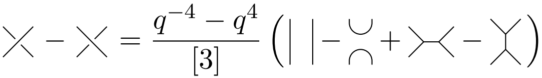
Diagratification



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Slides: alistairsavage.ca/talks

Papers

- arXiv:2107.12464 (with R. Gandhi and K. Zainoulline)
- arXiv:2204.11976 (with B. Westbury)

Outline

Goal: Describe a procedure for constructing diagrammatic presentations of categories of representations of Lie algebras and their quantum analogues.

Example: Type F_4

Overview:

1. Background: centralizers in classical types
2. String diagrams for monoidal categories
3. Example: The Brauer category
4. The diagrammatic F_4 category
5. The quantum diagrammatic F_4 category

Centralizers in type A

For $n \in \mathbb{N}$, let $V = \mathbb{C}^n$ be the natural $\mathrm{GL}(n)$-module.

The natural permutation action yields a surjective algebra homomorphism

$$\mathbb{C}\mathfrak{S}_r \twoheadrightarrow \mathrm{End}_{\mathrm{GL}(n)}(V^{\otimes r}).$$

This is part of Schur–Weyl duality.

If we want to include the dual module V^*, we have

$$\mathrm{OB}_{r,s}(n) \twoheadrightarrow \mathrm{End}_{\mathrm{GL}(n)}\left(V^{\otimes r} \otimes (V^*)^{\otimes s}\right),$$

where $\mathrm{OB}_{r,s}(n)$ is the walled Brauer algebra (or oriented Brauer algebra).

Centralizers in other types

Types BCD

The natural $\mathrm{O}(n)$-module V is self-dual. Let $n = \dim V$.

We have

$$\mathrm{B}_r(n) \twoheadrightarrow \mathrm{End}_{\mathrm{O}(n)}(V^{\otimes r}),$$

where $\mathrm{B}_r(n)$ is the Brauer algebra.

Similarly statement holds in the symplectic case.

Exceptional types

Less is known in exceptional types.

We will focus on the Lie algebra $\mathfrak{f}$ of type F_4.

If V is the natural $\mathfrak{f}$-module, we want

$$?? \twoheadrightarrow \mathrm{End}_{\mathfrak{f}}(V^{\otimes r})$$

Bigger picture

In fact, things become much more natural if we describe more.

General idea

Instead of only considering endomorphisms of $V^{\otimes r}$, we consider all morphisms $V^{\otimes r} \to V^{\otimes s}$ (and include dual V^* in type A).

Want a "nice" monoidal category $\mathcal{C}$, together with a full and essentially surjective monoidal functor

$$\mathcal{C} \to \mathfrak{g}\text{-mod}.$$

Desired features

1. $\mathcal{C}$ should be easy to describe, e.g. want a presentation in terms of generators and relations.
2. Should recover centralizer statements when restricting to appropriate endomorphism spaces.

Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.

Linear monoidal categories

For simplicity, we work over the ground field $\mathbb{C}$.

A strict linear monoidal category is a strict monoidal category such that

- each morphism space is a $\mathbb{C}$-module,
- composition of morphisms is $\mathbb{C}$-bilinear,
- tensor product of morphisms is $\mathbb{C}$-bilinear.

The interchange law

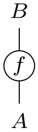
The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

$$\begin{array}{ccc} A_1 \otimes B_1 & \xrightarrow{1 \otimes g} & A_1 \otimes B_2 \\ {\scriptstyle f \otimes 1} \downarrow & \searrow^{f \otimes g} & \downarrow {\scriptstyle f \otimes 1} \\ A_2 \otimes B_1 & \xrightarrow[1 \otimes g]{} & A_2 \otimes B_2 \end{array}$$

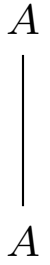
String diagrams

Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:



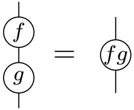
The identity map $1_A\colon A \to A$ is a string with no label:

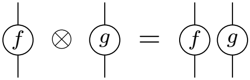


We sometimes omit the object labels when they are clear or unimportant.

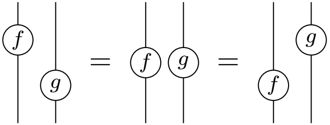
String diagrams

Composition is vertical stacking and tensor product is horizontal juxtaposition:

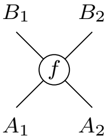




The interchange law then becomes:



A morphism $f \colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



Example: Brauer category

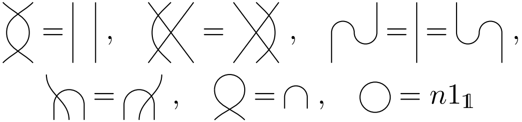
Fix $n \in \mathbb{C}$. The Brauer category $\mathcal{B}(n)$ is the linear monoidal category defined as follows.

One generating object: I

Three generating morphisms:

 $: \mathbb{1} \to \mathsf{I}^{\otimes 2}$,  $: \mathsf{I}^{\otimes 2} \to \mathbb{1}$,  $: \mathsf{I}^{\otimes 2} \to \mathsf{I}^{\otimes 2}$.

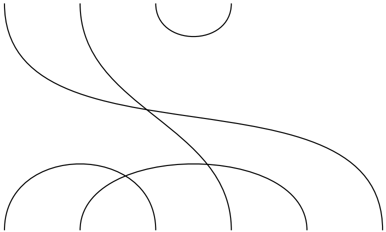
Relations:



(We'll motivate these relations soon.)

Example: Brauer category

An arbitrary morphism in $\mathcal{B}(n)$ is a linear combination of Brauer diagrams. E.g.

 $: \mathsf{I}^{\otimes 6} \to \mathsf{I}^{\otimes 4}.$

Composition: vertical “gluing”, replace closed components by a factor of n.

Tensor product: horizontal concatenation.

Example: Brauer category

Universal property

Any linear symmetric monoidal category $\mathcal{C}$ with self-dual object V of dimension n admits a linear monoidal functor

$$\mathcal{B}(n) \to \mathcal{C}, \quad | \mapsto V.$$

Corollary

We have a linear monoidal functor ($V =$ natural module)

$$\mathcal{B}(n) \to \mathrm{O}(n)\text{-mod}, \qquad | \mapsto V.$$

$$\begin{aligned}
\times &\mapsto \left(V^{\otimes 2} \to V^{\otimes 2}, \quad v \otimes w \mapsto w \otimes v\right), \\
\cap &\mapsto \left(V^{\otimes 2} \to \mathbb{C}, \quad v \otimes w \mapsto \langle v, w \rangle\right), \\
\cup &\mapsto \left(\mathbb{C} \to V^{\otimes 2}, \quad 1 \mapsto \textstyle\sum_{v \in \mathbf{B}_V} v \otimes v\right),
\end{aligned}$$

where $\langle\ ,\ \rangle$ is the bilinear form, and $\mathbf{B}_V$ is an orthonormal basis.

Example: Brauer category

In fact, this functor is full. In particular, we recover the surjection

$$\mathrm{B}_r(n) \cong \mathrm{End}_{\mathcal{B}(n)}(\mathsf{I}^{\otimes r}) \twoheadrightarrow \mathrm{End}_{\mathrm{O}(n)}(V^{\otimes r}),$$

where $\mathrm{B}_r(n)$ is the Brauer algebra.

Fact: Every f.d. $\mathrm{O}(n)$-module is a summand of $V^{\otimes r}$ for some r.

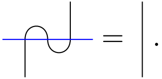
Philosophy

All homomorphisms between f.d. $\mathrm{O}(n)$-modules are built from the "flip" symmetry, the bilinear pairing, and its dual.

Similar statements are true in the symplectic case.

Understanding the relations

Nondegeneracy of the bilinear form gives the relation

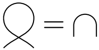


The left-hand side is the composition

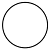
$$V \cong V \otimes \mathbb{C} \xrightarrow{1_V \otimes \bigcup} V \otimes V \otimes V \xrightarrow{\bigcap \otimes 1_V} \mathbb{C} \otimes V \cong V,$$

$$w \mapsto w \otimes 1 \mapsto \sum_{v \in \mathbf{B}_V} w \otimes v \otimes v \mapsto \sum_{v \in \mathbf{B}_V} \langle w, v \rangle \otimes v \mapsto \sum_{v \in \mathbf{B}_V} \langle w, v \rangle v = w.$$

Symmetry of the bilinear form gives the relation:



We also have



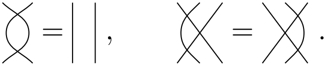
$$: \mathbb{C} \to \mathbb{C}, \quad 1 \mapsto \sum_{v \in \mathbf{B}_V} \langle v, v \rangle = \dim V.$$

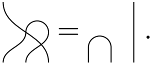
So the "bubble" corresponds to the dimension of the object V.

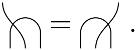
Understanding the relations

The remaining relations come from the fact that the flip map endows the category with the structure of a symmetric monoidal category.

Flip satisfies the relations of the symmetric group:



Naturality gives 

Composing on the bottom with  gives 

A small endomorphism space ($n \geq 4$)

We have $V = V_{\omega_1}$ and $V^{\otimes 2} = V_{2\omega_1} \oplus V_{\omega_2} \oplus \mathbb{C}$.

Then $\mathrm{End}_{\mathrm{O}(n)}(V^{\otimes 2})$ is 3-dimensional, spanned by

Goal

Goal

We want analogous results in type F_4.

Let $\mathfrak{f}$ be the simple complex Lie algebra of type F_4.

We want a linear monoidal category $\mathcal{F}$ and a functor

$$\Phi\colon \mathcal{F} \to \mathfrak{f}\text{-mod},$$

such that

- $\mathcal{F}$ has a simple presentation,
- Φ is full (surjective on morphisms),
- Φ is essentially surjective (surjective on isomorphism classes) after adding in summands.

Question

How do we find a presentation of $\mathcal{F}$ using as little information about the representation theory of $\mathfrak{f}$ as possible?

Important observations about F_4

Let $V = V_{\omega_4}$ be the smallest nontrivial simple $\mathfrak{f}$-module $(\dim V = 26)$.

Facts

- V is self-dual.
- Every f.d. simple $\mathfrak{f}$-module is contained in $V^{\otimes r}$ for some $r \in \mathbb{N}$.
- We have

$$\dim \operatorname{Hom}_{\mathfrak{f}}(V, V) = 1, \qquad \dim \operatorname{Hom}_{\mathfrak{f}}(V^{\otimes 2}, V) = 1,$$
$$\dim \operatorname{Hom}_{\mathfrak{f}}(V^{\otimes 2}, V^{\otimes 2}) = 5, \qquad \dim \operatorname{Hom}_{\mathfrak{f}}(V^{\otimes 2}, V^{\otimes 3}) = 15.$$

  Note: $\operatorname{Hom}_{\mathfrak{f}}(V^{\otimes m}, V^{\otimes n}) \cong \operatorname{Hom}_{\mathfrak{f}}(V^{\otimes (m+n)}, \mathbb{C})$ for all $m, n \in \mathbb{N}$.
- We have a decomposition

$$V^{\otimes 2} = \mathbb{C} \oplus V \oplus V_{\omega_1} \oplus V_{\omega_3} \oplus V_{2\omega_4}.$$

Generators of the diagrammatic category

Generating object

We have one generating object |, corresponding to V.

Generating morphisms

$\mathrm{Hom}_{\mathfrak{f}}(V^{\otimes 2}, \mathbb{C})$ and $\mathrm{Hom}_{\mathfrak{f}}(\mathbb{C}, V^{\otimes 2})$ are spanned by

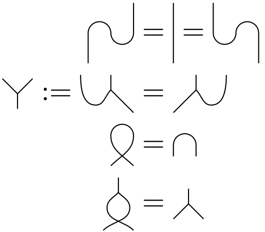
$$\cap \colon V \otimes V \to \mathbb{C} \quad \text{and} \quad \cup \colon \mathbb{C} \to V \otimes V.$$

$\mathrm{Hom}_{\mathfrak{f}}(V^{\otimes 2}, V)$ is spanned by  $\colon V^{\otimes 2} \to V$.

We have the flip map

 $\colon V^{\otimes 2} \to V^{\otimes 2}, \qquad v \otimes w \mapsto w \otimes v.$

We have



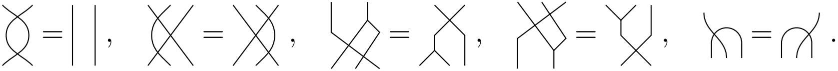
nondegeneracy of the bilinear form on V

$\dim \mathrm{Hom}_{\mathfrak{f}}(V, V^{\otimes 2}) = 1$

symmetry of the bilinear form

symmetry of the map 

The flip map  endows $\mathfrak{f}$-mod with the structure of a symmetric monoidal category:



More initial relations

Since $\dim \mathrm{Hom}_{\mathfrak{f}}(V, V) = 1$, we have

 $= \alpha \Big|$ for some $\alpha \in \mathbb{C}$.

Since $\dim \mathrm{Hom}_{\mathfrak{f}}(\mathbb{C}, \mathbb{C}) = 1$, we have

 $= n 1_{\mathbb{1}}$ for some $n \in \mathbb{C}$.

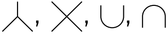
(In fact, $n = \dim V = 26$.)

Since $\dim \mathrm{Hom}_{\mathfrak{f}}(\mathbb{C}, V) = 0$, we have

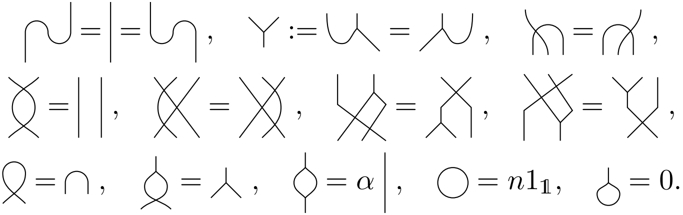
 $= 0$.

A preliminary category

For $\alpha, n \in \mathbb{C}$, $\mathcal{T} = \mathcal{T}_{\alpha,n}$ is the linear monoidal category defined as follows.

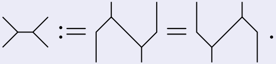
One generating object: I   Four generating morphisms: 

Relations:



Notes

- Can scale ∪ and ∩ to reduce to case $\alpha = 1$.
- Category is strict pivotal (isotopy invariance), so we can define



Additional relations

For simplicity, we will assume throughout that $n \neq -2$.

One can already show that we have a full functor

$$\mathcal{T} \to \mathfrak{f}\text{-mod}.$$

But the functor is certainly not faithful. So we need to impose more relations on morphisms in $\mathcal{T}$.

Let's focus on the space $\mathrm{Hom}_{\mathfrak{f}}(V^{\otimes 2}, V^{\otimes 2})$.

Use two facts

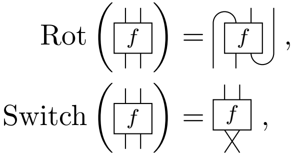
1. The kernel of the functor $\mathcal{T} \to \mathfrak{f}$-mod is a tensor ideal (i.e. closed under tensor product and composition).
2. We know $\dim \mathrm{Hom}_{\mathfrak{f}}(V^{\otimes 2}, V^{\otimes 2}) = 5$.

Rotation and switch

Consider the linear operators

$$\mathrm{Rot}\colon \mathrm{Hom}(\mathsf{I}^{\otimes 2}, \mathsf{I}^{\otimes 2}) \to \mathrm{Hom}(\mathsf{I}^{\otimes 2}, \mathsf{I}^{\otimes 2}),$$

$$\mathrm{Switch}\colon \mathrm{Hom}(\mathsf{I}^{\otimes 2}, \mathsf{I}^{\otimes 2}) \to \mathrm{Hom}(\mathsf{I}^{\otimes 2}, \mathsf{I}^{\otimes 2}),$$



In other words,

$$\mathrm{Rot}(f) = \left(\bigcap \otimes 1_{\mathsf{I}}^{\otimes 2}\right) \circ (1_{\mathsf{I}} \otimes f \otimes 1_{\mathsf{I}}) \circ (1_{\mathsf{I}}^{\otimes 2} \otimes \bigcup),$$

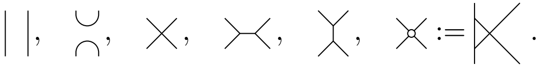
$$\mathrm{Switch}(f) = f \circ \times.$$

Any tensor ideal of $\mathcal{T}$ or $\mathcal{F}$ is invariant under Rot and Switch.

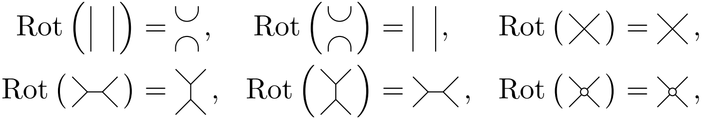
In particular, the kernel of the functor $\mathcal{T} \to \mathfrak{f}$-mod is invariant under Rot and Switch.

Rotation and switch

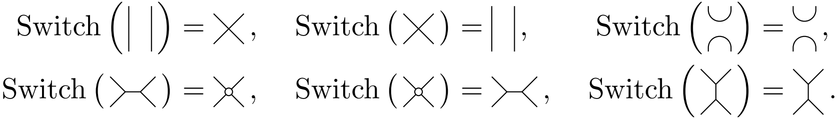
The operators Rot and Switch generate an action of $\mathfrak{S}_3$ on the 6-dimensional space spanned by



More precisely, one can show (using only relations in $\mathcal{T}$) that

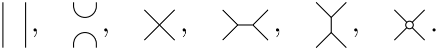


and



Deducing the relations

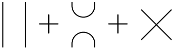
Since $\dim \mathrm{Hom}(V^{\otimes 2}, V^{\otimes 2}) = 5$, we want a linear dependence relation on the morphisms



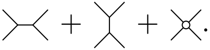
Key question: How do we figure out this relation without doing explicit computations in $\mathfrak{f}$-mod?

Key idea: Since the kernel of the functor $\mathcal{F} \to \mathfrak{f}$-mod is invariant under Rot and Switch, our relation must be an eigenvector for these operators. (Otherwise our space of relations would be too big.)

The space of simultaneous eigenvectors for both Rot and Switch is spanned by



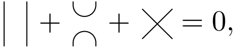
and



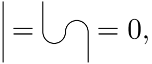
So our relation must be some linear combination of these two.

Deducing the relation

If

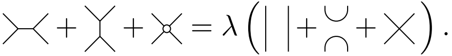


then composing on bottom with $\cup$ yields $(n+2)\cup = 0$. But then



and so the category becomes trivial.

So we want a relation of the form

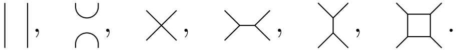


Composing on the bottom with $\cup$ gives

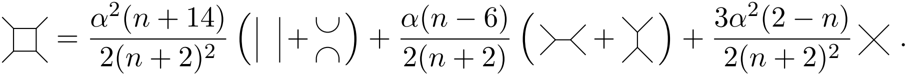
$$2\alpha \cup = \lambda(n+2)\cup \qquad \text{and so} \qquad \lambda = \frac{2\alpha}{n+2}.$$

Additional relations

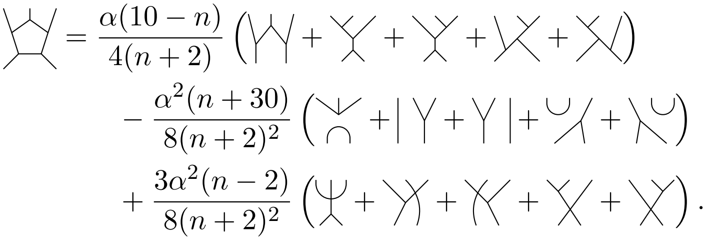
Since $\dim \mathrm{Hom}_{\mathfrak{f}}(V^{\otimes 2}, V^{\otimes 2}) = 5$, we also want a linear dependence relation on the morphisms



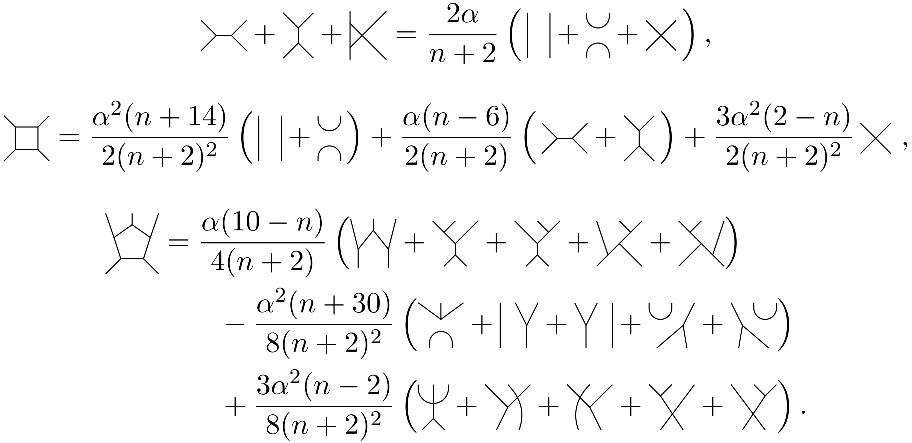
A similar analysis shows that the relation must be



Finally, using $\dim \mathrm{Hom}_{\mathfrak{f}}(V^{\otimes 2}, V^{\otimes 3}) = 15$, we can deduce the relation



The category $\mathcal{F} = \mathcal{F}_{\alpha,n}$ is obtained from $\mathcal{T}_{\alpha,n}$ by imposing three additional relations:



Theorem (Gandhi–S.–Zaynullin 2021)

1. There is a monoidal functor

$$\Phi\colon \mathcal{F}_{7/3,26} \to \mathfrak{f}\text{-mod}, \qquad \mathsf{I} \mapsto V.$$

2. The functor Φ is full (surjective on morphisms).

3. Passing to the additive Karoubi envelope (idempotent completion of the additive envelope) yields a functor

$$\mathrm{Kar}(\Phi)\colon \mathrm{Kar}(\mathcal{F}_{7/3,26}) \to \mathfrak{f}\text{-mod}.$$

that is essentially surjective (i.e. every f.d. $\mathfrak{f}$-module is isomorphic to an object in the image of $\mathrm{Kar}(\Phi)$).

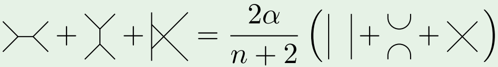
Remarks

Remark

The functor Φ can be described explicitly using the Albert algebra, which is the unique exceptional Jordan algebra.

Remark

The relation



corresponds to the Cayley–Hamilton equation for traceless 3×3 octonionic matrices.

The upshot

Since

$$\mathrm{Kar}(\Phi)\colon \mathrm{Kar}(\mathcal{F}_{7/3,26}) \to \mathfrak{f}\text{-mod}$$

is full and essentially surjective,

$$\mathfrak{f}\text{-mod is a quotient of } \mathrm{Kar}(\mathcal{F}_{7/3,26}).$$

Philosophy

All homomorphisms between f.d. $\mathfrak{f}$-modules are built from the "flip" symmetry, the bilinear form (and its dual), and a trilinear form (or the multiplication in the Albert algebra).

Question

Is the functor $\mathrm{Kar}(\Phi)$ faithful, and hence an equivalence of categories, or are there further relations?

Quantization

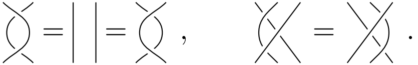
One can repeat the strategy for the quantized enveloping algebra $U_q(\mathfrak{f})$.

The main idea is that we move from symmetric monoidal categories to braided monoidal categories.

So we replace the flip  with two morphisms

,  $: \mathsf{I} \otimes \mathsf{I} \to \mathsf{I} \otimes \mathsf{I}$,

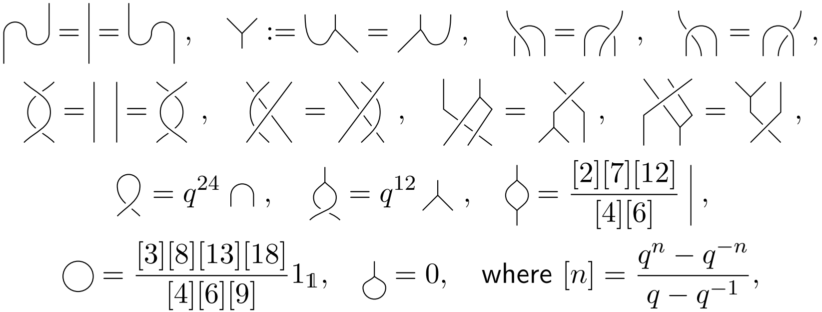
that are mutually inverse and satisfy the braid relations:



Quantum diagrammatic category

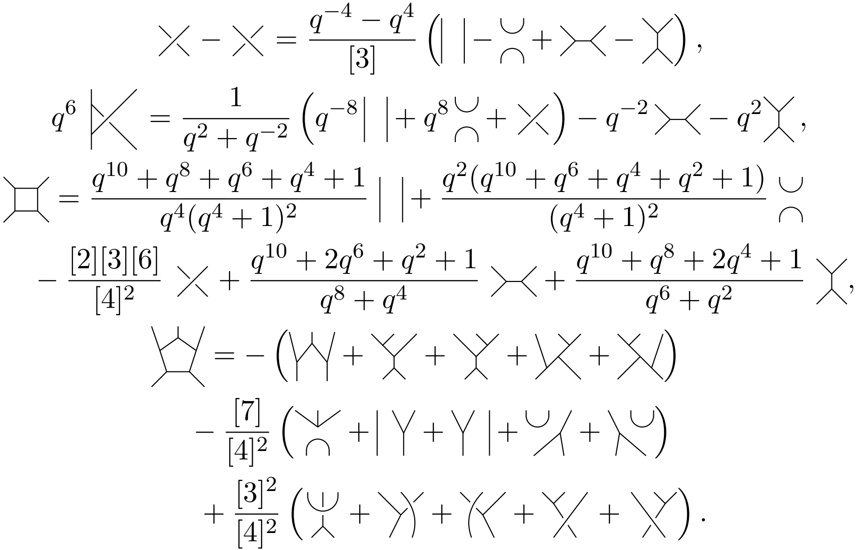
Let $\mathcal{F}_q$ be the $\mathbb{Q}(q)$-linear strict monoidal category generated by the object I and generating morphisms

subject to the following relations:



(cont. on next page)

Quantum diagrammatic category (cont.)



Theorem (S.–Westbury 2022)

1. There is a monoidal functor

$$\Phi_q \colon \mathcal{F}_q \to U_q(\mathfrak{f})\text{-mod}, \qquad \mathsf{I} \mapsto V_q,$$

where V_q is the quantum analogue of V.

2. The functor Φ_q is full (surjective on morphisms).

3. Passing to the additive Karoubi envelope (idempotent completion of the additive envelope) yields a functor

$$\mathrm{Kar}(\Phi_q) \colon \mathrm{Kar}(\mathcal{F}_q) \to U_q(\mathfrak{f})\text{-mod}.$$

that is essentially surjective (i.e. every type I f.d. $U_q(\mathfrak{f})$-module is isomorphic to an object in the image of $\mathrm{Kar}(\Phi_q)$).

Summary

Comparison to other types

- $\mathcal{F}$ is a type F_4 analogue of the Brauer category.
- $\mathcal{F}_q$ is a type F_4 analogue of the framed HOMFLYPT skein category (type A), the Kauffman category (types BCD), and Kuperberg's G_2 category.

Endomorphism algebras

$\mathrm{End}_{\mathcal{F}}(\mathsf{I}^{\otimes r})$ is a type F_4 analogue of the Brauer algebra.

We now have diagrammatic tools to study representation theory in type F_4.