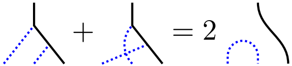
# The spin Brauer category

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Paper: arXiv:2312.11766

# Outline

Goal: Study the representation theory of the spin and pin groups using diagrammatic techniques.

Overview:

1. The Clifford algebra
2. The spin and pin groups and their modules
3. String diagrams for monoidal categories
4. The Brauer category
5. The spin Brauer category

# The Clifford algebra

Let $V$ be a finite-dimensional vector space of dimension $N$ and let

$$\Phi_V \colon V \times V \to \Bbbk$$

be a nondegenerate symmetric bilinear form.

Define

$$n = \left\lfloor \frac{N}{2} \right\rfloor \in \mathbb{N}, \quad \text{so that} \quad N = \begin{cases} 2n & \text{if } N \text{ is even,} \\ 2n+1 & \text{if } N \text{ is odd.} \end{cases}$$

Let

$$\mathrm{Cl} = \mathrm{Cl}(V) := T(V) / \left( vw + wv - 2\Phi_V(v,w) : v, w \in V \right)$$

denote the Clifford algebra associated to $V$. Here $T(V)$ is the tensor algebra on $V$.

# The Clifford algebra

Let $e_1, \dots, e_N$ be an orthonormal basis. For $1 \le j \le n$, define

$$\psi_j := \tfrac{1}{2}\left(e_{2j-1} + \sqrt{-1}e_{2j}\right), \qquad \psi_j^\dagger := \tfrac{1}{2}\left(e_{2j-1} - \sqrt{-1}e_{2j}\right).$$

Then we have

$$\Phi_V(\psi_i, \psi_j) = 0, \quad \Phi_V(\psi_i^\dagger, \psi_j^\dagger) = 0, \quad \Phi_V(\psi_i, \psi_j^\dagger) = \tfrac{1}{2}\delta_{ij}.$$

Hence

$$\psi_i\psi_j + \psi_j\psi_i = 0 = \psi_i^\dagger\psi_j^\dagger + \psi_j^\dagger\psi_i^\dagger, \qquad \psi_i\psi_j^\dagger + \psi_j^\dagger\psi_i = \delta_{ij}. \tag{1}$$

When $N$ is even, (1) gives a presentation of $\mathrm{Cl}$. When $N$ is odd, we need to include the additional relations

$$\psi_i e_N + e_N\psi_i = 0 = \psi_i^\dagger e_N + e_N\psi_i^\dagger, \qquad e_N^2 = 1,$$

to obtain a presentation of $\mathrm{Cl}$.

# The spin Clifford module

Let

$$S := \Lambda(W) = \bigoplus_{r=0}^{n} \Lambda^r(W), \qquad \text{where} \quad W = \mathrm{Span}_{\Bbbk}\{\psi_i^\dagger : 1 \le i \le n\}.$$

As a $\Bbbk$-module, $S$ has basis

$$x_I := \psi_{i_1}^\dagger \wedge \psi_{i_2}^\dagger \wedge \cdots \wedge \psi_{i_k}^\dagger,$$
$$I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad i_1 < i_2 < \dots < i_k.$$

In particular,

$$\dim_{\Bbbk}(S) = 2^n.$$

# The spin Clifford module

If $N$ is even, we turn $S$ into a $\mathrm{Cl}$-module by defining

$$\begin{aligned} \psi_i^\dagger x_J &= \psi_i^\dagger \wedge x_J, \\ \psi_i x_J &= \begin{cases} (-1)^{|\{j \in J \mid j < i\}|} x_{J \setminus \{i\}} & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases} \end{aligned} \tag{2}$$

If $N$ is odd, then we define two $\mathrm{Cl}$-module structures on $S$, depending on a choice of $\varepsilon \in \{\pm 1\}$.

We again use the action defined in (2) and additionally define

$$e_N x_I = \varepsilon (-1)^{|I|} x_I.$$

We call $S$ the spin module.

# The pin and spin groups

Define the pin group

$$\{v_1 v_2 \cdots v_k : k \in \mathbb{N},\ v_i \in V,\ \Phi_V(v_i, v_i) = 1\ \forall\ i\} \subseteq \mathrm{Cl}(V)^\times$$

and the spin group

$$\{v_1 v_2 \cdots v_k : k \in 2\mathbb{N},\ v_i \in V,\ \Phi_V(v_i, v_i) = 1\ \forall\ i\} \subseteq \mathrm{Pin}(V).$$

The group $\mathrm{Pin}(V)$ acts on $V$ by

$$g \cdot v = (-1)^{\deg g} g v g^{-1}.$$

This yields a short exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Pin}(V) \to \mathrm{O}(V) \to 1.$$

Restricting to $\mathrm{Spin}(V)$ yields another short exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to 1.$$

# The pin and spin groups

When $N$ is odd, $\mathrm{Pin}(V)$ is generated by $\mathrm{Spin}(V)$ and the central element $e_1 e_2 \cdots e_N$.

So the difference between the representation theory of $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ is not significant when $N$ is odd.

Define

$$\mathrm{G}(V) := \begin{cases} \mathrm{Pin}(V) & \text{if } N \text{ is even,} \\ \mathrm{Spin}(V) & \text{if } N \text{ is odd.} \end{cases}$$

**Goal**

Study the representation theory of $\mathrm{G}(V)$ using diagrammatic techniques.

# The spin and vector modules

## The spin module

Restriction of the $\mathrm{Cl}(V)$-action on $S$ gives natural actions of $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ on $S$.

We call this the spin module.

## The vector module

We view $V$ as a $\mathrm{Pin}(V)$-module with action

$$g \cdot v = gvg^{-1}.$$

We call this the vector module.

## Building blocks

All simple finite-dimensional $\mathrm{G}(V)$-modules are summands of tensor products of $S$.

# Bilinear form

Define a bilinear form on $S$ by

$$\Phi_S(x_I, x_J) = \begin{cases} (-1)^{\binom{|I|}{2} + nN|I| + |\{(i,j) \in I \times I^{\complement} : i > j\}|} & \text{if } J = I^{\complement}, \\ 0 & \text{otherwise.} \end{cases}$$

This form is $\mathrm{G}(V)$-invariant,

$$\Phi_S(gx, gy) = \Phi_S(x, y), \qquad g \in \mathrm{G}(V),\ x, y \in S,$$

and hence yields a homomorphism of $\mathrm{G}(V)$-modules

$$S \otimes S \to \text{trivial module.}$$

It is also (skew-)symmetric,

$$\Phi_S(x, y) = (-1)^{\binom{n}{2} + nN} \Phi_S(y, x).$$

# Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for objects $A$, $B$, $C$ and morphisms $f$, $g$, $h$,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

**Remark: Non-strict monoidal categories**

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.

# Linear monoidal categories

For simplicity, we work over the ground field $\mathbb{C}$.

A strict linear monoidal category is a strict monoidal category such that

- each morphism space is a $\mathbb{C}$-module,
- composition of morphisms is $\mathbb{C}$-bilinear,
- tensor product of morphisms is $\mathbb{C}$-bilinear.

## The interchange law

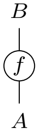
The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

$$\begin{array}{ccc} A_1 \otimes B_1 & \xrightarrow{1 \otimes g} & A_1 \otimes B_2 \\ {\scriptstyle f \otimes 1} \downarrow & \overset{f \otimes g}{\searrow} & \downarrow {\scriptstyle f \otimes 1} \\ A_2 \otimes B_1 & \xrightarrow[1 \otimes g]{} & A_2 \otimes B_2 \end{array}$$

# String diagrams

Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:

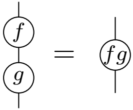


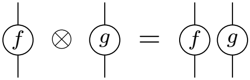

The identity map $1_A\colon A \to A$ is a string with no label:


We sometimes omit the object labels when they are clear or unimportant.

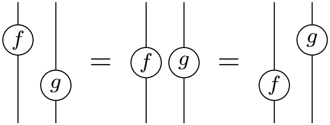
# String diagrams

Composition is vertical stacking and tensor product is horizontal juxtaposition:

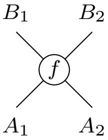



The interchange law then becomes:




A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



# The Brauer category

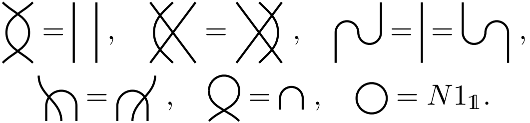
Fix $N \in \mathbb{C}$. The Brauer category $\mathcal{B}(N)$ is the strict linear monoidal category defined as follows.

One generating object: $\mathsf{I}$

Three generating morphisms:

$$\cup \colon \mathbb{1} \to \mathsf{I}^{\otimes 2}, \qquad \cap \colon \mathsf{I}^{\otimes 2} \to \mathbb{1}, \qquad \times \colon \mathsf{I}^{\otimes 2} \to \mathsf{I}^{\otimes 2}.$$

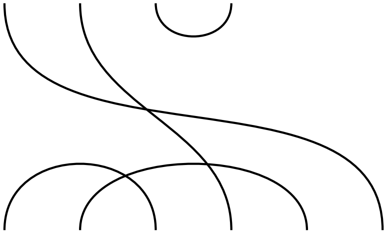
Relations:




(We'll motivate these relations soon.)

# The Brauer category

An arbitrary morphism in $\mathcal{B}(N)$ is a linear combination of Brauer diagrams. E.g.

 $: \mathsf{I}^{\otimes 6} \to \mathsf{I}^{\otimes 4}.$

Composition: vertical "gluing", replace closed components by a factor of $N$.

Tensor product: horizontal juxtaposition.

# The incarnation functor

## Universal property

Any linear symmetric monoidal category $\mathcal{C}$ with a (skew-)symmetrically self-dual object $V$ of dimension $N$ admits a linear monoidal functor

$$\mathcal{B}(N) \to \mathcal{C}, \quad | \mapsto V.$$

## Corollary

We have a linear monoidal incarnation functor

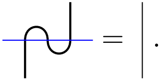
$$\mathcal{B}(N) \to \mathrm{O}(N)\text{-mod}, \qquad | \mapsto V = \text{natural module}.$$

$$\times \mapsto \left(V^{\otimes 2} \to V^{\otimes 2}, \quad v \otimes w \mapsto w \otimes v\right),$$
$$\cap \mapsto \left(V^{\otimes 2} \to \mathbb{C}, \quad v \otimes w \mapsto \langle v, w \rangle\right),$$
$$\cup \mapsto \left(\mathbb{C} \to V^{\otimes 2}, \quad 1 \mapsto \textstyle\sum_{v \in \mathbf{B}_V} v \otimes v\right),$$

where $\langle\ ,\ \rangle$ is the bilinear form, and $\mathbf{B}_V$ is an orthonormal basis.

# Understanding the relations

Nondegeneracy of the bilinear form gives the relation

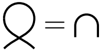


The left-hand side is the composition

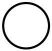
$$V \cong V \otimes \mathbb{C} \xrightarrow{1_V \otimes \bigcup} V \otimes V \otimes V \xrightarrow{\bigcap \otimes 1_V} \mathbb{C} \otimes V \cong V,$$

$$w \mapsto w \otimes 1 \mapsto \sum_{v \in \mathbf{B}_V} w \otimes v \otimes v \mapsto \sum_{v \in \mathbf{B}_V} \langle w, v \rangle \otimes v \mapsto \sum_{v \in \mathbf{B}_V} \langle w, v \rangle v = w.$$

Symmetry of the bilinear form gives the relation:



We also have



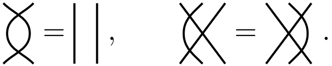
$$: \mathbb{C} \to \mathbb{C}, \quad 1 \mapsto \sum_{v \in \mathbf{B}_V} \langle v, v \rangle = \dim V.$$

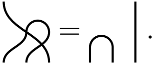
So the "bubble" corresponds to the dimension of the object $V$.

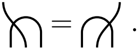
# Understanding the relations

The remaining relations come from the fact that the flip map endows the category with the structure of a symmetric monoidal category.

Flip satisfies the relations of the symmetric group:



Naturality gives 

Composing on the bottom with  gives 

## A small endomorphism space ($N \geq 4$)

We have $V = V_{\omega_1}$ and $V^{\otimes 2} = V_{2\omega_1} \oplus V_{\omega_2} \oplus \mathbb{C}$.

Then $\mathrm{End}_{\mathrm{O}(N)}(V^{\otimes 2})$ is 3-dimensional, spanned by

# Fullness

The incarnation functor is full. In particular, we have a surjection

$$\mathrm{B}_r(N) \cong \mathrm{End}_{\mathcal{B}(N)}(\mathsf{I}^{\otimes r}) \twoheadrightarrow \mathrm{End}_{\mathrm{O}(N)}(V^{\otimes r}),$$

where $\mathrm{B}_r(N)$ is the Brauer algebra.

What about surjectivity on objects?

**Fact**

Every f.d. $\mathrm{O}(N)$-module is a summand of $V^{\otimes r}$ for some $r$.

So we'd like to formally add in summands (and sums) of objects in the Brauer category.

# Additive envelope

The additive envelope $\mathrm{Add}(\mathcal{C})$ of a linear category $\mathcal{C}$ has:

Objects: Formal finite direct sums $\bigoplus_{i=1}^n X_i$, $X_i \in \mathcal{C}$.

Morphisms: Morphisms

$$\bigoplus_{i=1}^{n} X_i \to \bigoplus_{j=1}^{m} Y_j$$

are $m \times n$ matrices, where the $(j,i)$-entry is a morphism $f_{ij} \colon X_i \to Y_j$ in $\mathcal{C}$. Composition is matrix multiplication.

We can extend our incarnation functor to a functor

$$\mathrm{Add}(\mathcal{B}(n)) \to \mathrm{O}(n)\text{-mod}.$$

The objects in the image are direct sums of $V^{\otimes r}$, $r \in \mathbb{N}$.

# Idempotent completion

**Definition: idempotent completion**

The idempotent completion of a category $\mathcal{C}$ is the category whose

- objects are pairs $(A, e)$ where $A \in \operatorname{Ob}\mathcal{C}$ and $e$ is an idempotent $(e^2 = e)$ endomorphism of $A$, and
- morphisms from $(A, e)$ to $(B, f)$ are elements of

$$f \operatorname{Hom}_{\mathcal{C}}(A, B)e.$$

Intuition: One thinks of passing to the idempotent completion as adding in objects such that the idempotents correspond to projections onto direct summands.

$$A \cong X \oplus Y \twoheadrightarrow X \hookrightarrow X \oplus Y \cong A \qquad (\text{with } e : X \oplus Y \to X \oplus Y \text{ the composite})$$

Example: If $R$ is a ring, the idempotent completion of the category of free $R$-modules is the category of projective $R$-modules.

# The incarnation functor

Let $\mathrm{Kar}(\mathcal{B}(N))$ denote the idempotent completion of the additive envelope of $\mathcal{B}(N)$.

We have an induced incarnation functor

$$\mathrm{Kar}(\mathcal{B}(N)) \to \mathrm{O}(N)\text{-mod}.$$

This functor is

- full (surjective on morphism spaces) and
- essentially surjective (surjective on isomorphism classes of objects).

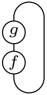
The category $\mathrm{Kar}(\mathcal{B}(N))$ is Deligne's interpolating category.

It is defined for any value of $N$, even $N \notin \mathbb{Z}$.

Question: What about the kernel of the induced incarnation functor?

# Negligible morphisms

A morphism $f\colon X \to Y$ is negligible if




$$= 0 \qquad \text{for all } g\colon Y \to X.$$

The negligible morphisms form a tensor ideal $\mathcal{I}$.

This ideal is the kernel of the induced incarnation functor

$$\mathrm{Kar}(\mathcal{B}(N)) \to \mathrm{O}(N)\text{-mod}.$$

Thus, we have an equivalence of categories

$$\mathrm{O}(N)\text{-mod} \simeq \mathrm{Kar}(\mathcal{B}(N))/\mathcal{I}.$$

The quotient $\mathrm{Kar}(\mathcal{B}(N))/\mathcal{I}$ is called the semisimplification of $\mathrm{Kar}(\mathcal{B}(N))$.

# What's missing?

Pulling back representations along the homomorphism

$$\mathrm{Pin}(V) \twoheadrightarrow \mathrm{O}(V)$$

yields a functor

$$\mathrm{O}(V)\text{-mod} \to \mathrm{Pin}(V)\text{-mod}.$$

Thus, we have a functor

$$\mathcal{B}(N) \to \mathrm{Pin}(V)\text{-mod}.$$

Problem: The above functor is no longer essentially surjective.

What's missing? The spin module!

**Goal**

Enlarge the Brauer category so that the corresponding incarnation functor to $\mathrm{Pin}(V)$-mod is essentially surjective. Add the spin module!

# The spin Brauer category

Fix $d, D \in \Bbbk$ and $\kappa \in \{\pm 1\}$.

The spin Brauer category $\mathcal{SB}(d, D; \kappa)$ is the strict $\Bbbk$-linear monoidal category presented as follows.

The generating objects are $\mathsf{S}$ and $\mathsf{V}$, with identity morphisms

 $:= 1_{\mathsf{S}}$,  $:= 1_{\mathsf{V}}$.

The generating morphisms are

 $: \mathsf{S} \otimes \mathsf{S} \to \mathbb{1}$,  $: \mathbb{1} \to \mathsf{S} \otimes \mathsf{S}$,

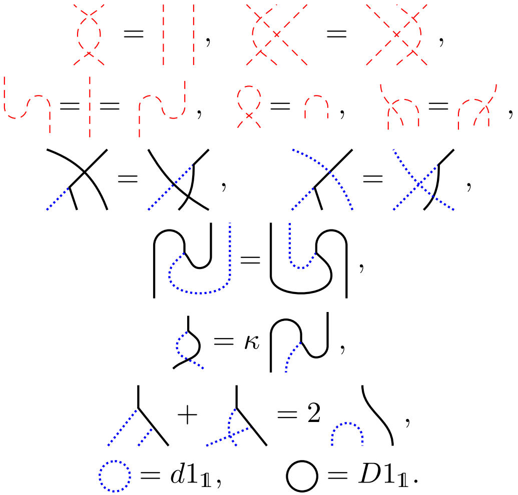
 $: \mathsf{V} \otimes \mathsf{V} \to \mathbb{1}$,  $: \mathbb{1} \to \mathsf{V} \otimes \mathsf{V}$,

 $: \mathsf{S} \otimes \mathsf{S} \to \mathsf{S} \otimes \mathsf{S}$,  $: \mathsf{V} \otimes \mathsf{V} \to \mathsf{V} \otimes \mathsf{V}$,

 $: \mathsf{V} \otimes \mathsf{S} \to \mathsf{S} \otimes \mathsf{V}$,  $: \mathsf{S} \otimes \mathsf{V} \to \mathsf{V} \otimes \mathsf{S}$,

 $: \mathsf{V} \otimes \mathsf{S} \to \mathsf{S}$.

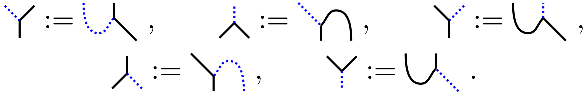
The defining relations on morphisms are:



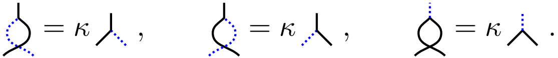

Here, the dashed red strands denote either | or ⋮.

# The spin Brauer category

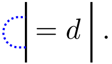
We introduce other trivalent morphisms by successive rotation:



Then one can show that



We can also show that



# The incarnation functor: ingredients

Fix an inner product space $(V, \Phi_V)$ of finite dimension $N$, and let

$$n = \left\lfloor \frac{N}{2} \right\rfloor, \qquad \text{so that} \quad N = 2n \quad \text{or} \quad N = 2n+1.$$

Recall

$$\mathrm{G}(V) := \begin{cases} \mathrm{Pin}(V) & \text{if } N \text{ is even,} \\ \mathrm{Spin}(V) & \text{if } N \text{ is odd.} \end{cases}$$

Let

$$\sigma_N := (-1)^{\binom{n}{2} + nN} \qquad \text{and} \qquad \kappa_N := (-1)^{nN},$$
$$\mathcal{SB}(V) := \mathcal{SB}(\underbrace{N}_{d}, \underbrace{\sigma_N 2^n}_{D}; \kappa_N).$$

(Recall that $\sigma_N$ is the sign describing the symmetry of the form $\Phi_S$.)

# The incarnation functor: ingredients

Fix a basis $\mathbf{B}_S$ of $S$, and let $\mathbf{B}_S^\vee = \{x^\vee : x \in \mathbf{B}_S\}$ denote the left dual basis with respect to $\Phi_S$, defined by

$$\Phi_S(x^\vee, y) = \delta_{x,y}, \qquad x, y \in \mathbf{B}_S.$$

We fix a basis $\mathbf{B}_V$ of $V$ and define the left dual basis $\mathbf{B}_V^\vee = \{v^\vee : v \in V\}$ similarly.

Then we have $\mathrm{G}(V)$-module homomorphisms

$$\Phi_S^\vee \colon \mathbb{C} \to S \otimes S, \qquad \lambda \mapsto \lambda \sum_{x \in \mathbf{B}_S} x \otimes x^\vee, \quad \lambda \in \mathbb{C},$$

$$\Phi_V^\vee \colon \mathbb{C} \to V \otimes V, \qquad \lambda \mapsto \lambda \sum_{v \in \mathbf{B}_V} v \otimes v^\vee, \quad \lambda \in \mathbb{C}.$$

Let

$$\tau \colon V \otimes S \to S, \quad v \otimes x \mapsto vx,$$

denote the homomorphism of $\mathrm{G}(V)$-modules induced by multiplication in the Clifford algebra $\mathrm{Cl}(V)$.

# The incarnation functor

**Theorem (McNamara–S.)**

There is a unique monoidal functor

$$\mathbf{F} \colon \mathcal{SB}(V) \to \mathrm{G}(V)\text{-mod}$$

given on objects by S $\mapsto S$, V $\mapsto V$, and on morphisms by

 $\mapsto \Phi_S$,  $\mapsto \Phi_V$,  $\mapsto \tau$,

 $\mapsto \sigma_N \, \mathrm{flip}_{S,S}$,  $\mapsto \mathrm{flip}_{S,V}$,  $\mapsto \mathrm{flip}_{V,S}$,  $\mapsto \mathrm{flip}_{V,V}$ .

Furthermore, we have

 $\mapsto \Phi_S^\vee$,  $\mapsto \Phi_V^\vee$.

We call $\mathbf{F}$ the incarnation functor.

# Properties of the incarnation functor

**Theorem (McNamara–S.)**

1. The functor $\mathbf{F}$ is full (surjective on morphism spaces).
2. After passing to the Karoubi envelope (formally adding in summands of objects), $\mathbf{F}$ is essentially surjective (surjective on isomorphism classes of objects).
3. When $N$ is even, the kernel of $\mathbf{F}$ is the tensor ideal of negligible morphisms. Thus, $\mathrm{G}(V)$-mod is equivalent to the semisimplification of $\mathrm{Kar}(\mathcal{SB}(V))$.
4. When $N$ is odd, the same is true if we impose one additional relation in $\mathcal{SB}(V)$.

# Incarnation

## Corollary

When $N$ is even, we have an equivalence of categories

$$\mathrm{G}(V)\text{-mod} \simeq \mathrm{Kar}(\mathcal{SB}(V))/\mathcal{I},$$

where $\mathcal{I}$ is the tensor ideal of negligible morphisms of $\mathrm{Kar}(\mathcal{SB}(V))$.

When $N$ is odd, we have an analogous statement after we add in the additional relation.

## More incarnation!

There exist other possible incarnation functors:

$$\mathcal{SB}(N, \sigma_N(m-2k)2^n; \kappa_N) \to (\mathrm{G}(V) \times \mathrm{OSp}(m|2k))\text{-mod},$$
$$\mathsf{S} \mapsto S \otimes W, \qquad \mathsf{V} \mapsto V,$$

where $W$ is the natural $\mathrm{OSp}(m|2k)$-supermodule.

# The affine spin Brauer category

There exists an affine spin Brauer category $\mathcal{ASB}(d, D; \kappa)$, obtained from the spin Brauer category by adding morphisms

 $: \mathsf{S} \to \mathsf{S}$,     $: \mathsf{V} \to \mathsf{V}$,

subject to additional relations.

Then we have an affine incarnation functor

$$\mathcal{ASB}(V) := \mathcal{ASB}(N, \sigma_N 2^n; \kappa_N) \to \mathcal{End}(\mathrm{G}(V)\text{-mod}),$$
$$\mathsf{S} \mapsto S \otimes -, \qquad \mathsf{V} \mapsto V \otimes -.$$

This induces an algebra homomorphism

$$\mathrm{End}_{\mathcal{ASB}(V)}(\mathbb{1}) \twoheadrightarrow \mathcal{End}(\mathrm{id}) \cong Z(U(\mathfrak{so}(V)))$$

whose image is $Z(U(\mathfrak{so}(V)))^{\mathrm{G}(V)}$.

# Further directions

## Basis theorem

It would be nice to describe an explicit basis for the morphism spaces of the spin Brauer category and the affine spin Brauer category.

## Description of the kernel

We know the kernel of the incarnation functor is the tensor ideal of negligible morphisms.

It would be nice to find explicit generators for this tensor ideal.

Such a description is known for the Brauer category.

# Quantum version

## Kauffman skein category

The Kauffman skein category is a quantum version of the Brauer category.

Its endomorphism algebras are BMW algebras.

There is a natural functor to $U_q(\mathfrak{so}(N))$-mod.

But this functor is not essentially surjective; it misses the quantum spin module.

There should exist a quantum version of the spin Brauer category—a spin Kauffman skein category.