

The quantum isomeric supercategory

(“isomeric” = “queer”)

$$\begin{array}{c} \uparrow \\ \bullet \\ \bullet \end{array} = - \begin{array}{c} \uparrow \\ | \\ | \end{array} = \begin{array}{c} \uparrow \\ \circ \\ \circ \end{array}, \quad \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array}, \quad \begin{array}{c} \nearrow \\ \circ \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \circ \\ \searrow \end{array}$$

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Outline

Goal: Introduce an isomeric analogue of the (affine) HOMFLYPT skein category.

Overview:

- 1 Background and motivation
- 2 The quantum isomeric superalgebra
- 3 The quantum isomeric supercategory
- 4 The incarnation superfunctor
- 5 The quantum affine isomeric supercategory
- 6 The affine action superfunctor

Schur's lemma

Schur's lemma (non-super case)

If V is a f.d. simple module over an associative \mathbb{C} -algebra A , then

$$\text{End}_A(V) \cong \mathbb{C}.$$

Schur's lemma (super case)

If V is a f.d. simple **super**module over an associative \mathbb{C} -**super**algebra A , then there are two possibilities:

- V is of **type M** : $\text{End}_A(V) \cong \mathbb{C}$.
- V is of **type Q** : $\text{End}_A(V) \cong \text{Cl}$, the 2-dimensional Clifford superalgebra.

Type Q simple supermodules are those that are isomorphic to their own parity shift. This parity shift is the odd generator of Cl .

Type M

Every f.d. superspace V is isomorphic to $\mathbb{C}^{m|n}$ for some $m, n \in \mathbb{N}$.

Let $M(m|n)$ denote the superalgebra of complex $(m|n) \times (m|n)$ supermatrices

$$\begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix}.$$

If V is a f.d. simple A -supermodule of **type M** , then the action gives a surjective homomorphism of superalgebras

$$A \twoheadrightarrow M(m|n).$$

The **general linear Lie superalgebra** $\mathfrak{gl}_{m|n}$ is $M(m|n)$ with superbracket

$$[X, Y] = XY - (-1)^{\bar{X}\bar{Y}} YX.$$

Type Q

Consider the Clifford superalgebra

$$\text{Cl} = \mathbb{C}[c]/(c^2 + 1), \quad c \text{ odd.}$$

Then Cl acts on $\mathbb{C}^{n|n}$, with c acting as

$$J := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Define $Q(n) = Q(n)_0 \oplus Q(n)_1$, where

$$Q(n)_i = \{C \in M(n|n)_i : CJ = (-1)^i JC\}.$$

If $\mathbb{C}^{n|n}$ is a f.d. simple A -supermodule of type Q , then the action gives a surjective homomorphism of superalgebras

$$A \twoheadrightarrow Q(n).$$

The isomeric Lie superalgebra \mathfrak{q}_n is $Q(n)$ with superbracket

$$[X, Y] = XY - (-1)^{\bar{X}\bar{Y}} YX.$$

Background: $\mathfrak{gl}_{m|n}$ case

$\mathcal{OB}(t)$ = oriented Brauer category depending on a dimension parameter t .

There is a full monoidal functor

$$\mathcal{OB}(m - n) \rightarrow \mathfrak{gl}_{m|n}\text{-smod}$$

sending the generating object of $\mathcal{OB}(m - n)$ to the natural $\mathfrak{gl}_{m|n}$ -supermodule $\mathbb{C}^{m|n}$.

Additive Karoubi envelope of $\mathcal{OB}(t)$ is Deligne's interpolating category $\underline{\text{Rep}}(\text{GL}_t)$.

Orthosymplectic case

$\mathcal{OB}(t)$ is replaced by the Brauer category (no longer oriented).

Background: $\mathfrak{gl}_{m|n}$ case

Any monoidal category acts on itself by the tensor product.

In particular, $\mathfrak{gl}_{m|n}$ -smod acts on itself by **translation functors** $V \otimes -$.

This can be extended to a monoidal functor

$$\mathcal{AOB}(m - n) \rightarrow \mathcal{E}nd(\mathfrak{gl}_{m|n}\text{-smod}),$$

where $\mathcal{AOB}(t)$ is the **affine oriented Brauer category** [Brundan–Comes–Nash–Reynolds 2017].

Benefits:

- study natural transformations between translation functors,
- tool to study cyclotomic Hecke algebras,
- yields natural elements in the center of $U(\mathfrak{gl}_{m|n})$.

Background: quantum $\mathfrak{gl}_{m|n}$ case

In the quantum setting, $\mathcal{OB}(t)$ and $\mathcal{AOB}(t)$ are replaced by

- the **HOMFLYPT skein category** $\mathcal{OS}(z, t)$ [Turaev 1989], and
- the **affine HOMFLYPT skein category** $\mathcal{AOS}(z, t)$ [Brundan 2017].

We have monoidal functors

$$\begin{aligned}\mathcal{OS}(q - q^{-1}, q^n) &\rightarrow U_q(\mathfrak{gl}_n)\text{-mod}, \\ \mathcal{AOS}(q - q^{-1}, q^n) &\rightarrow \mathcal{E}nd(U_q(\mathfrak{gl}_n)\text{-mod}).\end{aligned}$$

(Can likely be generalized to the super setting $U_q(\mathfrak{gl}_{m|n})$.)

Background: isomeric case

In the isomeric case, we have [Brundan–Comes–Kujawa 2019]:

- the **oriented Brauer–Clifford supercategory** OBC ,
- the **affine oriented Brauer–Clifford supercategory** $AOBC$.

There are monoidal superfunctors

$$OBC \rightarrow \mathfrak{q}_n\text{-smod} \quad \text{and} \quad AOBC \rightarrow \mathcal{SEnd}(\mathfrak{q}_n\text{-smod}).$$

Supercategories

We need to move to the setting of **supercategories** because there is an **odd** morphism in OBC and $AOBC$ that corresponds to the parity shift endomorphism of the natural \mathfrak{q}_n -supermodule.

Goal

We want to develop a similar picture for the quantum isomeric superalgebra $U_q(\mathfrak{q}_n)$.

| | Type M | Type Q |
|-----------------------|--|-----------------------------|
| degenerate quantum | $OB(t), \mathcal{AOB}(t)$ (affine) HOMFLYPT skein | OBC, \mathcal{AOBC} ?? |

The quantum isomeric superalgebra

The **quantum** analogue of \mathfrak{q}_n was initially defined by Olshanski using a super analogue of the FRT construction.

Let $V = \mathbb{C}(q)^{n|n}$, and define an explicit invertible element

$$\Theta = \sum_{i,j} \Theta_{ij} \otimes E_{ij} \in \text{End}(V)^{\otimes 2}$$

such that

$$\Theta \circ (J \otimes 1) = (J \otimes 1) \circ \Theta$$

and Θ satisfies the **Yang–Baxter equation**

$$\Theta^{12} \Theta^{13} \Theta^{23} = \Theta^{23} \Theta^{13} \Theta^{12}.$$

The quantum isomeric superalgebra

The **quantum isomeric superalgebra** $U_q = U_q(\mathfrak{q}_n)$ is the unital associative superalgebra over $\mathbb{C}(q)$ generated by

$$u_{ij}, \quad i, j \in \mathbf{I} = \{-1, -2, \dots, -n, 1, 2, \dots, n\}, \quad i \leq j,$$

subject to the relations

$$u_{ii}u_{-i,-i} = 1 = u_{-i,-i}u_{ii},$$
$$L^{12}L^{13}\Theta^{23} = \Theta^{23}L^{13}L^{12}, \quad L := \sum_{i,j \in \mathbf{I}, i \leq j} u_{ij} \otimes E_{ij},$$

where the second-to-last equality takes place in $U_q \otimes \text{End}_{\mathbb{k}}(V)^{\otimes 2}$. The parity of u_{ij} is $(-1)^{\text{sgn}(i)+\text{sgn}(j)}$.

Comments

- 1 A Drinfeld–Jimbo type presentation of U_q has been given by Grantcharov–Jung–Kang–Kim (2010).
- 2 U_q is a Hopf superalgebra, but it is **not** quasitriangular.

The natural representation

The map

$$U_q \rightarrow \text{End}(V), \quad u_{ij} \mapsto \Theta_{ij}, \quad i, j \in \mathbf{I}, \quad i \leq j,$$

defines the **natural representation** of U_q on V .

Objective

We want to develop a graphical calculus describing the full monoidal sub-supercategory of U_q -smod generated by V and its dual V^* .

Note: This sub-supercategory is **not** semisimple.

Strict monoidal supercategories

Fix a ground field \mathbb{k} .

A **strict monoidal supercategory** is a category \mathcal{C} enriched in the category of vector superspaces:

- each morphism space is a \mathbb{k} -supermodule,
- composition of morphisms is \mathbb{k} -bilinear and parity preserving,

together with

- a bifunctor (the **tensor product**) $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, and
- a **unit object** $\mathbb{1}$,

such that, for objects A, B, C and morphisms f, g, h ,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$,
- tensor product of morphisms is \mathbb{k} -bilinear and parity preserving.

String diagrams

Fix a strict monoidal supercategory \mathcal{C} .

We will denote a morphism $f: A \rightarrow B$ by:



The **identity map** $1_A: A \rightarrow A$ is a string with no label:



We sometimes omit the object labels when they are clear or unimportant.

String diagrams

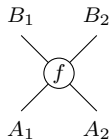
Composition is **vertical stacking** and tensor product is **horizontal juxtaposition**:

The diagram shows two equations. The first equation shows two circles labeled 'f' and 'g' stacked vertically on a single line, followed by an equals sign, and then a single circle labeled 'fg' on a single line. The second equation shows a circle labeled 'f' on a line, followed by a circle with a cross symbol '⊗', followed by a circle labeled 'g' on a line, followed by an equals sign, and then two circles labeled 'f' and 'g' side-by-side on separate lines.

The **super interchange law** is:

The diagram shows an equation between three string diagrams. The first diagram has a circle labeled 'f' on the left line and a circle labeled 'g' on the right line. The second diagram has two circles labeled 'f' and 'g' side-by-side. The third diagram has a circle labeled 'f' on the left line and a circle labeled 'g' on the right line, but the 'g' circle is positioned higher than the 'f' circle. The equation is: $(f \text{ on left, } g \text{ on right}) = (f \text{ and } g \text{ side-by-side}) = (-1)^{\bar{f}\bar{g}} (f \text{ on left, } g \text{ on right})$

A morphism $f: A_1 \otimes A_2 \rightarrow B_1 \otimes B_2$ can be depicted:



The quantum isomeric supercategory

The **quantum isomeric supercategory** $Q(z)$ is the strict monoidal supercategory generated by objects \uparrow and \downarrow and morphisms

$$\begin{aligned} \nearrow, \nwarrow: \uparrow \otimes \uparrow &\rightarrow \uparrow \otimes \uparrow, & \searrow, \swarrow: \uparrow \otimes \downarrow &\rightarrow \downarrow \otimes \uparrow, \\ \cap: \downarrow \otimes \uparrow &\rightarrow \mathbb{1}, & \cup: \mathbb{1} &\rightarrow \uparrow \otimes \downarrow, & \uparrow: \uparrow &\rightarrow \uparrow, \end{aligned}$$

subject to the relations

$$\begin{aligned} \begin{array}{c} \nearrow \\ \searrow \end{array} &= \uparrow \uparrow = \begin{array}{c} \nwarrow \\ \swarrow \end{array}, & \begin{array}{c} \nwarrow \\ \swarrow \end{array} &= \uparrow \downarrow, & \begin{array}{c} \swarrow \\ \nwarrow \end{array} &= \downarrow \uparrow, & \begin{array}{c} \nearrow \\ \nwarrow \end{array} &= \begin{array}{c} \nwarrow \\ \swarrow \end{array}, \\ \begin{array}{c} \nearrow \\ \swarrow \end{array} - \begin{array}{c} \nwarrow \\ \searrow \end{array} &= z \uparrow \uparrow, & \begin{array}{c} \uparrow \\ \cup \end{array} &= \uparrow, & \begin{array}{c} \downarrow \\ \cap \end{array} &= \downarrow, \\ \begin{array}{c} \uparrow \\ \uparrow \end{array} &= -\uparrow, & \begin{array}{c} \nearrow \\ \swarrow \end{array} &= \begin{array}{c} \nwarrow \\ \swarrow \end{array}, & \begin{array}{c} \circlearrowleft \\ \bullet \end{array} &= 0 = \begin{array}{c} \circlearrowright \\ \bullet \end{array}. \end{aligned}$$

In the above, we have used left crossings and a right cap defined by

$$\begin{array}{c} \nwarrow \\ \swarrow \end{array} := \begin{array}{c} \nearrow \\ \cup \end{array}, \quad \begin{array}{c} \cap \\ \downarrow \end{array} := \begin{array}{c} \nwarrow \\ \swarrow \end{array}.$$

The **Clifford token** \uparrow is odd; all the other generating morphisms are even.

Relation to the HOMFLYPT skein category

Given $z, t \in \mathbb{k}^\times$, the **HOMFLYPT skein category** $OS(z, t)$ is the quotient of the category of framed oriented tangles by the relations

$$\begin{array}{l} \nearrow \searrow - \searrow \nearrow = z \uparrow \uparrow, \quad \uparrow \circlearrowleft = t \uparrow, \quad \circlearrowleft = \frac{t - t^{-1}}{z} \mathbf{1}. \end{array}$$

The quantum isomeric supercategory $Q(z)$ is obtained from $OS(z, 1)$ by adjoining the Clifford token $\uparrow \bullet$, subject to the relations

$$\uparrow \bullet = -\uparrow, \quad \begin{array}{l} \nearrow \bullet \\ \bullet \searrow \end{array} = \begin{array}{l} \nearrow \bullet \\ \searrow \end{array}, \quad \bullet \circlearrowleft = 0.$$

In particular, **all relations in $OS(z, 1)$ hold in $Q(z)$.**

Properties of $Q(z)$

Pivotal

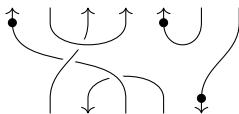
The supercategory $Q(z)$ is **pivotal**, with duality superfunctor given by rotation through 180° (with appropriate signs).

Intuitively, this means that morphisms are invariant under signed isotopy.

Degenerate ($z = 0$) case

$Q(0)$ is isomorphic to the **oriented Brauer–Clifford supercategory** introduced by Brundan–Comes–Kujawa (2019).

Basis theorem



Bases of morphism spaces in $Q(z)$ are given by diagrams satisfying:

- no closed strings,
- no self-intersections,
- no two strings cross each other more than once,
- all crossings are positive,
- strands can carry one (or zero) Clifford tokens near their terminus.

Note: Ignoring Clifford tokens, these diagrams give a bases for the HOMFLYPT skein category.

Endomorphism superalgebras

Proposition

The endomorphism superalgebra

$$\text{End}_{Q(z)}(\uparrow^{\otimes r})$$

is isomorphic to the **Hecke–Clifford superalgebra** of rank r .

More generally, we have the following.

Proposition

The endomorphism superalgebra

$$\text{End}_{Q(z)}(\uparrow^{\otimes r} \otimes \downarrow^{\otimes s})$$

is isomorphic to a **quantum walled Brauer–Clifford superalgebra** of Benkart–Guay–Jung–Kang–Wilcox (2016).

Evaluation and coevaluation

Define the **evaluation map**

$$\text{ev}: V^* \otimes V \rightarrow \mathbb{C}(q), \quad f \otimes v \mapsto f(v),$$

and the **coevaluation map**

$$\text{coev}: \mathbb{C}(q) \rightarrow V \otimes V^*, \quad 1 \mapsto \sum_{v \in \mathbf{B}_V} v \otimes v^*,$$

where \mathbf{B}_V is a basis of V and $\{v^* : v \in \mathbf{B}_V\}$ is the dual basis.

These are both U_q -supermodule homomorphisms.

The incarnation superfunctor

Incarnation Theorem (S. 2022)

For each $n \in \mathbb{N}$, there exists a unique monoidal superfunctor

$$\mathbf{F}_n: Q(q - q^{-1}) \rightarrow U_q(\mathfrak{q}_n)\text{-smod}$$

such that

$$\begin{aligned} \mathbf{F}_n(\uparrow) &= V, & \mathbf{F}_n(\downarrow) &= V^*, \\ \mathbf{F}_n(\nearrow \searrow) &= \text{flip} \circ \Theta, & \mathbf{F}_n(\searrow \nearrow) &= \text{ev}, & \mathbf{F}_n(\updownarrow) &= J. \end{aligned}$$

Furthermore, $\mathbf{F}_n(\searrow \nearrow) = \Theta^{-1} \circ \text{flip}$ and $\mathbf{F}_n(\updownarrow) = \text{coev}$.

The superfunctor \mathbf{F}_n is **full** for all $n \in \mathbb{N}$.

The induced map on morphism spaces is an **isomorphism** when the number of strands is $\leq n$.

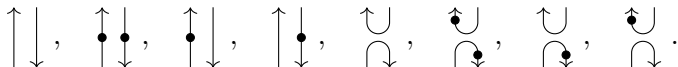
Example

Recall: The full monoidal sub-supercategory of U_q -smod generated by V and V^* is not semisimple.

For $n \geq 2$, it follows from the incarnation and basis theorems that

$$\text{End}_{U_q}(V \otimes V^*) \cong \text{End}_{Q(z)}(\uparrow \otimes \downarrow)$$

has basis given by



Using the relations

$$\circlearrowleft = \circlearrowright = \bullet \circlearrowleft = \circlearrowright \bullet = 0,$$

we see that the span of the last four elements is a nilpotent ideal. Hence $\text{End}_{U_q}(V \otimes V^*)$ is not semisimple.

Remarks

Note that $Q(z)$ does not depend on n .

However, the kernel of the incarnation superfunctor

$$Q(z) \rightarrow U_q(\mathfrak{q}_n)\text{-smod}$$

does depend on n .

The **semisimplification** of $Q(z)$ (i.e. the quotient by the tensor ideal of negligible morphisms) is the **trivial supercategory** since

$$1_{\uparrow} \quad \text{and} \quad 1_{\downarrow}$$

are negligible morphisms; so only the trivial object survives.

The chiral braiding

Since U_q is not quasitriangular, U_q -smod is **not** a braided monoidal supercategory.

Diagrammatically, this is because

$$\begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \bullet \\ \nearrow \end{array} \quad \text{but} \quad \begin{array}{c} \nwarrow \\ \bullet \\ \searrow \end{array} \neq \begin{array}{c} \nwarrow \\ \bullet \\ \nwarrow \end{array}.$$

However, for each U_q -supermodule M , we have a U_q -supermodule homomorphism

$$\begin{array}{c} \nearrow \\ \searrow \\ M \end{array} = \text{flip} \circ L: M \otimes V \rightarrow V \otimes M, \quad L := \sum_{i,j \in I, i \leq j} u_{ij} \otimes E_{ij}.$$

These satisfy

$$\begin{array}{c} \nearrow \\ \searrow \end{array} = \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} \nearrow \\ | \\ \nearrow \end{array}, \quad \begin{array}{c} \nwarrow \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ | \\ \nearrow \end{array} \begin{array}{c} | \\ | \\ | \end{array}, \quad \begin{array}{c} \nearrow \\ \searrow \end{array} = \begin{array}{c} | \\ | \\ \searrow \end{array}, \quad \begin{array}{c} \nwarrow \\ \searrow \end{array} = \begin{array}{c} \searrow \\ | \\ \searrow \end{array}, \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \neq \begin{array}{c} \nearrow \\ \searrow \end{array}.$$

where

$$\begin{array}{c} \searrow \\ \searrow \end{array} := \begin{array}{c} \curvearrowright \\ \searrow \end{array} \quad \text{and} \quad \begin{array}{c} \nwarrow \\ \searrow \end{array} := \begin{array}{c} \curvearrowleft \\ \searrow \end{array}.$$

The chiral braiding

Furthermore, we have

and

where f is any string diagram in $Q(z)$ not containing Clifford tokens,

for all $f \in \text{Hom}_{U_q}(M, N)$.

and

.

So we have something like a braiding, but only natural in **one** argument.

The quantum affine isomeric supercategory

The **quantum affine isomeric supercategory** $\mathcal{AQ}(z)$ is the strict monoidal supercategory obtained from $\mathcal{Q}(z)$ by adjoining an additional odd morphism (the **open Clifford token**)

$$\uparrow\circlearrowleft: \uparrow \rightarrow \uparrow$$

subject to the relations

$$\uparrow\circlearrowleft = -\uparrow, \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \circlearrowleft = \begin{array}{c} \nearrow \\ \searrow \end{array}, \quad \circlearrowleft = 0.$$

Symmetry

There is an isomorphism of monoidal supercategories

$$\mathcal{AQ}(z) \rightarrow \mathcal{AQ}(-z)$$

that interchanges open and closed Clifford tokens and flips crossings:

$$\uparrow\circlearrowleft \mapsto \uparrow\circlearrowright, \quad \uparrow\circlearrowright \mapsto \uparrow\circlearrowleft, \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \mapsto \begin{array}{c} \searrow \\ \nearrow \end{array}, \quad \begin{array}{c} \searrow \\ \nearrow \end{array} \mapsto \begin{array}{c} \nearrow \\ \searrow \end{array}.$$

Affine endomorphism algebras

For $r \in \mathbb{N}$, there is a homomorphism of associative superalgebras

$$\text{AHC}_r(z) \rightarrow \text{End}_{\mathcal{A}Q(z)}(\uparrow^{\otimes r}),$$

where $\text{AHC}_r(z)$ is the **affine Hecke–Clifford superalgebra** defined by Jones–Nazarov (1999).

Under the functor

$$\mathcal{A}Q(z) \rightarrow Q(z), \quad \left| \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \right| \mapsto \left| \begin{array}{c} \curvearrowright \\ \curvearrowleft \\ \bullet \end{array} \right|,$$

where $\left| \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \right|$ and $\left| \begin{array}{c} \curvearrowright \\ \curvearrowleft \\ \bullet \end{array} \right|$ denote some sequences of strands, we have

$$\uparrow^{\otimes(r-i)} \otimes \left| \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \right| \otimes \uparrow^{\otimes(i-1)} \mapsto -J_i,$$

where $J_i \in \text{End}_{Q(z)}(\uparrow^{\otimes r})$ corresponds to the i -th **Jucys–Murphy element** under the isomorphism

$$\text{HC}_r(z) \cong \text{End}_{Q(z)}(\uparrow^{\otimes r}).$$

Affinization

Even affinization (Mousaaid–S. 2021)

There exists a general notion of the **affinization** of a braided monoidal category corresponding to considering string diagrams on a cylinder.

Gives Jucys–Murphy elements



E.g. these are the JM elements in the Iwahori–Hecke algebra of type A in terms of string diagrams.

Above results suggest a theory of **odd affinization**, where above diagram is replaced by



The affine action superfunctor

$Q(z)$ acts on itself by the tensor product. We extend this action to $\mathcal{A}Q(z)$.

Theorem (S. 2022)

There is a unique monoidal superfunctor

$$\widehat{\mathbf{F}}_n: \mathcal{A}Q(z) \rightarrow \mathcal{SEnd}(U_q(\mathfrak{q}_n)\text{-smod}),$$

such that

$$\left(\widehat{\mathbf{F}}_n|_{Q(z)}\right)_M = - \otimes M, \quad \text{and} \quad \widehat{\mathbf{F}}_n(\uparrow)_M = \left(\begin{array}{c} \uparrow \\ \text{red crossing} \\ \downarrow \\ M \end{array} \right).$$

In particular

$$\widehat{\mathbf{F}}_n(\uparrow) = V \otimes - \quad \text{and} \quad \widehat{\mathbf{F}}_n(\downarrow) = V^* \otimes -$$

are translation endosuperfunctors of U_q -smod.

Central elements

Proposition (S. 2022)

The supercommutative superalgebra $\text{End}_{\mathcal{A}Q(z)}(\mathbb{1})$ is generated by

$$\textcircled{\bullet} 2k, \quad k \in \mathbb{Z}_{>0},$$

where

$$2k \uparrow \bullet := \left. \begin{array}{c} \uparrow \\ \circ \\ \vdots \\ \bullet \\ \circ \end{array} \right\} 2k \text{ tokens}$$

Under the action superfunctor $\widehat{\mathbf{F}}_n$, elements of $\text{End}_{\mathcal{A}Q(z)}(\mathbb{1})$ correspond to elements in the center of U_q .

We expect that the central elements obtained in this way from $\textcircled{\bullet} 2k$, $k \in \mathbb{Z}_{>0}$, almost generate the center of U_q .

Future directions and open problems

Quantum isomeric Heisenberg supercategory

To each **central charge** $k \in \mathbb{Z}$, one can associate a **quantum Heisenberg category** (Brundan–S.–Webster 2020).

The central charge $k = 0$ quantum Heisenberg category is the **affine HOMFLYPT skein category** ($U_q(\mathfrak{gl}_n)$ case).

One should be able to define a **quantum isomeric Heisenberg supercategory** depending on a central charge $k \in \mathbb{Z}$, such that $k = 0$ recovers $\mathcal{AQ}(z)$.

Quantum isomeric Frobenius Heisenberg supercategory

More generally, one can associate a **quantum Frobenius Heisenberg supercategory** to every central charge $k \in \mathbb{Z}$ and Frobenius superalgebra A .

One should be able to define a **quantum isomeric Frobenius Heisenberg supercategory** depending on A and k , such that $A = \mathbb{k}$ recovers the quantum isomeric Heisenberg supercategory.