Diagrammatics for real supergroups

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Outline

Goal: Develop a simple and intuitive graphical calculus for real representations of real supergroups

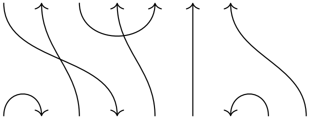
Overview:

1. Background: oriented and unoriented Brauer categories
2. Motivation: Schur's lemma and division superalgebras
3. Superhermitian forms
4. Real Lie superalgebras
5. Graphical calculus

The oriented Brauer category

The oriented Brauer category $\mathcal{OB}(d)$ is the free rigid symmetric $\mathbb{C}$-linear monoidal category on a generating object $\uparrow$ of dimension d.

Morphisms are linear combinations of oriented Brauer diagrams:



There is a full monoidal functor

$$\mathcal{OB}(m) \to \mathrm{GL}(m,\mathbb{C})\text{-mod}, \qquad \uparrow \mapsto V = \mathbb{C}^m.$$

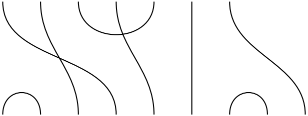
In particular, there is a surjective algebra homomorphism (half of Schur–Weyl duality)

$$\mathbb{C}\mathfrak{S}_r \cong \mathrm{End}_{\mathcal{OB}(m)}(\uparrow^{\otimes r}) \twoheadrightarrow \mathrm{End}_{\mathrm{GL}(m,\mathbb{C})}(V^{\otimes r}).$$

The unoriented Brauer category

The unoriented Brauer category $\mathcal{B}(d)$ is the free rigid symmetric $\mathbb{C}$-linear monoidal category on a symmetrically self-dual object I of dimension d.

Morphisms are linear combinations of unoriented Brauer diagrams:



There are full monoidal functors

$$\mathcal{B}(m) \to \mathrm{O}(m, \mathbb{C})\text{-mod} \qquad \text{and} \qquad \mathcal{B}(-2m) \to \mathrm{Sp}(2m, \mathbb{C})\text{-mod}.$$

Here the endomorphism algebras are Brauer algebras.

Observations

Super unifies

In fact, there are full functors

$$\mathcal{OB}(m-n) \to \mathrm{GL}(m|n,\mathbb{C})\text{-smod} \qquad \text{and}$$
$$\mathcal{B}(m-2n) \to \mathrm{OSp}(m|2n,\mathbb{C})\text{-smod}.$$

Trivial yet important observation

Functors induce isomorphisms

$$\mathbb{C} \cong \mathrm{Span}_{\mathbb{C}}\{\uparrow\} = \mathrm{End}_{\mathcal{OB}(m-n)}(\uparrow) \xrightarrow{\cong} \mathrm{End}_{\mathrm{GL}(m|n,\mathbb{C})}(V)$$

and

$$\mathbb{C} \cong \mathrm{Span}_{\mathbb{C}}\{|\} = \mathrm{End}_{\mathcal{B}(m-2n)}(|) \xrightarrow{\cong} \mathrm{End}_{\mathrm{OSp}(m|2n,\mathbb{C})}(V)$$

Schur's lemma

Fix a ground field $\mathbb{k}$. All supermodules are assumed to be finite dimensional over $\mathbb{k}$.

Let R be an associative superalgebra or Lie superalgebra over $\mathbb{k}$.

Schur's lemma

If V is a simple R-supermodule, then $\mathrm{End}_R(V)$ is a finite-dimensional division $\mathbb{k}$-superalgebra.

Non-super world

- There is one complex division algebra: $\mathbb{C}$.
- There are three real division algebras: $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

Complex division superalgebras

If $\Bbbk = \mathbb{C}$, then there are two complex division superalgebras:

- the complex numbers $\mathbb{C}$,
- the complex Clifford superalgebra $\mathrm{Cl}(\mathbb{C}) := \mathbb{C} \oplus \varepsilon\mathbb{C}$, with $\bar{\varepsilon} = 1$,

$$\varepsilon^2 = -1 \qquad \text{and} \qquad z\varepsilon = \varepsilon z \quad \forall\, z \in \mathbb{C}.$$

Consequence

When $\Bbbk = \mathbb{C}$, there are two types of simple supermodule V over a superalgebra R:

- Type M: $\mathrm{End}_R(V) = \mathbb{C}$,
- Type Q: $\mathrm{End}_R(V) = \mathrm{Cl}(\mathbb{C})$.

Real division superalgebras

Theorem (Wall 1964)

Every real division superalgebra is isomorphic to exactly one of the following, where $\bar{\varepsilon} = 1$, and $\star$ denotes complex conjugation:

- $\mathrm{Cl}_0(\mathbb{R}) = \mathbb{R}$;
- $\mathrm{Cl}_1(\mathbb{R}) := \mathbb{R} \oplus \varepsilon\mathbb{R}$, with $\varepsilon^2 = 1$;
- $\mathrm{Cl}_2(\mathbb{R}) := \mathbb{C} \oplus \varepsilon\mathbb{C}$, with $\varepsilon^2 = 1$ and $z\varepsilon = \varepsilon z^\star$ for all $z \in \mathbb{C}$;
- $\mathrm{Cl}_3(\mathbb{R}) := \mathbb{H} \oplus \varepsilon\mathbb{H}$, with $\varepsilon^2 = -1$ and $z\varepsilon = \varepsilon z$ for all $z \in \mathbb{H}$;
- $\mathrm{Cl}_4(\mathbb{R}) := \mathbb{H}$;
- $\mathrm{Cl}_5(\mathbb{R}) := \mathbb{H} \oplus \varepsilon\mathbb{H}$, with $\varepsilon^2 = 1$ and $z\varepsilon = \varepsilon z$ for all $z \in \mathbb{H}$;
- $\mathrm{Cl}_6(\mathbb{R}) := \mathbb{C} \oplus \varepsilon\mathbb{C}$, with $\varepsilon^2 = -1$ and $z\varepsilon = \varepsilon z^\star$ for all $z \in \mathbb{C}$;
- $\mathrm{Cl}_7(\mathbb{R}) := \mathbb{R} \oplus \varepsilon\mathbb{R}$, with $\varepsilon^2 = -1$;
- $\mathbb{C}$;
- $\mathrm{Cl}(\mathbb{C})$.

Remarks

The $\mathrm{Cl}_r(\mathbb{R})$, $0 \leq r \leq 7$, are real Clifford superalgebras. They are the only central real division superalgebras (i.e. with even center $\mathbb{R}$).

The notation $\mathrm{Cl}_r(\mathbb{R})$ is inspired by the fact that (subscripts mod 8)

$$\mathrm{Cl}_r(\mathbb{R}) \otimes \mathrm{Cl}_s(\mathbb{R}) \quad \text{is Morita equivalent to} \quad \mathrm{Cl}_{r+s}(\mathbb{R}).$$

The opposite superalgebra of an associative superalgebra A is

$$A^{\mathsf{op}} := \{a^{\mathsf{op}} : a \in A\}$$

with multiplication

$$a^{\mathsf{op}} b^{\mathsf{op}} = (-1)^{\bar{a}\bar{b}} (ba)^{\mathsf{op}}.$$

We have

- $\mathrm{Cl}_r(\mathbb{R})^{\mathsf{op}} \cong \mathrm{Cl}_{-r}(\mathbb{R})$, with subscripts considered modulo 8,
- $\mathrm{Cl}(\mathbb{C})^{\mathsf{op}} \cong \mathrm{Cl}(\mathbb{C})$, $\varepsilon \mapsto \varepsilon i$,
- $\mathbb{C}^{\mathsf{op}} \cong \mathbb{C}$.

Motivating idea

The tenfold way

Suppose R is a real associative superalgebra or a real Lie superalgebra.

There are ten types of simple R-supermodule:

$$\mathrm{End}_R(V) \in \{\mathrm{Cl}_r(\mathbb{R}), \mathbb{C}, \mathrm{Cl}(\mathbb{C}) : 0 \le r \le 7\}.$$

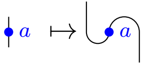
We want to modify the oriented Brauer category so that

$$\mathrm{End}(\uparrow) \text{ is a division superalgebra } A.$$

This amounts to adding morphisms

$$, \qquad a \in A.$$

For the unoriented Brauer category, we'll need an anti-involution on A corresponding to



Real general linear Lie superalgebras

Let's look at $\mathfrak{gl}(m|n, \mathbb{D})$ for $\mathbb{D}$ a real division superalgebra.

Simplification

- If $\mathbb{D}_1 \neq 0$, then $\mathfrak{gl}(m|n, \mathbb{D}) \cong \mathfrak{gl}(m+n, \mathbb{D})$.
- We have $\mathfrak{gl}(m|n, \mathbb{D}) \cong \mathfrak{gl}(m|n, \mathbb{D}^{\mathrm{op}})$.

Thus, the general linear Lie superalgebras over real division superalgebras are:

- $\mathfrak{gl}(m, \mathrm{Cl}_1(\mathbb{R})) = \mathfrak{q}(m, \mathbb{R})$ is the split real isomeric Lie superalgebra (a.k.a. the split real queer Lie superalgebra),
- $\mathfrak{gl}(m, \mathrm{Cl}(\mathbb{C})) = \mathfrak{q}(m, \mathbb{C})$ is the complex isomeric Lie superalgebra,
- $\mathfrak{gl}(m, \mathrm{Cl}_2(\mathbb{R}))$,
- $\mathfrak{gl}(m, \mathrm{Cl}_3(\mathbb{R}))$,
- $\mathfrak{gl}(m|n, \mathbb{R})$, $\mathfrak{gl}(m|n, \mathbb{C})$, $\mathfrak{gl}(m|n, \mathbb{H})$.

Complexification of general linear Lie superalgebras

The complexifications of all central real division superalgebras are

$$\mathbb{R}^{\mathbb{C}} \cong \mathbb{C}, \qquad \mathbb{H}^{\mathbb{C}} \cong \mathrm{Mat}_2(\mathbb{C}),$$
$$\mathrm{Cl}_1(\mathbb{R})^{\mathbb{C}} \cong \mathrm{Cl}_7(\mathbb{R})^{\mathbb{C}} \cong \mathrm{Cl}(\mathbb{C}),$$
$$\mathrm{Cl}_2(\mathbb{R})^{\mathbb{C}} \cong \mathrm{Cl}_6(\mathbb{R})^{\mathbb{C}} \cong \mathrm{Mat}_{1|1}(\mathbb{C}),$$
$$\mathrm{Cl}_3(\mathbb{R})^{\mathbb{C}} \cong \mathrm{Cl}_5(\mathbb{R})^{\mathbb{C}} \cong \mathrm{Mat}_2(\mathrm{Cl}(\mathbb{C})).$$

If $\mathbb{D}$ is a real division superalgebra, we have

$$\mathfrak{gl}(m|n, \mathbb{D})^{\mathbb{C}} \cong \mathfrak{gl}(m|n, \mathbb{D}^{\mathbb{C}}).$$

Hence

- $\mathfrak{gl}(m|n, \mathbb{R})^{\mathbb{C}} \cong \mathfrak{gl}(m|n, \mathbb{C})$,
- $\mathfrak{gl}(m|n, \mathbb{H})^{\mathbb{C}} \cong \mathfrak{gl}(2m|2n, \mathbb{C})$,
- $\mathfrak{gl}(m, \mathrm{Cl}_1(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(m, \mathrm{Cl}_7(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(m, \mathrm{Cl}(\mathbb{C})) = \mathfrak{q}(m, \mathbb{C})$,
- $\mathfrak{gl}(m, \mathrm{Cl}_2(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(m, \mathrm{Cl}_6(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(m|m, \mathbb{C})$,
- $\mathfrak{gl}(m, \mathrm{Cl}_3(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(m, \mathrm{Cl}_5(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(2m, \mathrm{Cl}(\mathbb{C})) = \mathfrak{q}(2m, \mathbb{C})$.

Motivation for anti-involutions

Fix an associative $\mathbb{R}$-superalgebra A and a right A-supermodule V.

Then the dual $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ is a left A-supermodule, with action

$$(af)(v) = (-1)^{\bar{a}\bar{f} + \bar{a}\bar{v}} f(va), \quad a \in A,\ f \in V^*,\ v \in V.$$

We want to examine the situation where V is self dual:

$$V \cong V^* \qquad \text{as right } A\text{-supermodules.}$$

In order for this to make sense, we need to turn V^* into a right A-supermodule.

Recall that a right A-supermodule is the same as a left A^{op}-supermodule.

So, if we have an isomorphism $A^{\mathsf{op}} \cong A$, we can convert left A-supermodules into right A-supermodules.

Involutive superalgebras

Definition

An involutive superalegbra is a pair $(A, \star)$, where

- A is an associative superalgebra, and
- $\star\colon A \to A$, $a \mapsto a^\star$, is an anti-involution:

$$(a^\star)^\star = a, \qquad (ab)^\star = (-1)^{\bar{a}\bar{b}} b^\star a^\star.$$

An anti-involution $\star$ gives an isomorphism $A^{\mathsf{op}} \cong A$.

So, if V is a right supermodule over an involutive superalgebra $(A, \star)$, then V^* is a right A-supermodule via

$$(fa)(v) = (-1)^{\bar{a}\bar{v}} f(va^\star), \quad a \in A,\ f \in V^*,\ v \in V.$$

Involutive real division superalgebras

Recall that

$$\mathbb{C}^{\mathsf{op}} \cong \mathbb{C}, \quad \mathrm{Cl}(\mathbb{C})^{\mathsf{op}} \cong \mathrm{Cl}(\mathbb{C}), \quad \mathrm{Cl}_r(\mathbb{R})^{\mathsf{op}} \cong \mathrm{Cl}_{-r}(\mathbb{R}).$$

So the real division superalgebras admitting anti-involutions are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathrm{Cl}(\mathbb{C})$.

In particular, we have

- $(\mathbb{R}, \mathrm{id})$,
- $(\mathbb{C}, \mathrm{id})$,
- $(\mathbb{C}, \star)$, where $\star$ is complex conjugation,
- $(\mathbb{H}, \star)$, where $\star$ is quaternionic conjugation,
- $(\mathrm{Cl}(\mathbb{C}), \star)$, where

$$(a + \varepsilon bi)^\star = a^\star + \varepsilon b^\star i, \quad a, b \in \mathbb{C}.$$

Superhermitian forms

Let V be a right supermodule over an involutive real division superalgebra $(\mathbb{D}, \star)$.

Definition

Let $\nu \in \{\pm 1\}$. A $(\nu, \star)$-superhermitian form on V is a homogeneous $\mathbb{R}$-bilinear map

$$\varphi \colon V \times V \to \mathbb{D}$$

such that

- $\varphi(va, wb) = (-1)^{\bar{a}(\bar{\varphi}+\bar{v})} a^\star \varphi(v,w) b$ for all $a, b \in A$, $v, w \in V$,
- $\varphi(v,w) = \nu(-1)^{\bar{v}\bar{w}} \varphi(w,v)^\star$ for all $v, w \in V$.

Remarks

- A superhermitian form gives an isomorphism $V \cong V^*$.
- A superhermitian form can be even ($\bar{\varphi} = 0$) or odd ($\bar{\varphi} = 1$).

Examples

Assume everything is even (i.e., all odd parts are zero).

Example

If $(\mathbb{D}, \star) = (\mathbb{C}, \mathrm{id})$, then

- an $(1, \mathrm{id})$-superhermitian form is a symmetric form,
- an $(-1, \mathrm{id})$-superhermitian form is skew-symmetric form.

Example

If $(\mathbb{D}, \star) = (\mathbb{C}, \star)$, where $\star$ is complex conjugation, then

- a $(1, \star)$-superhermitian form is a hermitian form in the usual sense,
- a $(-1, \star)$-superhermitian form is a skew-hermitian form.

Lie superalgebras associated to a superhermitian form

Suppose

- $(\mathbb{D}, \star)$ is an involutive real division superalgebra,
- φ is a $(\nu, \star)$-superhermitian form.

Let

$$\mathfrak{g}(\varphi) = \{X \in \mathfrak{gl}(V) : \varphi(Xv, w) = -(-1)^{\bar{X}\bar{v}}\varphi(v, Xw)\ \forall\ v, w \in V\}.$$

be the Lie sub-superalgebra of $\mathfrak{gl}(V)$ preserving φ.

Let $G(\varphi)$ be the supergroup preserving φ.

Real forms

A real Lie superalgebra $\mathfrak{g}$ is a real form of the complex Lie superalgebra $\mathfrak{g}^{\mathbb{C}}$.

Example ($\Bbbk = \mathbb{C}$)

The Lie superalgebras of the form $\mathfrak{g}(\varphi)$ are:

- the orthosymplectic Lie superalgebras $\mathfrak{osp}(m|2n, \mathbb{C})$ (when φ is even),
- the periplectic Lie superalgebras $\mathfrak{p}(m, \mathbb{C})$ (when φ is odd).

Now suppose $\Bbbk = \mathbb{R}$. We have

- the Lie superalgebras $\mathfrak{gl}(m|n, \mathbb{D})$ for a real division superalgebra $\mathbb{D}$,
- the Lie superalgebras $\mathfrak{g}(\varphi)$ for φ a $(\nu, \star)$-superhermitian form for an involutive real division superalgebra $(\mathbb{D}, \star)$.

These correspond to all real forms of the complex Lie superalgebras

$$\mathfrak{gl}(m|n, \mathbb{C}), \quad \mathfrak{osp}(m|2n, \mathbb{C}), \quad \mathfrak{p}(m, \mathbb{C}), \quad \mathfrak{q}(m, \mathbb{C}).$$

Strict monoidal supercategories

Fix a ground field $\Bbbk$.

A strict monoidal supercategory is a category $\mathcal{C}$ enriched in the category of vector superspaces:

- each morphism space is a $\Bbbk$-supermodule,
- composition of morphisms is $\Bbbk$-bilinear and parity preserving,

together with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

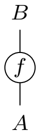
such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$,
- tensor product of morphisms is $\Bbbk$-bilinear and parity preserving.

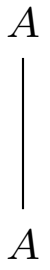
String diagrams

Fix a strict monoidal supercategory $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:



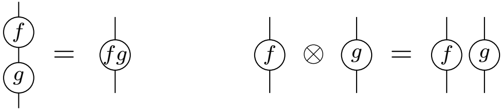
The identity map $1_A\colon A \to A$ is a string with no label:



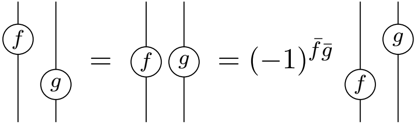
We sometimes omit the object labels when they are clear.

String diagrams

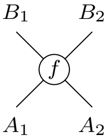
Composition is vertical stacking and tensor product is horizontal juxtaposition:



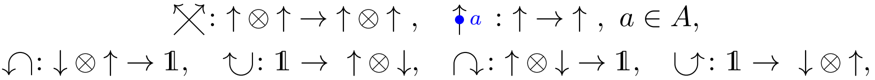
The super interchange law is:



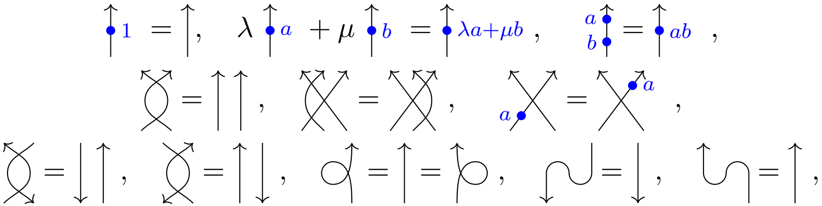
A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



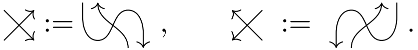
For an associative superalgebra A, we define $\mathcal{OB}_{\Bbbk}(A)$ to be the strict monoidal supercategory generated by objects $\uparrow$ and $\downarrow$ and morphisms



subject to the relations



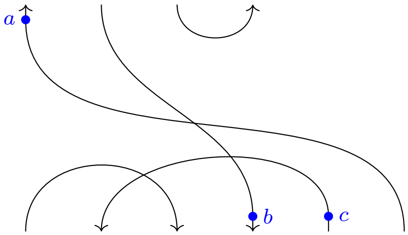
for all $a, b \in A$ and $\lambda, \mu \in \Bbbk$. In the above, the left and right crossings are defined by



The parity of  is $\bar{a}$, and all the other generating morphisms are even.

The oriented supercategory

Morphisms in $\mathcal{OB}_\Bbbk(A)$ are $\Bbbk$-linear combinations of diagrams such as



$a, b, c \in A.$

Composition is vertical stacking; tensor product is horizontal juxtaposition.

Example

- $\mathcal{OB}_\Bbbk(\Bbbk)$ is the oriented Brauer category.
- $\mathcal{OB}_\mathbb{C}(\mathrm{Cl}(\mathbb{C}))$ is the oriented Brauer–Clifford supercategory (Brundan–Comes–Kujawa).

The oriented incarnation superfunctor

Suppose that $\mathbb{D}$ is a real division superalgebra. Let $V = \mathbb{D}^{m|n}$.

Theorem (Samchuck–Schnarch–S.)

There exists a unique monoidal superfunctor

$$\mathsf{G}\colon \mathcal{OB}_{\mathbb{R}}(\mathbb{D}^{\mathrm{op}}) \to \mathfrak{gl}(m|n,\mathbb{D})\text{-smod}$$

such that $\mathsf{G}(\uparrow) = V$, $\mathsf{G}(\downarrow) = V^*$, and

$$\begin{aligned}
\mathsf{G}(\nwarrow\!\!\!\!\nearrow)&\colon V \otimes V \to V \otimes V, & v \otimes w &\mapsto (-1)^{\bar{v}\bar{w}} w \otimes v,\\
\mathsf{G}(\curvearrowleft)&\colon V^* \otimes V \to \mathbb{R}, & f \otimes v &\mapsto f(v),\\
\mathsf{G}(\uparrow\!\!\bullet_{a^{\mathrm{op}}})&\colon V \to V, & v &\mapsto (-1)^{\bar{a}\bar{v}} va.
\end{aligned}$$

The superfunctor G is full.

Remark

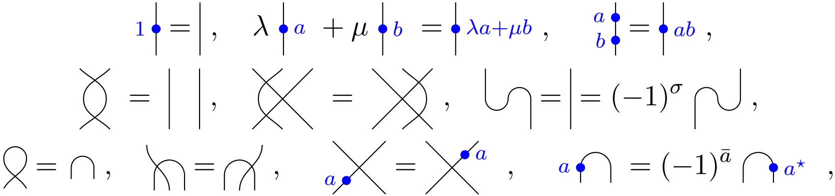
When $\Bbbk = \mathbb{C}$, the analogous theorem was known.

The unoriented supercategory

Let $(\mathbb{D}, \star)$ be an involutive division superalgebra over $\Bbbk$, and let $\sigma \in \mathbb{Z}_2$.

We define $\mathscr{B}_{\Bbbk}^{\sigma}(\mathbb{D}, \star)$ to be the strict monoidal supercategory generated by one object I and morphisms

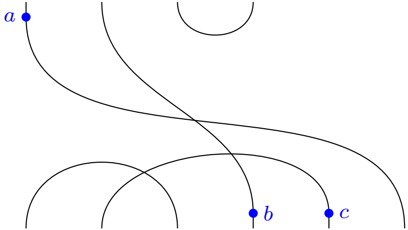
subject to the relations



for all $a, b \in \mathbb{D}$ and $\lambda, \mu \in \Bbbk$. The parity of  is $\bar{a}$, the morphisms  and  both have parity σ, and  is even.

The unoriented supercategory

Morphisms in $\mathcal{B}^{\sigma}_{\Bbbk}(\mathbb{D}, \star)$ are $\Bbbk$-linear combinations of diagrams such as



$a, b, c \in \mathbb{D}$.

Composition is vertical stacking; tensor product is horizontal juxtaposition.

Example

- $\mathcal{B}^{0}_{\Bbbk}(\Bbbk, \mathrm{id})$ is the Brauer category (Lehrer–Zhang).
- $\mathcal{B}^{1}_{\Bbbk}(\Bbbk, \mathrm{id})$ is the periplectic Brauer supercategory (Kujawa–Tharp).

The unoriented incarnation superfunctor

Let $(\mathbb{D}, \star)$ be an involutive real division superalgebra, let $V = \mathbb{D}^{m|n}$, and let φ be a nondegenerate $(\nu, \star)$-superhermitian form of parity σ on V.

Theorem (Samchuck–Schnarch–S.)

There exists a unique monoidal superfunctor

$$\mathsf{F}_\varphi \colon \mathcal{B}_{\mathbb{R}}^{\sigma}(\mathbb{D}, \star) \to G(\varphi)\text{-smod}$$

such that $\mathsf{F}_\varphi(\mathsf{I}) = V$ and

$$\begin{aligned}
\mathsf{F}_\varphi\left(\times\right) &\colon V \otimes V \to V \otimes V, && v \otimes w \mapsto (-1)^{\bar{v}\bar{w}} w \otimes v,\\
\mathsf{F}_\varphi\left(\cap\right) &\colon V \otimes V \to \mathbb{R}, && v \otimes w \mapsto \mathrm{Re}_0(\varphi(v, w)),\\
\mathsf{F}_\varphi\left(\,|\!\bullet a\,\right) &\colon V \to V, && v \mapsto (-1)^{\bar{a}\bar{v}} v a^\star,
\end{aligned}$$

where $\mathrm{Re}_0(a)$ is the real part of the even part of $a \in \mathbb{D}$.

The superfunctor F_φ is full.

The unoriented incarnation superfunctor

Previous results ($\Bbbk = \mathbb{C}$)

- When $\sigma = 0$, so $G(\varphi) = \mathrm{OSp}(m|2n, \mathbb{C})$, the result is due to Lehrer–Zhang, Deligne–Lehrer–Zhang.
- When $\sigma = 1$, so $G(\varphi) = \mathrm{P}(m, \mathbb{C})$, the result is due to Coulembier–Ehrig, with key step by Deligne–Lehrer–Zhang.

Over $\mathbb{R}$, the proof of fullness is split into cases and involves mapping the complexification $\mathcal{B}_{\mathbb{R}}^{\sigma}(\mathbb{D}, \star)^{\mathbb{C}}$ into other supercategories:

- For $(\mathbb{D}, \star) = (\mathbb{R}, \mathrm{id})$, $\mathcal{B}_{\mathbb{R}}^{\sigma}(\mathbb{R}, \mathrm{id})^{\mathbb{C}} \cong \mathcal{B}_{\mathbb{C}}^{\sigma}(\mathbb{C}, \mathrm{id})$,
- For $(\mathbb{D}, \star) \in \{(\mathbb{C}, \star), (\mathrm{Cl}(\mathbb{C}), \star)\}$, we embed $\mathcal{B}_{\mathbb{R}}^{\sigma}(\mathbb{D}, \star)^{\mathbb{C}}$ in the superadditive envelope of $\mathcal{OB}_{\mathbb{C}}(\mathbb{D})$.
- For $(\mathbb{D}, \star) = (\mathbb{H}, \star)$, we embed $\mathcal{B}_{\mathbb{R}}^{\sigma}(\mathbb{H}, \star)^{\mathbb{C}}$ in the superadditive envelope of $\mathcal{B}_{\mathbb{C}}^{\sigma}(\mathbb{C}, \mathrm{id})$.

Then we use the known fullness results in the $\Bbbk = \mathbb{C}$ cases.

Corollary

Corollary of incarnation theorems

If $p, p', q, q' \in \mathbb{N}$ satisfy $p + q = p' + q'$, then we have equivalences of monoidal categories

$$\mathrm{O}(p,q)\text{-tmod}_{\mathbb{R}} \simeq \mathrm{O}(p',q')\text{-tmod}_{\mathbb{R}},$$
$$\mathrm{U}(p,q)\text{-tmod}_{\mathbb{R}} \simeq \mathrm{U}(p',q')\text{-tmod}_{\mathbb{R}},$$
$$\mathrm{Sp}(p,q)\text{-tmod}_{\mathbb{R}} \simeq \mathrm{Sp}(p',q')\text{-tmod}_{\mathbb{R}},$$

sending the natural supermodule to the natural supermodule, where tmod denotes the category of tensor modules.

Above corollary is false if we replace O by SO or U by SU. E.g.

$$\mathrm{SU}(1,1)\text{-tmod}_{\mathbb{R}} \qquad \text{and} \qquad \mathrm{SU}(2)\text{-tmod}_{\mathbb{R}}$$

are not equivalent.

Final remarks

Non-super cases

We obtain a diagrammatic calculus for real forms (including the compact forms) of the classical Lie groups $\mathrm{GL}_m(\mathbb{C})$, $\mathrm{O}_m(\mathbb{C})$, and $\mathrm{Sp}_{2m}(\mathbb{C})$.

Schur–Weyl type duality

We obtain Schur–Weyl-type duality statements for real Lie superalgebras/supergroups.

Quantum versions

There exist quantum versions of the diagrammatic categories introduced here.

These will provide a diagrammatic calculus for real quantum groups analogous to the existing diagrammatics for complex quantum groups.