

# Diagrammatics for real supergroups



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Slides: [alistairsavage.ca/talks](http://alistairsavage.ca/talks)

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# Outline

**Goal:** Develop a simple and intuitive graphical calculus for real representations of real supergroups

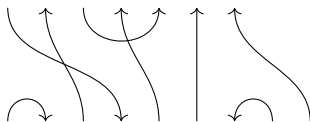
## Overview:

- 1 Background: oriented and unoriented Brauer categories
- 2 Motivation: Schur's lemma and division superalgebras
- 3 Superhermitian forms
- 4 Real Lie superalgebras
- 5 Graphical calculus

# The oriented Brauer category

The **oriented Brauer category**  $\mathcal{OB}(d)$  is the free rigid symmetric  $\mathbb{C}$ -linear monoidal category on a generating object  $\uparrow$  of dimension  $d$ .

Morphisms are linear combinations of **oriented Brauer diagrams**:



There is a **full** monoidal functor

$$\mathcal{OB}(m) \rightarrow \mathrm{GL}(m, \mathbb{C})\text{-mod}, \quad \uparrow \mapsto V = \mathbb{C}^m.$$

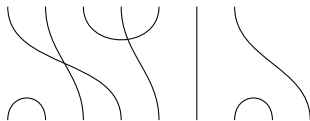
In particular, there is a surjective algebra homomorphism (half of Schur–Weyl duality)

$$\mathbb{C}\mathfrak{S}_r \cong \mathrm{End}_{\mathcal{OB}(m)}(\uparrow^{\otimes r}) \twoheadrightarrow \mathrm{End}_{\mathrm{GL}(m, \mathbb{C})}(V^{\otimes r}).$$

# The unoriented Brauer category

The **unoriented Brauer category**  $\mathcal{B}(d)$  is the free rigid symmetric  $\mathbb{C}$ -linear monoidal category on a symmetrically self-dual object  $I$  of dimension  $d$ .

Morphisms are linear combinations of unoriented Brauer diagrams:



There are **full** monoidal functors

$$\mathcal{B}(m) \rightarrow \mathrm{O}(m, \mathbb{C})\text{-mod} \quad \text{and} \quad \mathcal{B}(-2m) \rightarrow \mathrm{Sp}(2m, \mathbb{C})\text{-mod}.$$

Here the endomorphism algebras are **Brauer algebras**.

# Observations

## Super unifies

In fact, there are full functors

$$\begin{aligned} \mathcal{OB}(m-n) &\rightarrow \mathrm{GL}(m|n, \mathbb{C})\text{-smod} && \text{and} \\ \mathcal{B}(m-2n) &\rightarrow \mathrm{OSp}(m|2n, \mathbb{C})\text{-smod}. \end{aligned}$$

## Trivial yet important observation

Functors induce isomorphisms

$$\mathbb{C} \cong \mathrm{Span}_{\mathbb{C}}\{\uparrow\} = \mathrm{End}_{\mathcal{OB}(m-n)}(\uparrow) \xrightarrow{\cong} \mathrm{End}_{\mathrm{GL}(m|n, \mathbb{C})}(V)$$

and

$$\mathbb{C} \cong \mathrm{Span}_{\mathbb{C}}\{|\} = \mathrm{End}_{\mathcal{B}(m-2n)}(|) \xrightarrow{\cong} \mathrm{End}_{\mathrm{OSp}(m|2n, \mathbb{C})}(V)$$

# Schur's lemma

Fix a ground field  $\mathbb{k}$ . All supermodules are assumed to be **finite dimensional** over  $\mathbb{k}$ .

Let  $R$  be an associative superalgebra or Lie superalgebra over  $\mathbb{k}$ .

## Schur's lemma

If  $V$  is a **simple**  $R$ -supermodule, then  $\text{End}_R(V)$  is a finite-dimensional division  $\mathbb{k}$ -superalgebra.

## Non-super world

- There is one **complex** division algebra:  $\mathbb{C}$ .
- There are three **real** division algebras:  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$ .

# Complex division superalgebras

If  $\mathbb{k} = \mathbb{C}$ , then there are two **complex division superalgebras**:

- the complex numbers  $\mathbb{C}$ ,
- the complex Clifford superalgebra  $\text{Cl}(\mathbb{C}) := \mathbb{C} \oplus \varepsilon\mathbb{C}$ , with  $\bar{\varepsilon} = 1$ ,

$$\varepsilon^2 = -1 \quad \text{and} \quad z\varepsilon = \varepsilon z \quad \forall z \in \mathbb{C}.$$

## Consequence

When  $\mathbb{k} = \mathbb{C}$ , there are two types of simple supermodule  $V$  over a superalgebra  $R$ :

- **Type  $M$** :  $\text{End}_R(V) = \mathbb{C}$ ,
- **Type  $Q$** :  $\text{End}_R(V) = \text{Cl}(\mathbb{C})$ .

# Real division superalgebras

## Theorem (Wall 1964)

Every **real** division superalgebra is isomorphic to exactly one of the following, where  $\bar{\varepsilon} = 1$ , and  $\star$  denotes complex conjugation:

- $\text{Cl}_0(\mathbb{R}) = \mathbb{R}$ ;
- $\text{Cl}_1(\mathbb{R}) := \mathbb{R} \oplus \varepsilon\mathbb{R}$ , with  $\varepsilon^2 = 1$ ;
- $\text{Cl}_2(\mathbb{R}) := \mathbb{C} \oplus \varepsilon\mathbb{C}$ , with  $\varepsilon^2 = 1$  and  $z\varepsilon = \varepsilon z^\star$  for all  $z \in \mathbb{C}$ ;
- $\text{Cl}_3(\mathbb{R}) := \mathbb{H} \oplus \varepsilon\mathbb{H}$ , with  $\varepsilon^2 = -1$  and  $z\varepsilon = \varepsilon z$  for all  $z \in \mathbb{H}$ ;
- $\text{Cl}_4(\mathbb{R}) := \mathbb{H}$ ;
- $\text{Cl}_5(\mathbb{R}) := \mathbb{H} \oplus \varepsilon\mathbb{H}$ , with  $\varepsilon^2 = 1$  and  $z\varepsilon = \varepsilon z$  for all  $z \in \mathbb{H}$ ;
- $\text{Cl}_6(\mathbb{R}) := \mathbb{C} \oplus \varepsilon\mathbb{C}$ , with  $\varepsilon^2 = -1$  and  $z\varepsilon = \varepsilon z^\star$  for all  $z \in \mathbb{C}$ ;
- $\text{Cl}_7(\mathbb{R}) := \mathbb{R} \oplus \varepsilon\mathbb{R}$ , with  $\varepsilon^2 = -1$ ;
- $\mathbb{C}$ ;
- $\text{Cl}(\mathbb{C})$ .



## Remarks

The  $\text{Cl}_r(\mathbb{R})$ ,  $0 \leq r \leq 7$ , are real Clifford superalgebras. They are the only **central** real division superalgebras (i.e. with even center  $\mathbb{R}$ ).

The notation  $\text{Cl}_r(\mathbb{R})$  is inspired by the fact that (subscripts mod 8)

$$\text{Cl}_r(\mathbb{R}) \otimes \text{Cl}_s(\mathbb{R}) \quad \text{is Morita equivalent to} \quad \text{Cl}_{r+s}(\mathbb{R}).$$

The **opposite superalgebra** of an associative superalgebra  $A$  is

$$A^{\text{op}} := \{a^{\text{op}} : a \in A\}$$

with multiplication

$$a^{\text{op}}b^{\text{op}} = (-1)^{\bar{a}\bar{b}}(ba)^{\text{op}}.$$

We have

- $\text{Cl}_r(\mathbb{R})^{\text{op}} \cong \text{Cl}_{-r}(\mathbb{R})$ , with subscripts considered modulo 8,
- $\text{Cl}(\mathbb{C})^{\text{op}} \cong \text{Cl}(\mathbb{C})$ ,  $\varepsilon \mapsto \varepsilon i$ ,
- $\mathbb{C}^{\text{op}} \cong \mathbb{C}$ .

# Motivating idea

## The tenfold way

Suppose  $R$  is a **real** associative superalgebra or a real Lie superalgebra.

There are **ten types** of simple  $R$ -supermodule:

$$\text{End}_R(V) \in \{\text{Cl}_r(\mathbb{R}), \mathbb{C}, \text{Cl}(\mathbb{C}) : 0 \leq r \leq 7\}.$$

We want to modify the oriented Brauer category so that

$\text{End}(\uparrow)$  is a division superalgebra  $A$ .

This amounts to adding morphisms

$$\uparrow a, \quad a \in A.$$

For the **unoriented Brauer category**, we'll need an anti-involution on  $A$  corresponding to

$$\downarrow a \mapsto \uparrow a.$$

# Real general linear Lie superalgebras

Let's look at  $\mathfrak{gl}(m|n, \mathbb{D})$  for  $\mathbb{D}$  a real division superalgebra.

## Simplification

- If  $\mathbb{D}_1 \neq 0$ , then  $\mathfrak{gl}(m|n, \mathbb{D}) \cong \mathfrak{gl}(m+n, \mathbb{D})$ .
- We have  $\mathfrak{gl}(m|n, \mathbb{D}) \cong \mathfrak{gl}(m|n, \mathbb{D}^{\text{op}})$ .

Thus, the general linear Lie superalgebras over real division superalgebras are:

- $\mathfrak{gl}(m, \text{Cl}_1(\mathbb{R})) = \mathfrak{q}(m, \mathbb{R})$  is the **split real isomeric Lie superalgebra** (a.k.a. the **split real queer Lie superalgebra**),
- $\mathfrak{gl}(m, \text{Cl}(\mathbb{C})) = \mathfrak{q}(m, \mathbb{C})$  is the **complex isomeric Lie superalgebra**,
- $\mathfrak{gl}(m, \text{Cl}_2(\mathbb{R}))$ ,
- $\mathfrak{gl}(m, \text{Cl}_3(\mathbb{R}))$ ,
- $\mathfrak{gl}(m|n, \mathbb{C})$ ,
- $\mathfrak{gl}(m|n, \mathbb{H})$ .

# Complexification of general linear Lie superalgebras

The complexifications of all central real division superalgebras are

$$\begin{aligned}\mathbb{R}^{\mathbb{C}} &\cong \mathbb{C}, & \mathbb{H}^{\mathbb{C}} &\cong \text{Mat}_2(\mathbb{C}), \\ \text{Cl}_1(\mathbb{R})^{\mathbb{C}} &\cong \text{Cl}_7(\mathbb{R})^{\mathbb{C}} \cong \text{Cl}(\mathbb{C}), \\ \text{Cl}_2(\mathbb{R})^{\mathbb{C}} &\cong \text{Cl}_6(\mathbb{R})^{\mathbb{C}} \cong \text{Mat}_{1|1}(\mathbb{C}), \\ \text{Cl}_3(\mathbb{R})^{\mathbb{C}} &\cong \text{Cl}_5(\mathbb{R})^{\mathbb{C}} \cong \text{Mat}_2(\text{Cl}(\mathbb{C})).\end{aligned}$$

If  $\mathbb{D}$  is a real division superalgebra, we have

$$\mathfrak{gl}(m|n, \mathbb{D})^{\mathbb{C}} \cong \mathfrak{gl}(m|n, \mathbb{D}^{\mathbb{C}}).$$

Hence

- $\mathfrak{gl}(m|n, \mathbb{R})^{\mathbb{C}} \cong \mathfrak{gl}(m|n, \mathbb{C})$ ,
- $\mathfrak{gl}(m|n, \mathbb{H})^{\mathbb{C}} \cong \mathfrak{gl}(2m|2n, \mathbb{C})$ ,
- $\mathfrak{gl}(m, \text{Cl}_1(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(m, \text{Cl}_7(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(m, \text{Cl}(\mathbb{C})) = \mathfrak{q}(m, \mathbb{C})$ ,
- $\mathfrak{gl}(m, \text{Cl}_2(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(m, \text{Cl}_6(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(m|m, \mathbb{C})$ ,
- $\mathfrak{gl}(m, \text{Cl}_3(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(m, \text{Cl}_5(\mathbb{R}))^{\mathbb{C}} \cong \mathfrak{gl}(2m, \text{Cl}(\mathbb{C})) = \mathfrak{q}(2m, \mathbb{C})$ .

## Motivation for anti-involutions

Fix an associative  $\mathbb{R}$ -superalgebra  $A$  and a **right**  $A$ -supermodule  $V$ .

Then the **dual**  $V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$  is a **left**  $A$ -supermodule, with action

$$(af)(v) = (-1)^{\bar{a}\bar{f} + \bar{a}\bar{v}} f(va), \quad a \in A, f \in V^*, v \in V.$$

We want to examine the situation where  $V$  is **self dual**:

$$V \cong V^* \quad \text{as **right** } A\text{-supermodules.}$$

In order for this to make sense, we need to turn  $V^*$  into a **right**  $A$ -supermodule.

Recall that a **right**  $A$ -supermodule is the same as a **left**  $A^{\text{op}}$ -supermodule.

So, if we have an isomorphism  $A^{\text{op}} \cong A$ , we can convert left  $A$ -supermodules into right  $A$ -supermodules.

# Involutive superalgebras

## Definition

An **involutive superalgebra** is a pair  $(A, \star)$ , where

- $A$  is an associative superalgebra, and
- $\star: A \rightarrow A$ ,  $a \mapsto a^\star$ , is an **anti-involution**:

$$(a^\star)^\star = a, \quad (ab)^\star = (-1)^{\bar{a}\bar{b}} b^\star a^\star.$$

An anti-involution  $\star$  gives an isomorphism  $A^{\text{op}} \cong A$ .

So, if  $V$  is a **right** supermodule over an involutive superalgebra  $(A, \star)$ , then  $V^\star$  is a **right**  $A$ -supermodule via

$$(fa)(v) = (-1)^{\bar{a}\bar{v}} f(va^\star), \quad a \in A, f \in V^\star, v \in V.$$

# Involutive real division superalgebras

Recall that

$$\mathbb{C}^{\text{op}} \cong \mathbb{C}, \quad \text{Cl}(\mathbb{C})^{\text{op}} \cong \text{Cl}(\mathbb{C}), \quad \text{Cl}_r(\mathbb{R})^{\text{op}} \cong \text{Cl}_{-r}(\mathbb{R}).$$

So the real division superalgebras admitting anti-involutions are  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$ , and  $\text{Cl}(\mathbb{C})$ .

In particular, we have

- $(\mathbb{R}, \text{id})$ ,
- $(\mathbb{C}, \text{id})$ ,
- $(\mathbb{C}, \star)$ , where  $\star$  is complex conjugation,
- $(\mathbb{H}, \star)$ , where  $\star$  is quaternionic conjugation,
- $(\text{Cl}(\mathbb{C}), \star)$ , where

$$(a + \varepsilon b)^\star = a^\star + \varepsilon b^\star i, \quad a, b \in \mathbb{C}.$$

# Superhermitian forms

Let  $V$  be a right supermodule over an involutive real division superalgebra  $(\mathbb{D}, \star)$ .

## Definition

Let  $\nu \in \{\pm 1\}$ . A  $(\nu, \star)$ -superhermitian form on  $V$  is a homogeneous  $\mathbb{R}$ -bilinear map

$$\varphi: V \times V \rightarrow \mathbb{D}$$

such that

- $\varphi(va, wb) = (-1)^{\bar{a}(\bar{\varphi} + \bar{v})} a^* \varphi(v, w) b$  for all  $a, b \in A$ ,  $v, w \in V$ ,
- $\varphi(v, w) = \nu (-1)^{\bar{v}\bar{w}} \varphi(w, v)^*$  for all  $v, w \in V$ .

## Remarks

- A superhermitian form gives an isomorphism  $V \cong V^*$ .
- A superhermitian form can be even ( $\bar{\varphi} = 0$ ) or odd ( $\bar{\varphi} = 1$ ).



# Examples

Assume everything is even (i.e., all odd parts are zero).

## Example

If  $(\mathbb{D}, \star) = (\mathbb{C}, \text{id})$ , then

- an  $(1, \text{id})$ -superhermitian form is a symmetric form,
- an  $(-1, \text{id})$ -superhermitian form is skew-symmetric form.

## Example

If  $(\mathbb{D}, \star) = (\mathbb{C}, \star)$ , where  $\star$  is complex conjugation, then

- a  $(1, \star)$ -superhermitian form is a hermitian form in the usual sense,
- a  $(-1, \star)$ -superhermitian form is a skew-hermitian form.

# Lie superalgebras associated to a superhermitian form

Suppose

- $(\mathbb{D}, \star)$  is an involutive real division superalgebra,
- $\varphi$  is a  $(\nu, \star)$ -superhermitian form.

Let

$$\mathfrak{g}(\varphi) = \{X \in \mathfrak{gl}(V) : \varphi(Xv, w) = -(-1)^{\bar{X}\bar{v}} \varphi(v, Xw) \forall v, w \in V\}.$$

be the Lie sub-superalgebra of  $\mathfrak{gl}(V)$  preserving  $\varphi$ .

Let  $G(\varphi)$  be the supergroup preserving  $\varphi$ .

# Real forms

A real Lie algebra  $\mathfrak{g}$  is a **real form** of the complex Lie algebra  $\mathfrak{g}^{\mathbb{C}}$ .

## Example ( $\mathbb{k} = \mathbb{C}$ )

The Lie superalgebras of the form  $\mathfrak{g}(\varphi)$  are:

- the **orthosymplectic Lie superalgebras**  $\mathfrak{osp}(m|2n, \mathbb{C})$  (when  $\varphi$  is even),
- the **periplectic Lie superalgebras**  $\mathfrak{p}(m, \mathbb{C})$  (when  $\varphi$  is odd).

Now suppose  $\mathbb{k} = \mathbb{R}$ . We have

- the Lie superalgebras  $\mathfrak{gl}(m|n, \mathbb{D})$  for a real division superalgebra  $\mathbb{D}$ ,
- the Lie superalgebras  $\mathfrak{g}(\varphi)$  for  $\varphi$  a  $(\nu, \star)$ -superhermitian form for an involutive real division superalgebra  $(\mathbb{D}, \star)$ .

These correspond to **all real forms** of the complex Lie superalgebras

$$\mathfrak{gl}(m|n, \mathbb{C}), \quad \mathfrak{osp}(m|2n, \mathbb{C}), \quad \mathfrak{p}(m, \mathbb{C}), \quad \mathfrak{q}(m, \mathbb{C}).$$

# Strict monoidal supercategories

Fix a ground field  $\mathbb{k}$ .

A **strict monoidal supercategory** is a category  $\mathcal{C}$  enriched in the category of vector superspaces:

- each morphism space is a  $\mathbb{k}$ -supermodule,
- composition of morphisms is  $\mathbb{k}$ -bilinear and parity preserving,

together with

- a bifunctor (the **tensor product**)  $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , and
- a **unit object**  $\mathbb{1}$ ,

such that, for objects  $A, B, C$  and morphisms  $f, g, h$ ,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ ,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$ ,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ ,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$ ,
- tensor product of morphisms is  $\mathbb{k}$ -bilinear and parity preserving.

# String diagrams

Fix a strict monoidal supercategory  $\mathcal{C}$ .

We will denote a morphism  $f: A \rightarrow B$  by:



The **identity map**  $1_A: A \rightarrow A$  is a string with no label:



We sometimes omit the object labels when they are clear.

# String diagrams

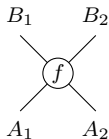
Composition is **vertical stacking** and tensor product is **horizontal juxtaposition**:

The diagram shows two equations. The first equation shows two circles labeled 'f' and 'g' stacked vertically on a single vertical line, followed by an equals sign, and then a single circle labeled 'fg' on a vertical line. The second equation shows a circle labeled 'f' on a vertical line, followed by a circle with a cross inside, followed by a circle labeled 'g' on a vertical line, followed by an equals sign, and then two vertical lines side-by-side, each with a circle labeled 'f' and 'g' respectively.

The **super interchange law** is:

The diagram shows the super interchange law. On the left, a vertical line with a circle 'f' is to the left of a vertical line with a circle 'g'. This is followed by an equals sign, then two vertical lines side-by-side with circles 'f' and 'g' respectively. This is followed by another equals sign, then a vertical line with a circle 'f' to the left of a vertical line with a circle 'g', with a  $(-1)^{\bar{f}\bar{g}}$  symbol between the two lines.

A morphism  $f: A_1 \otimes A_2 \rightarrow B_1 \otimes B_2$  can be depicted:



# The oriented supercategory

For an associative superalgebra  $A$ , we define  $\mathcal{OB}_{\mathbb{k}}(A)$  to be the strict monoidal supercategory generated by objects  $\uparrow$  and  $\downarrow$  and morphisms

$$\overrightarrow{\bowtie} : \uparrow \otimes \uparrow \rightarrow \uparrow \otimes \uparrow, \quad \uparrow_a : \uparrow \rightarrow \uparrow, \quad a \in A,$$

$$\downarrow : \downarrow \otimes \uparrow \rightarrow \mathbb{1}, \quad \uparrow : \mathbb{1} \rightarrow \uparrow \otimes \downarrow, \quad \downarrow : \uparrow \otimes \downarrow \rightarrow \mathbb{1}, \quad \uparrow : \mathbb{1} \rightarrow \downarrow \otimes \uparrow,$$

subject to the relations

$$\uparrow_1 = \uparrow, \quad \lambda \uparrow_a + \mu \uparrow_b = \uparrow_{\lambda a + \mu b}, \quad \begin{matrix} a \\ \uparrow \\ b \end{matrix} = \uparrow_{ab},$$

$$\overrightarrow{\bowtie} = \uparrow \uparrow, \quad \overleftarrow{\bowtie} = \overleftarrow{\bowtie}, \quad \begin{matrix} \nearrow \\ \searrow \\ a \end{matrix} = \begin{matrix} \nearrow \\ \searrow \\ \uparrow_a \end{matrix},$$

$$\overleftarrow{\bowtie} = \downarrow \uparrow, \quad \overleftarrow{\bowtie} = \uparrow \downarrow, \quad \uparrow = \uparrow = \downarrow, \quad \downarrow = \downarrow, \quad \uparrow = \uparrow,$$

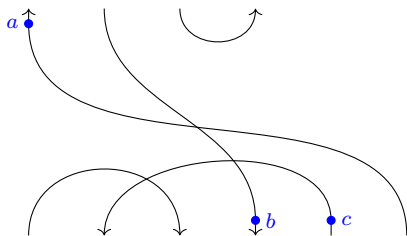
for all  $a, b \in A$  and  $\lambda, \mu \in \mathbb{k}$ . In the above, the left and right crossings are defined by

$$\overrightarrow{\bowtie} := \begin{matrix} \nearrow \\ \searrow \end{matrix}, \quad \overleftarrow{\bowtie} := \begin{matrix} \searrow \\ \nearrow \end{matrix}.$$

The parity of  $\uparrow_a$  is  $\bar{a}$ , and all the other generating morphisms are even.

# The oriented supercategory

Morphisms in  $\mathcal{OB}_{\mathbb{k}}(A)$  are  $\mathbb{k}$ -linear combinations of diagrams such as



$a, b, c \in A$ .

Composition is vertical stacking; tensor product is horizontal juxtaposition.

## Example

- $\mathcal{OB}_{\mathbb{k}}(\mathbb{k})$  is the **oriented Brauer category**.
- $\mathcal{OB}_{\mathbb{C}}(\text{Cl}(\mathbb{C}))$  is the **oriented Brauer–Clifford supercategory** (Brundan–Comes–Kujawa).



# The oriented incarnation superfunctor

Suppose that  $\mathbb{D}$  is a real division superalgebra. Let  $V = \mathbb{D}^{m|n}$ .

## Theorem (Samchuck–Schnarch–S.)

There exists a unique monoidal superfunctor

$$G: \mathcal{OB}_{\mathbb{R}}(\mathbb{D}^{\text{op}}) \rightarrow \mathfrak{gl}(m|n, \mathbb{D})\text{-smod}$$

such that  $G(\uparrow) = V$ ,  $G(\downarrow) = V^*$ , and

$$\begin{aligned} G(\begin{array}{c} \nearrow \times \nwarrow \\ \searrow \end{array}): V \otimes V \rightarrow V \otimes V, & v \otimes w \mapsto (-1)^{\bar{v}\bar{w}} w \otimes v, \\ G(\begin{array}{c} \downarrow \cap \\ \uparrow \end{array}): V^* \otimes V \rightarrow \mathbb{R}, & f \otimes v \mapsto f(v), \\ G(\begin{array}{c} \uparrow \\ \downarrow \end{array} \alpha^{\text{op}}): V \rightarrow V, & v \mapsto (-1)^{\bar{a}\bar{v}} va. \end{aligned}$$

The superfunctor  $G$  is **full**.

## Remark

When  $\mathbb{k} = \mathbb{C}$ , the analogous theorem was known.

# The unoriented supercategory

Let  $(\mathbb{D}, \star)$  be an involutive division superalgebra over  $\mathbb{k}$ , and let  $\sigma \in \mathbb{Z}_2$ .

We define  $\mathcal{B}_{\mathbb{k}}^{\sigma}(\mathbb{D}, \star)$  to be the strict monoidal supercategory generated by one object  $I$  and morphisms

$$\times: I^{\otimes 2} \rightarrow I^{\otimes 2}, \quad \cap: I^{\otimes 2} \rightarrow \mathbb{1}, \quad \cup: \mathbb{1} \rightarrow I^{\otimes 2}, \quad \bullet_a: I \rightarrow I, \quad a \in A,$$

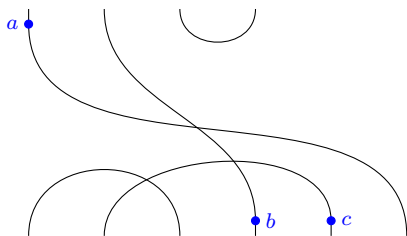
subject to the relations

$$\begin{aligned} 1 \bullet &= |, & \lambda \bullet_a + \mu \bullet_b &= \bullet_{\lambda a + \mu b}, & \begin{matrix} a \\ \bullet \\ b \\ \bullet \end{matrix} &= \bullet_{ab}, \\ \text{crossing} &= | |, & \text{crossing} &= \text{crossing}, & \cup &= | = (-1)^{\sigma} \cap, \\ \cap &= \cap, & \cup &= \cup, & \bullet_a \cap &= (-1)^{\bar{a}} \cap \bullet_{a^*}, \end{aligned}$$

for all  $a, b \in A$  and  $\lambda, \mu \in \mathbb{k}$ . The parity of  $\bullet_a$  is  $\bar{a}$ , the morphisms  $\cup$  and  $\cap$  both have parity  $\sigma$ , and  $\times$  is even.

# The unoriented supercategory

Morphisms in  $\mathcal{B}_{\mathbb{k}}^{\sigma}(A, \star)$  are  $\mathbb{k}$ -linear combinations of diagrams such as



$a, b, c \in A$ .

Composition is vertical stacking; tensor product is horizontal juxtaposition.

## Example

- $\mathcal{B}_{\mathbb{k}}^0(\mathbb{k}, \text{id})$  is the **Brauer category** (Lehrer–Zhang).
- $\mathcal{B}_{\mathbb{k}}^1(\mathbb{k}, \text{id})$  is the **periplectic Brauer supercategory** (Kujawa–Tharp).

# The unoriented incarnation superfunctor

Let  $(\mathbb{D}, \star)$  be an involutive real division superalgebra, let  $V = \mathbb{D}^{m|n}$ , and let  $\varphi$  be a nondegenerate  $(\nu, \star)$ -superhermitian form of parity  $\sigma$  on  $V$ .

## Theorem (Samchuck–Schnarch–S.)

There exists a unique monoidal superfunctor

$$F_\varphi: \mathcal{B}_{\mathbb{R}}^\sigma(\mathbb{D}, \star) \rightarrow G(\varphi)\text{-smod}$$

such that  $F_\varphi(\mathbb{1}) = V$  and

$$\begin{aligned} F_\varphi(\times) &: V \otimes V \rightarrow V \otimes V, & v \otimes w &\mapsto (-1)^{\bar{v}\bar{w}} w \otimes v, \\ F_\varphi(\cap) &: V \otimes V \rightarrow \mathbb{R}, & v \otimes w &\mapsto \operatorname{Re}_0(\varphi(v, w)), \\ F_\varphi(\bullet a) &: V \rightarrow V, & v &\mapsto (-1)^{\bar{a}\bar{v}} v a^\star, \end{aligned}$$

where  $\operatorname{Re}_0(a)$  is the real part of the even part of  $a \in \mathbb{D}$ .

The superfunctor  $F_\varphi$  is **full**.

# The unoriented incarnation superfunctor

## Previous results ( $\mathbb{k} = \mathbb{C}$ )

- When  $\sigma = 0$ , so  $G(\varphi) = \mathrm{OSp}(m|2n, \mathbb{C})$ , the result is due to Lehrer–Zhang, Deligne–Lehrer–Zhang.
- When  $\sigma = 1$ , so  $G(\varphi) = \mathrm{P}(m, \mathbb{C})$ , the result is due to Coulembier–Ehrig, with key step by Deligne–Lehrer–Zhang.

Over  $\mathbb{R}$ , the proof of fullness is split into cases and involves mapping the complexification  $\mathcal{B}_{\mathbb{R}}^{\sigma}(\mathbb{D}, \star)^{\mathbb{C}}$  into other supercategories:

- For  $(\mathbb{D}, \star) = (\mathbb{R}, \mathrm{id})$ ,  $\mathcal{B}_{\mathbb{R}}^{\sigma}(\mathbb{R}, \mathrm{id})^{\mathbb{C}} \cong \mathcal{B}_{\mathbb{C}}^{\sigma}(\mathbb{C}, \mathrm{id})$ ,
- For  $(\mathbb{D}, \star) \in \{(\mathbb{C}, \star), (\mathrm{Cl}(\mathbb{C}), \star)\}$ , we embed  $\mathcal{B}_{\mathbb{R}}^{\sigma}(\mathbb{D}, \star)^{\mathbb{C}}$  in the superadditive envelope of  $\mathcal{OB}_{\mathbb{C}}(\mathbb{D})$ .
- For  $(\mathbb{D}, \star) = (\mathbb{H}, \star)$ , we embed  $\mathcal{B}_{\mathbb{R}}^{\sigma}(\mathbb{H}, \star)^{\mathbb{C}}$  in the superadditive envelope of  $\mathcal{B}_{\mathbb{C}}^{\sigma}(\mathbb{C}, \mathrm{id})$ .

Then we use the known fullness results in the  $\mathbb{k} = \mathbb{C}$  cases.

# Corollary

## Corollary of incarnation theorems

If  $p, p', q, q' \in \mathbb{N}$  satisfy  $p + q = p' + q'$ , then we have equivalences of monoidal categories

$$\mathrm{O}(p, q)\text{-tmod}_{\mathbb{R}} \simeq \mathrm{O}(p', q')\text{-tmod}_{\mathbb{R}},$$

$$\mathrm{U}(p, q)\text{-tmod}_{\mathbb{R}} \simeq \mathrm{U}(p', q')\text{-tmod}_{\mathbb{R}},$$

$$\mathrm{Sp}(p, q)\text{-tmod}_{\mathbb{R}} \simeq \mathrm{Sp}(p', q')\text{-tmod}_{\mathbb{R}},$$

sending the natural supermodule to the natural supermodule, where  $\mathrm{tmod}$  denotes the category of **tensor modules**.

Above corollary is **false** if we replace  $\mathrm{O}$  by  $\mathrm{SO}$  or  $\mathrm{U}$  by  $\mathrm{SU}$ . E.g.

$$\mathrm{SU}(1, 1)\text{-tmod}_{\mathbb{R}} \quad \text{and} \quad \mathrm{SU}(2)\text{-tmod}_{\mathbb{R}}$$

are **not** equivalent.

# Final remarks

## Non-super cases

We obtain a diagrammatic calculus for real forms (including the compact forms) of the classical Lie groups  $GL_m(\mathbb{C})$ ,  $O_m(\mathbb{C})$ , and  $Sp_{2m}(\mathbb{C})$ .

## Schur–Weyl type duality

We obtain Schur–Weyl-type duality statements for real Lie superalgebras/supergroups.

## Quantum versions

There exist **quantum versions** of the diagrammatic categories introduced here.

These will provide a diagrammatic calculus for **real** quantum groups analogous to the existing diagrammatics for complex quantum groups.