{N}$
- Categories of 1-morphisms: $\mathfrak{B}(X_m, X_n) = (\mathrm{TL}_n, \mathrm{TL}_m)$-bimod

Let

$$\mathcal{M} := \bigoplus_{n \in \mathbb{N}} \mathrm{TL}_n\text{-mod}.$$

Then $\mathfrak{B}$ acts on $\mathcal{M}$ by tensor product.

We define 2-functors

$$\boxed{\text{diagrammatic } \mathfrak{D}} \xrightarrow{\mathbf{F}} \boxed{\text{TL bimodules } \mathfrak{B}} \xrightarrow{\mathbf{A}} \boxed{\text{functors on } \mathcal{M}}$$

On 1-morphisms:

$$\mathbf{F}(\mathtt{S}_+ 1_n) = {}_{n+1}(\mathrm{TL}_{n+1})_n, \qquad \mathbf{F}(\mathtt{S}_- 1_{n+1}) = {}_n(\mathrm{TL}_{n+1})_{n+1},$$
$$\mathbf{F}(\mathtt{D}_+ 1_n) = {}_{n+2}(\mathrm{TL}_{n+2} e_{n+1})_n, \qquad \mathbf{F}(\mathtt{D}_- 1_{n+2}) = {}_n(e_{n+1} \mathrm{TL}_{n+2})_{n+2},$$

$e_{n+1} :=$ 

The incarnation functor (cont.)

Generating 2-morphisms are sent to bimodule maps:

$$\mathbf{F}\left(\ n \right) \colon {}_n(\mathrm{TL}_n)_{n-1}(\mathrm{TL}_n)_n \to {}_n(\mathrm{TL}_n)_n, \qquad x \otimes y \mapsto xy,$$

$$\mathbf{F}\left(\ n \right) \colon {}_n(\mathrm{TL}_n)_n \to {}_n(\mathrm{TL}_{n+1})_n, \qquad x \mapsto x,$$

$$\mathbf{F}\left(\ n \right) \colon {}_n(\mathrm{TL}_{n+1})_n \to {}_n(\mathrm{TL}_n)_n, \qquad x \mapsto \mathrm{tr}_n^{n+1}(x),$$

$$\mathbf{F}\left(\ n \right) \colon {}_n(\mathrm{TL}_n e_{n-1})_{n-2}(e_{n-1}\mathrm{TL}_n)_n \to {}_n(\mathrm{TL}_n)_n, \qquad x \otimes y \mapsto xy,$$

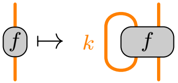
$$\mathbf{F}\left(\ n \right) \colon {}_n(\mathrm{TL}_n)_n \to {}_n(e_{n+1}\mathrm{TL}_{n+2}e_{n+1})_n, \qquad x \mapsto [2]^{-2} e_{n+1} x e_{n+1},$$

$$\mathbf{F}\left(\ n \right) \colon {}_n(e_{n+1}\mathrm{TL}_{n+2}e_{n+1})_n \to {}_n(\mathrm{TL}_n)_n, \qquad x \mapsto \mathrm{tr}_n^{n+2}(x),$$

$$\mathbf{F}\left(\ n \right) \colon {}_{n+2}(\mathrm{TL}_{n+2})_n \to {}_{n+2}(\mathrm{TL}_{n+2}e_{n+1})_n, \qquad x \mapsto x e_{n+1},$$

$$\mathbf{F}\left(\ n \right) \colon {}_{n+2}(\mathrm{TL}_{n+2}e_{n+1})_n \to {}_{n+2}(\mathrm{TL}_{n+2})_n, \qquad x \mapsto x,$$

Above we have used the partial trace map $\mathrm{tr}_n^{n+k} \colon \mathrm{TL}_{n+k} \to \mathrm{TL}_n$,



The diagrammatic monoidal category

It is convenient to package all the induction and restriction functors together:

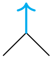
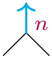
$$\mathrm{Ind} := \sum_{n\in\mathbb{N}} \mathrm{Ind}_n^{n+1}, \qquad \mathrm{Res} := \sum_{n\in\mathbb{N}} \mathrm{Res}_n^{n+1}.$$

Diagrammatically, we proceed as follows:

- Let $\mathcal{D}'$ be the strict monoidal category whose objects encode 1-endomorphisms of the formal sum $\bigoplus_{n\in\mathbb{N}} X_n$ (row-finite and column-finite infinite matrices).
- Let $\mathcal{D}$ be the full strict monoidal subcategory of $\mathcal{D}'$ generated by the objects

$$\mathtt{S}_+ := \sum_{n=0}^{\infty} \mathtt{S}_+ 1_n, \qquad \mathtt{S}_- := \sum_{n=1}^{\infty} \mathtt{S}_- 1_n, \qquad \mathtt{D}_+ := \sum_{n=0}^{\infty} \mathtt{D}_+ 1_n, \qquad \mathtt{D}_- := \sum_{n=2}^{\infty} \mathtt{D}_- 1_n.$$

A string diagram in $\mathfrak{D}$ with the region labels removed defines an element of $\mathcal{D}$ by summing over all labels of the rightmost region. For example

 $\in \mathcal{D}(\mathtt{S}_+\mathtt{S}_+, \mathtt{D}_+)$ is defined to be $\displaystyle\sum_{n\in\mathbb{N}}$ 

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Action on modules

$$\text{diagrammatic } \mathfrak{D} \text{ or } \mathcal{D} \xrightarrow{\mathbf{F}} \text{TL bimodules } \mathfrak{B} \xrightarrow{\mathbf{A}} \text{functors on } \bigoplus_{n\in\mathbb{N}} \mathrm{TL}_n\text{-mod}$$

The functor $\mathbf{A}$ is full by the Eilenberg–Watts Theorem.

The composition $\mathbf{AF}$ induces an action of $\mathcal{D}$ on $\mathcal{M}$, given on $M \in \mathrm{TL}_n$-mod by

$$\begin{aligned}
\mathtt{S}_+ \otimes M &= \mathbf{AF}(\mathtt{S}_+ 1_n)(M) = {}_{n+1}(\mathrm{TL}_{n+1})_n \otimes_n M, && n \in \mathbb{N},\\
\mathtt{S}_- \otimes M &= \mathbf{AF}(\mathtt{S}_- 1_n)(M) = {}_{n-1}(\mathrm{TL}_{n})_n \otimes_n M, && n \geq 1,\\
\mathtt{D}_+ \otimes M &= \mathbf{AF}(\mathtt{D}_+ 1_n)(M) = {}_{n+2}(\mathrm{TL}_{n+2} e_{n+1})_n \otimes_n M, && n \in \mathbb{N},\\
\mathtt{D}_- \otimes M &= \mathbf{AF}(\mathtt{D}_- 1_n)(M) = {}_{n-2}(e_{n-1}\mathrm{TL}_{n})_n \otimes_n M, && n \geq 2,
\end{aligned}$$

Action on modules

For $n, k \in \mathbb{N}$, $n - k \in 2\mathbb{N}$, we have

$$\mathtt{S}_+ \otimes L_k^n \cong L_{k+1}^{n+1} \oplus L_{k-1}^{n+1}, \qquad \mathtt{D}_+ \otimes L_k^n \cong L_k^{n+2},$$

$$\mathtt{S}_- \otimes L_k^n \cong L_{k+1}^{n-1} \oplus L_{k-1}^{n-1}, \quad n \geq 1, \qquad \mathtt{D}_- \otimes L_k^n \cong L_k^{n-2}, \quad n \geq 2.$$

Furthermore, $\mathbf{AF}(\circlearrowright_{r}\, n)$ and $\mathbf{AF}(\boxed{\sigma_r}\, n)$ are the natural transformations with components

$$\mathbf{AF}\left(\circlearrowright_{r}\, n\right)_{L_k^n} = \delta_{k \leq n-2r} 1_{L_k^n} \qquad \text{and} \qquad \mathbf{AF}\left(\boxed{\sigma_r}\, n\right)_{L_k^n} = \delta_{k,n-2r} 1_{L_k^n}.$$

$\boxed{\sigma_r}\, n$ corresponds to projection onto the isotypic component L_{n-2r}^n.

Decategorification

Theorem

The following isomorphisms hold in $\mathcal{D}$:

$$\mathtt{S}_+\mathtt{D}_+ \cong \mathtt{D}_+\mathtt{S}_+, \qquad \mathtt{S}_-\mathtt{D}_- \cong \mathtt{D}_-\mathtt{S}_-,$$
$$\mathtt{S}_-\mathtt{D}_+ \cong \mathtt{S}_+, \qquad \mathtt{D}_-\mathtt{S}_+ \cong \mathtt{S}_-.$$

In the additive envelope of $\mathcal{D}$, we also have

$$\mathtt{S}_-\mathtt{S}_+ \oplus \mathtt{D}_+\mathtt{D}_- \cong \mathtt{S}_+\mathtt{S}_- \oplus \mathbb{1}.$$

Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be the $\mathbb{Z}$-graded ring generated by s_-, s_+, d_+, d_-, with degrees

$$\deg(s_\pm) = \pm 1, \qquad \deg(d_\pm) = \pm 2,$$

and subject to the following relations:

$$s_+d_+ = d_+s_+, \qquad s_-d_- = d_-s_-,$$
$$d_-d_+ = 1, \qquad s_-d_+ = s_+, \qquad d_-s_+ = s_-, \qquad s_-s_+ = s_+s_- - d_+d_- + 1.$$

Decategorification: the ring and its polynomial module

Basis of R

We have an isomorphism of $\mathbb{Z}$-graded additive groups

$$R \cong \mathbb{Z}[s_+, d_+] \otimes_{\mathbb{Z}} \mathbb{Z}[s_-, d_-],$$

with the two factors corresponding to subrings of R.

Polynomial module

The polynomial module is the induced graded R-module

$$\mathrm{Poly} = \bigoplus_{n \in \mathbb{N}} \mathrm{Poly}_n := R \otimes_{\mathbb{Z}[s_-, d_-]} \text{triv}.$$

Chebyshev polynomials

The Chebyshev polynomials of the second kind are defined recursively by

$$U_{-1}(x) = 0, \qquad U_0(x) = 1, \qquad U_{k+1}(x) = xU_k(x) - U_{k-1}(x).$$

Define the homogenized Chebyshev polynomials

$$u_{n,k} := d_+^{n/2} U_k\left(\frac{s_+}{\sqrt{d_+}}\right) \in \mathrm{Poly}_n, \quad k \in \mathbb{N},\ n \in k + 2\mathbb{N}.$$

The elements

$$u_{n,k}, \qquad k \in \mathbb{N}, \quad n \in k + 2\mathbb{N},$$

form a basis for Poly. In this basis, the action of R on Poly is given by

$$\begin{aligned} d_+ u_{n,k} &= u_{n+2,k}, & s_+ u_{n,k} &= u_{n+1,k+1} + u_{n+1,k-1}, \\ d_- u_{n,k} &= u_{n-2,k}, & s_- u_{n,k} &= u_{n-1,k+1} + u_{n-1,k-1}, \end{aligned}$$

where $u_{n,k} := 0$ if $k < 0$ or $n \notin k + 2\mathbb{N}$. Compare to

$$\mathrm{Ind}_n^{n+1}(L_k^n) \cong L_{k+1}^{n+1} \oplus L_{k-1}^{n+1}, \qquad \mathrm{Res}_n^{n+1}(L_k^{n+1}) \cong L_{k+1}^n \oplus L_{k-1}^n.$$

Decategorification theorem

Let $K_\oplus(\mathcal{C})$ denote the split Grothendieck group of an additive category $\mathcal{C}$.

Theorem

We have an isomorphism of graded $\mathbb{Z}$-modules

$$\Phi \colon \mathrm{Poly} \to K_\oplus \left(\bigoplus_n \mathrm{TL}_n\text{-mod} \right), \qquad u_{n,k} \mapsto [L_k^n],$$

and an isomorphism of graded rings

$$\Psi \colon R \to K_\oplus(\mathrm{Add}(\mathcal{D})), \qquad s_\pm \mapsto [\mathtt{S}_\pm], \qquad d_\pm \mapsto [\mathtt{D}_\pm].$$

such that the following diagram commutes:

$$\begin{tikzcd}
R \times \mathrm{Poly} \arrow[r] \arrow[d, "\Psi\times\Phi"', "\cong"] & \mathrm{Poly} \arrow[d, "\Phi"', "\cong"] \\
K_\oplus(\mathrm{Add}(\mathcal{D})) \times K_\oplus \left(\bigoplus_n \mathrm{TL}_n\text{-mod} \right) \arrow[r] & K_\oplus \left(\bigoplus_n \mathrm{TL}_n\text{-mod} \right)
\end{tikzcd}$$

Idempotent completion

For a monoidal category $\mathcal{C}$, its additive Karoubi envelope (the idempotent completion of its additive envelope) $\mathrm{Kar}(\mathcal{C})$ has an object for each idempotent endomorphism in $\mathcal{C}$.

Example

If R is a ring, then Kar(category of f.g. free R-modules) $\simeq$ category of f.g. projective R-modules.

In most categorifications, one must pass to the Karoubi envelope to get the categorification statement:

- categorification of quantum groups,
- categorification of Heisenberg algebras.

Example

The Heisenberg category $\mathcal{Heis}$ is generated by two objects, $\uparrow$ and $\downarrow$. We have

$$\mathbb{C}S_n \subseteq \mathcal{Heis}(\uparrow^{\otimes n}, \uparrow^{\otimes n}),$$

and so the Young idempotents give an object in $\mathrm{Kar}(\mathcal{Heis})$ for each partition.

Idempotent completion

In our diagrammatic category $\mathcal{D}$, the situation is very different.

Proposition

The natural inclusion $\mathrm{Add}(\mathcal{D}) \hookrightarrow \mathrm{Kar}(\mathcal{D})$ induces an isomorphism of rings

$$K_\oplus(\mathrm{Add}(\mathcal{D})) \cong K_\oplus(\mathrm{Kar}(\mathcal{D})).$$

This makes the Grothendieck-ring computation much easier, since passing to the idempotent completion does not change the split Grothendieck ring

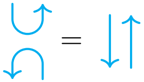
Why is idempotent completion unnecessary?

Why?

The fundamental ingredients of the idempotents in $\mathcal{D}$ are the bubbles:

which are idempotent:

However, the relation



implies that

$\mathtt{D}_+^r \mathtt{D}_-^r \cong$ object in $\mathrm{Kar}(\mathcal{D})$ corresponding to .

So, the objects in $\mathrm{Kar}(\mathcal{D})$ are already in $\mathrm{Add}(\mathcal{D})$!

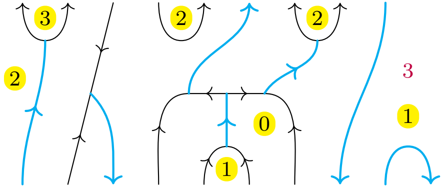
Remark

Previous work of Harper–Samuelson did not include the objects $\mathtt{D}_\pm$. So, it was necessary there to pass to the Karoubi envelope.

Basis theorem

We have explicit descriptions of bases for all 2-morphism spaces in the 2-category $\mathfrak{D}$.

Basis elements are indexed by bridges, which are closely related to Dyck paths. They look like:



, where $\quad \boxed{n-2r}\; n \; := \; (\sigma_r)\; n \quad .$

General idea:

- Forgetting all cyan strings, all orientations, and all $k\; n$ leaves a Temperley–Lieb diagram.
- There is a unique reduced way to add in the cyan strands so that orientations work out properly.
- There are rules about how one can add in the $k\; n$.
- The bridge is obtained by reading the $k\; n$ around the boundary of the diagram. It uniquely determines the basis element.

The incarnation functor is full and faithful

The basis theorem allows us to prove the following:

Theorem

The 2-functor

$$\mathrm{Kar}(\mathbf{F})\colon \mathrm{Kar}(\mathfrak{D}) \to \mathfrak{B}$$

induced by $\mathbf{F}$ is an 2-equivalence of 2-categories.

Result

This theorem implies that the 2-category $\mathfrak{D}$ is a complete diagrammatic calculus for bimodules over Temperley–Lieb algebras.

Summary

We defined a diagrammatic monoidal category $\mathcal{D}$ and functors

$$\text{diagrammatic } \mathcal{D} \xrightarrow{\mathbf{F}} \text{TL bimodules} \xrightarrow{\mathbf{A}} \text{functors on } \bigoplus_{n \in \mathbb{N}} \mathrm{TL}_n\text{-mod}$$

The functors $\mathbf{F}$ and $\mathbf{A}$ are full and faithful. $\mathrm{Kar}(\mathbf{F})$ is an equivalence.

We described $K_\oplus(\mathcal{D})$ and its action on $\bigoplus_n K_0(\mathrm{TL}_n\text{-mod})$ algebraically.

Classes of simples in TL_n-mod correspond to homogenized Chebyshev polynomials.

We have a basis theorem for 2-morphism spaces of $\mathfrak{D}$.

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