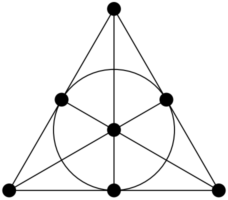
Mathematics at Play: Combinatorics and Geometry in Spot It!



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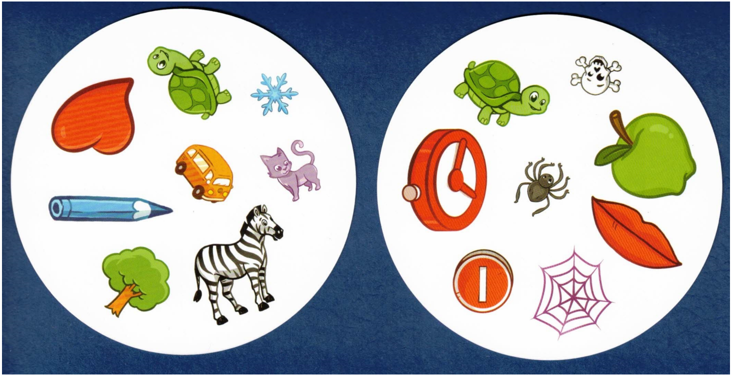
Slides: alistairsavage.ca/talks

Spot it!

What symbol do these two cards have in common?

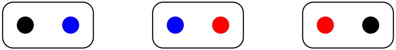


How does this work?

A very boring game

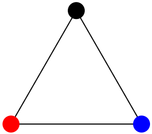
We could have three symbols: ●, ●, ●

Then we can have three cards, with two symbols on each card:

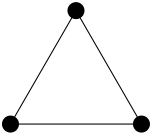


We can depict this in a different way:

- each symbol is a vertex,
- each card is a line through some vertices (the symbols on the card).



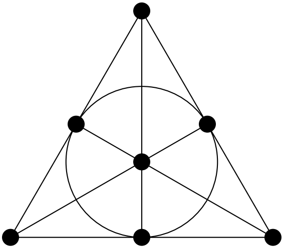
or, dropping colours,



A slightly boring game

Here's a game with

- 7 symbols
- 7 cards
- 3 symbols on each card



This is called the Fano plane.

Projective planes

Definition

A projective plane consists of a set of lines and a set of points, together with a relation between points and lines, called incidence, satisfying the following properties:

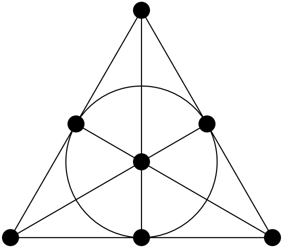
1. For any two distinct points, there is exactly one line incident with both of them.
2. For any two distinct lines, there is exactly one point incident with both of them.
3. There are four points such that no line is incident with more than two of them.

Remarks

- The definition is symmetric under the interchange of the words point and line.
- The last condition excludes "degenerate" cases.

The Fano plane (again)

The Fano plane is a projective plane.



1. For any two distinct points, there is exactly one line incident with both of them. ✓
2. For any two distinct lines, there is exactly one point incident with both of them. ✓
3. There are four points such that no line is incident with more than two of them. ✓

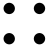
Constructing projective planes

Let's start with the Cartesian plane $\mathbb{R}^2$ with

- points being usual points of $\mathbb{R}^2$,
- lines being straight lines in $\mathbb{R}^2$.

Look at our conditions:

1. For any two distinct points, there is exactly one line incident with both of them. $\checkmark$
2. For any two distinct lines, there is exactly one point incident with both of them. Fail! Parallel lines!
3. There are four points such that no line is incident with more than two of them.



$\checkmark$

The problem of parallel lines

Image created with Dall-E 3.

Solution: Parallel lines intersect at infinity!

Adding infinity

Idea: Start with $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ and add "points at infinity".

Rule 1: All lines parallel to a given one should meet at the same point at infinity.

Image created with Dall-E 3.

Rule 2: The "point at infinity" in the forwards and backwards direction are the same.

Conclusion: We add one point at infinity θ_m for each slope m of a line in $\mathbb{R}^2$ (including infinite slope for vertical lines).

The real projective plane

The real projective plane is

$$\mathbb{RP}^2 = \mathbb{R}^2 \cup \{\theta_m : m \in \mathbb{R} \cup \{\infty\}\}.$$

Two distinct lines

$$y = m_1 x + b_1 \quad \text{and} \quad y = m_2 x + b_2$$

intersect at exactly one point:

- a point in $\mathbb{R}^2$ if $m_1 \neq m_2$ (linear algebra!),
- the point θ_{m_1} if $m_1 = m_2$.

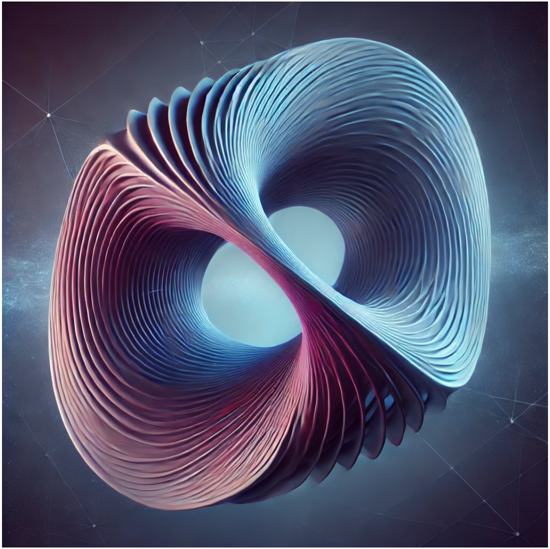
We also have the vertical lines ($m = \infty$)

$$x = a, \qquad a \in \mathbb{R}.$$

These meet the non-vertical lines at a point in $\mathbb{R}^2$ (linear algebra!) and they meet each other at θ_∞.

Note: There is also a line at infinity through all the points at infinity!

The real projective plane according to AI



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An infinite game of Spot It!

We now have a game of Spot It! with

- a symbol for each point in $\mathbb{RP}^2$,
- a card for each line in $\mathbb{RP}^2$.

With infinitely many symbols, this game would be very hard!

Spot It: Math Edition

Each card is the equation of a line in $\mathbb{R}^2$

The game is to flip over two cards and then find the point of intersection of the lines (which could be a point at infinity).

Another approach

Let's come with another way to create projective planes.

Let's think outside the box! The "points" of our projective plane don't need to be points in the usual sense.

Let's look inside $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$. We could say

- the projective points are *lines* in $\mathbb{R}^3$ through the origin,
- the projective lines are *planes* in $\mathbb{R}^3$ through the origin.

Look at the axioms of a projective plane:

1. For any two distinct projective points, there is exactly one projective line incident with both of them. ✓
2. For any two distinct projective lines, there is exactly one projective point incident with both of them. ✓
3. There are four projective points such that no projective line is incident with more than two of them. ✓

Projective coordinates

For $x, y, z \in \mathbb{R}$ not all zero, let

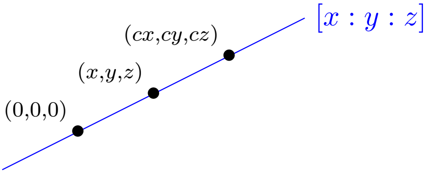
$$[x : y : z]$$

denote the projective point that is the line through the origin and (x, y, z).

Note that, for $c \neq 0$,

$$[x : y : z] = [cx : cy : cz]$$

since (x, y, z) and (cx, cy, cz) lie on the same line in $\mathbb{R}^3$ through the origin.



Projective coordinates

Let X denote the set of projective points $[x : y : z]$.

Claim: $X = \mathbb{RP}^2$

If $z \neq 0$, then we can rescale the projective point $[x : y : z]$ by z^{-1} so that the last coordinate is 1.

We have

$$[x_1 : y_1 : 1] = [x_2 : y_2 : 1] \iff x_1 = x_2 \text{ and } y_1 = y_2.$$

So we have an injective map

$$\mathbb{R}^2 \hookrightarrow X, \qquad (x, y) \mapsto [x : y : 1].$$

All the other points of X are of the form $[x : y : 0]$. We claim these are the points at infinity!

Points at infinity: a second look

Consider the line in $\mathbb{R}^2$ given by $y = mx + b$.

Under our injection $\mathbb{R}^2 \hookrightarrow X$, the points of this line are mapped to

$$[x : mx + b : 1].$$

Now let's go to infinity!

$$[x : mx + b : 1] = \left[1 : m + \tfrac{b}{x} : \tfrac{1}{x}\right] \xrightarrow{x \to \infty} [1 : m : 0].$$

$[1 : m : 0]$ is the point at infinity θ_m corresponding to lines with slope m.

We're missing vertical lines $x = a$. Under our injection $\mathbb{R}^2 \hookrightarrow X$, these points are mapped to

$$[a : y : 1] = \left[\tfrac{a}{y} : 1 : \tfrac{1}{y}\right] \xrightarrow{y \to \infty} [0 : 1 : 0].$$

$[0 : 1 : 0]$ is the point at infinity θ_∞ corresponding to vertical lines.

The projective plane: a second look

So we have

$$\begin{array}{rccccccc} \mathbb{RP}^2 & = & \mathbb{R}^2 & \cup & \{\theta_m : m \in \mathbb{R}\} & \cup & \{\theta_\infty\} \\ & & \updownarrow & & \updownarrow & & \updownarrow \\ & = & \{[x:y:1] : x, y \in \mathbb{R}\} & \cup & \{[1:m:0] : m \in \mathbb{R}\} & \cup & [0:1:0] \end{array}$$

In some sense,

$$\mathbb{RP}^2 = \mathbb{R}^2 \cup \mathbb{R}^1 \cup \mathbb{R}^0.$$

Back to card games

But we want finite projective planes!

Idea: Replace $\mathbb{R}$ by something finite!

We can do linear algebra over any field.

Definition

A field is a set F with two binary operations, addition and multiplication, such that

- addition and multiplication are associative: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- addition and multiplication are commutative: $a + b = b + a$ and $a \cdot b = b \cdot a$
- addition and multiplication have identities: $a + 0 = a$ and $a \cdot 1 = a$
- every $a \in F$ has an additive inverse: $a + (-a) = 0$
- every nonzero $a \in F$ has a multiplicative inverse: $a \cdot a^{-1} = 1$
- distributivity: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Fields

Examples of fields

$\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ are all fields.

Nonexamples

The following are not fields:

- $\mathbb{N}$ (e.g., no additive inverses)
- $\mathbb{Z}$ (no multiplicative inverses)

Finite fields

There are finite fields! For $n \in \mathbb{N}$, let

$$\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}.$$

Define addition and multiplication modulo n (only keep the remainder upon division by n).

Example: $n = 5$

In $\mathbb{Z}_5$,

$$2 \cdot 3 = 1, \quad 3 \cdot 3 = 4, \quad 4 + 4 = 3.$$

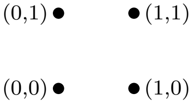
Proposition

$\mathbb{Z}_n$ is a field if and only if n is prime.

If n is prime, we can do linear algebra over $\mathbb{Z}_n$ and define finite projective planes.

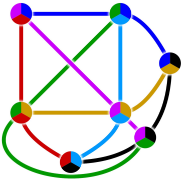
The projective plane of order 2

Let's work over the field $\mathbb{Z}_2$. The plane $\mathbb{Z}_2^2$ has 4 points:

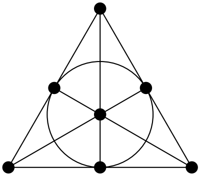


There are three possible slopes for lines 0, 1, ∞.

Adding in the 3 points at infinity and drawing all the lines in the projective plane, we get:

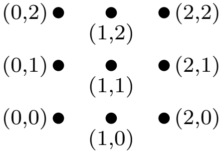


which is the Fano plane!



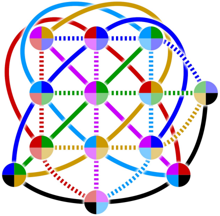
The projective plane of order 3

Now, let's work over $\mathbb{Z}_3$. The plane $\mathbb{Z}_3^2$ has 9 points:



There are four possible slopes for lines 0, 1, 2, ∞.

Adding in the 4 points at infinity and drawing all the lines in the projective plane, we get:



Counting projective points

Suppose n is prime.

Claim

The projective plane over $\mathbb{Z}_n$ has $n^2 + n + 1$ points.

Proof

Remember that we can view the points of the projective plane as lines through the origin in $\mathbb{Z}_n^3$.

Any line through the origin in $\mathbb{Z}_n^3$ is determined by a single nonzero point on the line. There are $n^3 - 1$ such points.

Two such points determine the same line iff they are nonzero scalar multiples of each other. The number of nonzero scalars is $n - 1$.

Thus, the number of lines through the origin in $\mathbb{Z}_n^3$ is

$$\frac{n^3 - 1}{n - 1} = n^2 + n + 1.$$

Counting projective lines

Claim

The projective plane over $\mathbb{Z}_n$ has $n^2 + n + 1$ lines (same as the number of points).

Proof

Recall that

- projective point = line through origin in $\mathbb{Z}_n^3$
- projective line = plane through origin in $\mathbb{Z}_n^3$

We have a natural bijection

$$\begin{aligned} \{\text{lines through origin in } \mathbb{Z}_n^3\} &\leftrightarrow \{\text{planes through origin in } \mathbb{Z}_n^3\}, \\ \text{line } L &\mapsto \text{plane orthogonal to } L, \\ \text{normal to } H &\leftarrow\!\shortmid \text{plane } H. \end{aligned}$$

Counting points on lines

Claim

1. Each projective line is incident to $n+1$ projective points.
2. Each projective point is incident to $n+1$ projective lines.

Proof

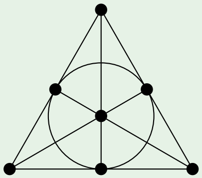
1. Each line in $\mathbb{Z}_n^2$ has n points, since there are n choices for the x coordinate (or the y coordinate if the line is vertical).

   Then we add one point at infinity.

2. Given a point, there are $n+1$ choices of the slope of a line passing through it:
   - n finite slopes,
   - plus one infinite slope (vertical line).

Examples

$n = 2$ (Fano plane)

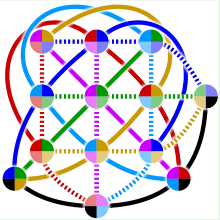


$7 = 2^2 + 2 + 1$ points (and lines!)

$3 = 2 + 1$ points on each line

$3 = 2 + 1$ lines through each point

$n = 3$



$13 = 3^2 + 3 + 1$ points (and lines!)

$4 = 3 + 1$ points on each line

$4 = 3 + 1$ lines through each point

Back to Spot It!

Spot It! corresponds to the finite projective plane over $\mathbb{Z}_7$

So, there should be

- $7^2 + 7 + 1 = 57$ symbols and cards,
- $7 + 1 = 8$ symbols on each card.

For some reason, the creators of Spot It! only created 55 cards.

Coordinatizing the Spot It! Plane: `https://ericmoorhouse.org/pg27/`

Spot It! Jr. corresponds to the finite projective plane over $\mathbb{Z}_5$

So, there should be

- $5^2 + 5 + 1 = 31$ symbols and cards,
- $5 + 1 = 6$ symbols on each card.

There are!

Extreme Spot It!

In fact, for any prime number p and any integer $k \geq 1$, there is a finite field with p^k elements. (But it is not $\mathbb{Z}_{p^k}$.)

So we can take n to be any power of a prime in our game design!

After 7, the next choice is $n = 8 = 2^3$. This game has

- $8^2 + 8 + 1 = 73$ cards,
- $8 + 1 = 9$ symbols on each card.

Next we have $n = 9 = 3^2$. This game has

- $9^2 + 9 + 1 = 91$ cards,
- $9 + 1 = 10$ symbols on each card.

Mathematical dessert

We started with a combinatorial definition of a projective plane.

Definition

A projective plane consists of a set of lines and a set of points, together with a relation between points and lines, called incidence, satisfying the following properties:

1. For any two distinct points, there is exactly one line incident with both of them.
2. For any two distinct lines, there is exactly one point incident with both of them.
3. There are four points such that no line is incident with more than two of them.

It can be shown that, for every projective plane, there is an integer $N \geq 2$ (the order of the plane) such that the plane has

- $N^2 + N + 1$ points and $N^2 + N + 1$ lines,
- $N + 1$ points on each line and $N + 1$ lines through each point.

Mathematical dessert

Using our finite field construction, we know there exists a projective plane of order $N = p^k$ for each prime power p^k.

All known finite projective planes have prime power order.

Open questions

- Do there exist projective planes whose order is not a prime power?
- Do there exist projective planes of prime order that are not finite field planes?

Generalization: Steiner systems

Definition

A Steiner system $S(t,k,n)$ is

- an n-element set S,
- a set of k-element subsets of S (called blocks),

such that

- each t-element subset of S is contained in exactly one block.

A finite projective plane of order N is an

$$S(2, N+1, N^2+N+1),$$

where

- S is the set of symbols,
- the blocks are the projective lines.

There is much more to learn!