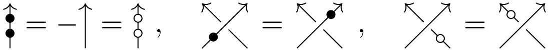
The quantum isomeric supercategory

("isomeric" = "queer")



Alistair Savage
University of Ottawa

Slides: alistairsavage.ca/talks

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Outline

Goal: Introduce an isomeric analogue of the (affine) HOMFLYPT skein category.

Overview:

1. Background and motivation
2. The quantum isomeric superalgebra
3. The quantum isomeric supercategory
4. The incarnation superfunctor
5. The quantum affine isomeric supercategory
6. The affine action superfunctor

Schur's lemma

Schur's lemma (non-super case)

If V is a f.d. simple module over an associative $\mathbb{C}$-algebra A, then

$$\mathrm{End}_A(V) \cong \mathbb{C}.$$

Schur's lemma (super case)

If V is a f.d. simple supermodule over an associative $\mathbb{C}$-superalgebra A, then there are two possibilities:

- V is of type M: $\mathrm{End}_A(V) \cong \mathbb{C}$.
- V is of type Q: $\mathrm{End}_A(V) \cong \mathrm{Cl}$, the 2-dimensional Clifford superalgebra.

Type Q simple supermodules are those that are isomorphic to their own parity shift. This parity shift is the odd generator of Cl.

Type M

Every f.d. superspace V is isomorphic to $\mathbb{C}^{m|n}$ for some $m, n \in \mathbb{N}$.

Let $M(m|n)$ denote the superalgebra of complex $(m|n) \times (m|n)$ supermatrices

$$\begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix}.$$

If V is a f.d. simple A-supermodule of type M, then the action gives a surjective homomorphism of superalgebras

$$A \twoheadrightarrow M(m|n).$$

The general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ is $M(m|n)$ with superbracket

$$[X, Y] = XY - (-1)^{\bar{X}\bar{Y}} YX.$$

Type Q

Consider the Clifford superalgebra

$$\mathrm{Cl} = \mathbb{C}[c]/(c^2+1), \quad c \text{ odd}.$$

Then Cl acts on $\mathbb{C}^{n|n}$, with c acting as

$$J := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Define $Q(n) = Q(n)_0 \oplus Q(n)_1$, where

$$Q(n)_i = \{C \in M(n|n)_i : CJ = (-1)^i JC\}.$$

If $\mathbb{C}^{n|n}$ is a f.d. simple A-supermodule of type Q, then the action gives a surjective homomorphism of superalgebras

$$A \twoheadrightarrow Q(n).$$

The isomeric Lie superalgebra $\mathfrak{q}_n$ is $Q(n)$ with superbracket

$$[X, Y] = XY - (-1)^{\bar{X}\bar{Y}} YX.$$

Background: $\mathfrak{gl}_{m|n}$ case

$\mathcal{OB}(t)$ = oriented Brauer category depending on a dimension parameter t.

There is a full monoidal functor

$$\mathcal{OB}(m-n) \to \mathfrak{gl}_{m|n}\text{-smod}$$

sending the generating object of $\mathcal{OB}(m-n)$ to the natural $\mathfrak{gl}_{m|n}$-supermodule $\mathbb{C}^{m|n}$.

Additive Karoubi envelope of $\mathcal{OB}(t)$ is Deligne's interpolating category $\underline{\mathrm{Rep}}(\mathrm{GL}_t)$.

Orthosymplectic case

$\mathcal{OB}(t)$ is replaced by the Brauer category (no longer oriented).

Background: $\mathfrak{gl}_{m|n}$ case

Any monoidal category acts on itself by the tensor product.

In particular, $\mathfrak{gl}_{m|n}$-smod acts on itself by translation functors $V \otimes -$.

This can be extended to a monoidal functor

$$\mathcal{AOB}(m-n) \to \mathcal{E}nd\,(\mathfrak{gl}_{m|n}\text{-smod}),$$

where $\mathcal{AOB}(t)$ is the affine oriented Brauer category [Brundan–Comes–Nash–Reynolds 2017].

Benefits:

- study natural transformations between translation functors,
- tool to study cyclotomic Hecke algebras,
- yields natural elements in the center of $U(\mathfrak{gl}_{m|n})$.

Background: quantum $\mathfrak{gl}_{m|n}$ case

In the quantum setting, $\mathcal{OB}(t)$ and $\mathcal{AOB}(t)$ are replaced by

- the HOMFLYPT skein category $\mathcal{OS}(z,t)$ [Turaev 1989], and
- the affine HOMFLYPT skein category $\mathcal{AOS}(z,t)$ [Brundan 2017].

We have monoidal functors

$$\mathcal{OS}(q-q^{-1},q^n) \to U_q(\mathfrak{gl}_n)\text{-mod},$$
$$\mathcal{AOS}(q-q^{-1},q^n) \to \mathcal{E}nd(U_q(\mathfrak{gl}_n)\text{-mod}).$$

(Can likely be generalized to the super setting $U_q(\mathfrak{gl}_{m|n})$.)

Background: isomeric case

In the isomeric case, we have [Brundan–Comes–Kujawa 2019]:

- the oriented Brauer–Clifford supercategory $\mathcal{OBC}$,
- the affine oriented Brauer–Clifford supercategory $\mathcal{AOBC}$.

There are monoidal superfunctors

$$\mathcal{OBC} \to \mathfrak{q}_n\text{-smod} \quad \text{and} \quad \mathcal{AOBC} \to \mathcal{SEnd}(\mathfrak{q}_n\text{-smod}).$$

Supercategories

We need to move to the setting of supercategories because there is an odd morphism in $\mathcal{OBC}$ and $\mathcal{AOBC}$ that corresponds to the parity shift endomorphism of the natural $\mathfrak{q}_n$-supermodule.

Goal

We want to develop a similar picture for the quantum isomeric superalgebra $U_q(\mathfrak{q}_n)$.

	Type M	Type Q
degenerate	$\mathcal{OB}(t)$, $\mathcal{AOB}(t)$	$\mathcal{OBC}$, $\mathcal{AOBC}$
quantum	(affine) HOMFLYPT skein	??

The quantum isomeric Lie superalgebra

The quantum analogue of $\mathfrak{q}_n$ was initially defined by Olshanski using a super analogue of the FRT construction.

Let $V = \mathbb{C}(q)^{n|n}$, and define an explicit invertible element

$$\Theta = \sum_{i,j} \Theta_{ij} \otimes E_{ij} \in \mathrm{End}(V)^{\otimes 2}$$

such that

$$\Theta \circ (J \otimes 1) = (J \otimes 1) \circ \Theta$$

and Θ satisfies the Yang–Baxter equation

$$\Theta^{12}\Theta^{13}\Theta^{23} = \Theta^{23}\Theta^{13}\Theta^{12}.$$

The quantum isomeric Lie superalgebra

The quantum isomeric superalgebra $U_q = U_q(\mathfrak{q}_n)$ is the unital associative superalgebra over $\mathbb{C}(q)$ generated by

$$u_{ij}, \qquad i, j \in \mathtt{I} = \{-1, -2, \dots, -n, 1, 2, \dots, n\}, \quad i \le j,$$

subject to the relations

$$u_{ii} u_{-i,-i} = 1 = u_{-i,-i} u_{ii},$$
$$L^{12} L^{13} \Theta^{23} = \Theta^{23} L^{13} L^{12}, \quad L := \sum_{i,j \in \mathtt{I},\ i \le j} u_{ij} \otimes E_{ij},$$

where the second-to-last equality takes place in $U_q \otimes \mathrm{End}_{\Bbbk}(V)^{\otimes 2}$. The parity of u_{ij} is $(-1)^{\mathrm{sgn}(i) + \mathrm{sgn}(j)}$.

Comments

1. A Drinfeld–Jimbo type presentation of U_q has been given by Grantcharov–Jung–Kang–Kim (2010).
2. U_q is a Hopf superalgebra, but it is not quasitriangular.

The natural representation

The map

$$U_q \to \mathrm{End}(V), \qquad u_{ij} \mapsto \Theta_{ij}, \quad i,j \in \mathtt{I}, \quad i \le j,$$

defines the natural representation of U_q on V.

Objective

We want to develop a graphical calculus describing the full monoidal sub-supercategory of U_q-smod generated by V and its dual V^*.

Note: This sub-supercategory is not semisimple.

Strict monoidal supercategories

Fix a ground field $\Bbbk$.

A strict monoidal supercategory is a category $\mathcal{C}$ enriched in the category of vector superspaces:

- each morphism space is a $\Bbbk$-supermodule,
- composition of morphisms is $\Bbbk$-bilinear and parity preserving,

together with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

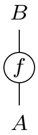
such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$,
- tensor product of morphisms is $\Bbbk$-bilinear and parity preserving.

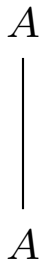
String diagrams

Fix a strict monoidal supercategory $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:



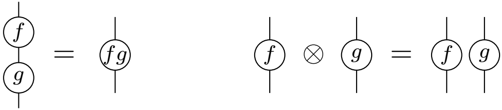
The identity map $1_A\colon A \to A$ is a string with no label:



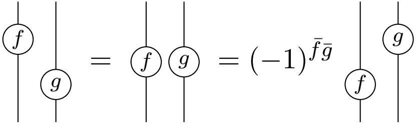
We sometimes omit the object labels when they are clear or unimportant.

String diagrams

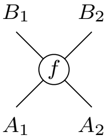
Composition is vertical stacking and tensor product is horizontal juxtaposition:



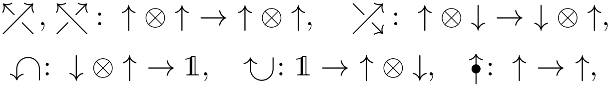
The super interchange law is:



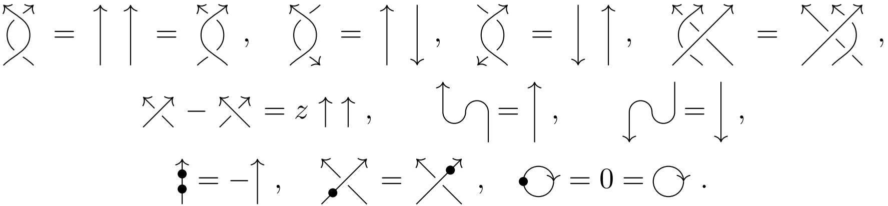
A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



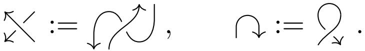
The quantum isomeric supercategory $Q(z)$ is the strict monoidal supercategory generated by objects $\uparrow$ and $\downarrow$ and morphisms



subject to the relations



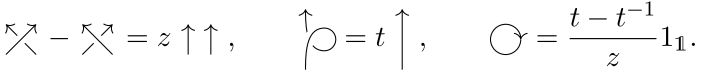
In the above, we have used left crossings and a right cap defined by



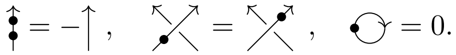
The Clifford token  is odd; all the other generating morphisms are even.

Relation to the HOMFLYPT skein category

Given $z, t \in \Bbbk^\times$, the HOMFLYPT skein category $\mathcal{OS}(z,t)$ is the quotient of the category of framed oriented tangles by the relations



The quantum isomeric supercategory $Q(z)$ is obtained from $\mathcal{OS}(z,1)$ by adjoining the Clifford token, subject to the relations

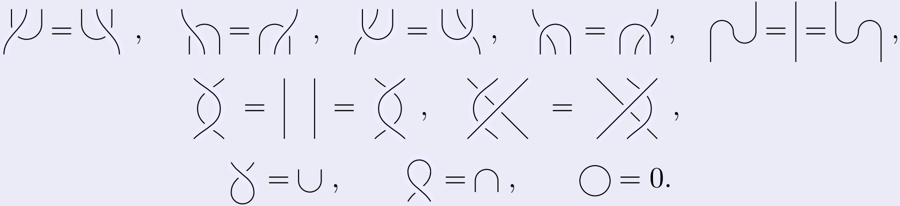


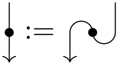
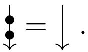
In particular, all relations in $\mathcal{OS}(z,1)$ hold in $Q(z)$.

Additional relations in $Q(z)$

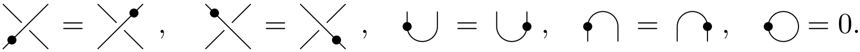
Example

The following relations hold in $Q(z)$ for all orientations of the strands:



Define , so that .

Then we have, for all orientations of the strands:



Properties of $Q(z)$

Pivotal

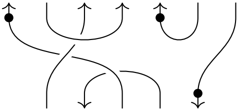
The supercategory $Q(z)$ is pivotal, with duality superfunctor given by rotation through $180°$ (with appropriate signs).

Intuitively, this means that morphism are invariant under signed isotopy.

Degenerate $(z = 0)$ case

$Q(0)$ is isomorphic to the oriented Brauer–Clifford supercategory introduced by Brundan–Comes–Kujawa (2019).

Basis theorem



Bases of morphism spaces in $Q(z)$ are given by diagrams satisfying:

- no closed strings,
- no self-intersections,
- no two strings cross each other more than once,
- all crossings are positive,
- strands can carry one (or zero) Clifford tokens near their terminus.

Note: Ignoring Clifford tokens, these diagrams give a bases for the HOMFLYPT skein category.

Endomorphism superalgebras

Proposition

The endomorphism superalgebra

$$\mathrm{End}_{Q(z)}(\uparrow^{\otimes r})$$

is isomorphic to the Hecke–Clifford superalgebra of rank r.

More generally, we have the following.

Proposition

The endomorphism superalgebra

$$\mathrm{End}_{Q(z)}(\uparrow^{\otimes r} \otimes \downarrow^{\otimes s})$$

is isomorphic to a quantum walled Brauer–Clifford superalgebra of Benkart–Guay–Jung–Kang–Wilcox (2016).

Evaluation and coevaluation

Define the evaluation map

$$\mathrm{ev}\colon V^* \otimes V \to \mathbb{C}(q), \qquad f \otimes v \mapsto f(v),$$

and the coevaluation map

$$\mathrm{coev}\colon \mathbb{C}(q) \to V \otimes V^*, \qquad 1 \mapsto \sum_{v \in \mathbf{B}_V} v \otimes v^*,$$

where $\mathbf{B}_V$ is a basis of V and $\{v^* : v \in \mathbf{B}_V\}$ is the dual basis.

These are both U_q-supermodule homomorphisms.

The incarnation superfunctor

Incarnation Theorem (S. 2022)

For each $n \in \mathbb{N}$, there exists a unique monoidal superfunctor

$$\mathbf{F}_n \colon Q(q - q^{-1}) \to U_q(\mathfrak{q}_n)\text{-smod}$$

such that

$$\mathbf{F}_n(\uparrow) = V, \qquad \mathbf{F}_n(\downarrow) = V^*,$$

$$\mathbf{F}_n(\nwarrow\!\!\!\!\!\nearrow) = \mathrm{flip} \circ \Theta, \qquad \mathbf{F}_n(\curvearrowleft) = \mathrm{ev}, \qquad \mathbf{F}_n(\dot{\uparrow}) = J.$$

Furthermore, $\mathbf{F}_n(\nwarrow\!\!\!\!\!\nearrow) = \Theta^{-1} \circ \mathrm{flip}$ and $\mathbf{F}_n(\circlearrowleft) = \mathrm{coev}$.

The superfunctor $\mathbf{F}_n$ is full for all $n \in \mathbb{N}$.

The induced map on morphism spaces is an isomorphism when the number of strands is $\leq n$.

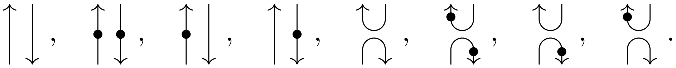
Example

Recall: The full monoidal sub-supercategory of U_q-smod generated by V and V^* is not semisimple.

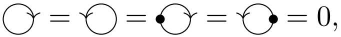
For $n \geq 2$, it follows from the incarnation and basis theorems that

$$\mathrm{End}_{U_q}(V \otimes V^*) \cong \mathrm{End}_{Q(z)}(\uparrow \otimes \downarrow)$$

has basis given by



Using the relations



we see that the span of the last four elements is a nilpotent ideal. Hence $\mathrm{End}_{U_q}(V \otimes V^*)$ is not semisimple.

Remarks

Note that $Q(z)$ does not depend on n.

However, the kernel of the incarnation superfunctor

$$Q(z) \to U_q(\mathfrak{q}_n)\text{-smod}$$

does depend on n.

The semisimplification of $Q(z)$ (i.e. the quotient by the tensor ideal of negligible morphisms) is the trivial supercategory since

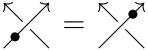
$$1_\uparrow \qquad \text{and} \qquad 1_\downarrow$$

are negligible morphisms; so only the trivial object survives.

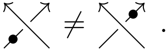
The chiral braiding

Since U_q is not quasitriangular, U_q-smod is not a braided monoidal supercategory.

Diagrammatically, this is because



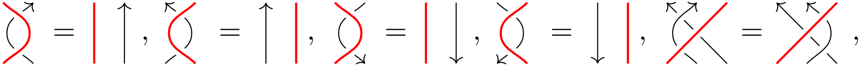
but



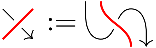
However, for each U_q-supermodule M, we have a U_q-supermodule homomorphism

$$= \mathrm{flip} \circ L \colon M \otimes V \to V \otimes M, \quad L := \sum_{i,j \in \mathtt{I},\ i \le j} u_{ij} \otimes E_{ij}.$$

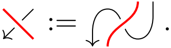
These satisfy



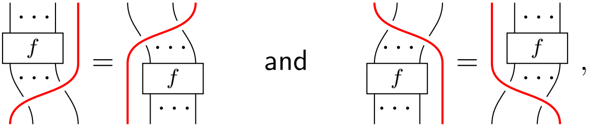
where



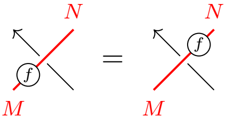
and



Furthermore, we have

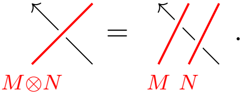


where f is any string diagram in $Q(z)$ not containing Clifford tokens,



for all $f \in \mathrm{Hom}_{U_q}(M, N)$.

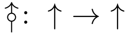
and



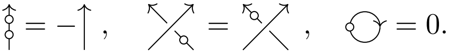
So we have something like a braiding, but only natural in one argument.

The quantum affine isomeric supercategory

The quantum affine isomeric supercategory $\mathcal{AQ}(z)$ is the strict monoidal supercategory obtained from $Q(z)$ by adjoining an additional odd morphism (the open Clifford token)



subject to the relations

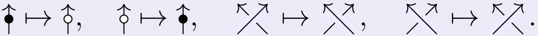


Symmetry

There is an isomorphism of monoidal supercategories

$$\mathcal{AQ}(z) \to \mathcal{AQ}(-z)$$

that interchanges open and closed Clifford tokens and flips crossings:



Affine endomorphism algebras

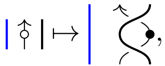
For $r \in \mathbb{N}$, there is a homomorphism of associative superalgebras

$$\mathrm{AHC}_r(z) \to \mathrm{End}_{\mathcal{AQ}(z)}(\uparrow^{\otimes r}),$$

where $\mathrm{AHC}_r(z)$ is the affine Hecke–Clifford superalgebra defined by Jones–Nazarov (1999).

Under the functor

$$\mathcal{AQ}(z) \to \mathcal{Q}(z),$$



where | and | denote some sequences of strands, we have

$\uparrow^{\otimes(r-i)} \otimes$  $\otimes \uparrow^{\otimes(i-1)} \mapsto -J_i,$

where $J_i \in \mathrm{End}_{\mathcal{Q}(z)}(\uparrow^{\otimes r})$ corresponds to the i-th Jucys–Murphy element under the isomorphism

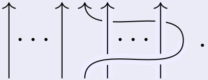
$$\mathrm{HC}_r(z) \cong \mathrm{End}_{\mathcal{Q}(z)}(\uparrow^{\otimes r}).$$

Affinization

Even affinization (Mousaaid–S. 2021)

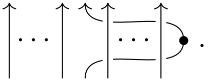
There exists a general notion of the affinization of a braided monoidal category corresponding to considering string diagrams on a cylinder.

Gives Jucys–Murphy elements



E.g. these are the JM elements in the Iwahori–Hecke algebra of type A in terms of string diagrams.

Above results suggest a theory of odd affinization, where above diagram is replaced by



The affine action superfunctor

$Q(z)$ acts on itself by the tensor product. We extend this action to $\mathcal{AQ}(z)$.

Theorem (S. 2022)

There is a unique monoidal superfunctor

$$\widehat{\mathbf{F}}_n \colon \mathcal{AQ}(z) \to \mathcal{SE}nd\,(U_q(\mathfrak{q}_n)\text{-smod}),$$

such that

$$\left(\widehat{\mathbf{F}}_n|_{Q(z)}\right)_M = - \otimes M, \qquad \text{and} \qquad \widehat{\mathbf{F}}_n(\mathring{\uparrow})_M = \left(\right).$$

In particular

$$\widehat{\mathbf{F}}_n(\uparrow) = V \otimes - \qquad \text{and} \qquad \widehat{\mathbf{F}}_n(\downarrow) = V^* \otimes -$$

are translation endosuperfunctors of U_q-smod.

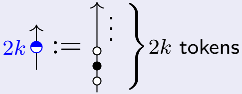
Central elements

Proposition (S. 2022)

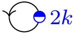
The supercommutative superalgebra $\mathrm{End}_{\mathcal{AQ}(z)}(\mathbb{1})$ is generated by

$$, \qquad k \in \mathbb{Z}_{>0},$$

where



Under the action superfunctor $\widehat{\mathbf{F}}_n$, elements of $\mathrm{End}_{\mathcal{AQ}(z)}(\mathbb{1})$ correspond to elements in the center of U_q.

We expect that the central elements obtained in this way from  , $k \in \mathbb{Z}_{>0}$, almost generate the center of U_q.

Future directions and open problems

Quantum isomeric Heisenberg supercategory

To each central charge $k \in \mathbb{Z}$, one can associate a quantum Heisenberg category (Brundan–S.–Webster 2020).

The central charge $k = 0$ quantum Heisenberg category is the affine HOMFLYPT skein category ($U_q(\mathfrak{gl}_n)$ case).

One should be able to define a quantum isomeric Heisenberg supercategory depending on a central charge $k \in \mathbb{Z}$, such that $k = 0$ recovers $\mathcal{AQ}(z)$.

Quantum isomeric Frobenius Heisenberg supercategory

More generally, one can associate a quantum Frobenius Heisenberg supercategory to every central charge $k \in \mathbb{Z}$ and Frobenius superalgebra A.

One should be able to define a quantum isomeric Frobenius Heisenberg supercategory depending on A and k, such that $A = \Bbbk$ recovers the quantum isomeric Heisenberg supercategory.