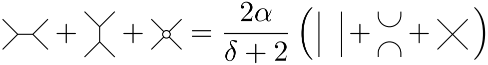
Diagrammatics for F_4



Alistair Savage
University of Ottawa

Slides: alistairsavage.ca/talks
Preprint: arXiv:2107.12464

Joint with Raj Gandhi and Kirill Zaynullin.

Outline

Goal: Define an F_4 analogue of the oriented and unoriented Brauer categories and algebras.

Overview:

1. Centralizers in classical types
2. Diagrammatic categories
3. Oriented and unoriented Brauer categories
4. The category $\mathcal{F}$
5. The functor $\mathcal{F} \to \mathfrak{f}$-mod
6. Further directions

Centralizers in type A

For $\delta \in \mathbb{N}$, let $V = \mathbb{C}^\delta$ be the natural $\mathfrak{gl}_\delta$-module.

The natural permutation action yields a surjective algebra homomorphism

$$\mathbb{C}\mathfrak{S}_r \twoheadrightarrow \mathrm{End}_{\mathfrak{gl}_\delta}(V^{\otimes r}).$$

This is part of Schur–Weyl duality.

If we want to include the dual module V^*, we have

$$\mathrm{OB}_{r,s}(\delta) \twoheadrightarrow \mathrm{End}_{\mathfrak{gl}_\delta}\left(V^{\otimes r} \otimes (V^*)^{\otimes s}\right),$$

where $\mathrm{OB}_{r,s}(\delta)$ is the walled Brauer algebra (or oriented Brauer algebra).

Centralizers in other types

Types BCD

Let $\mathfrak{g}$ be a simple Lie algebra of type BCD.

The natural module V is self-dual. Let $\delta = \dim V$.

We have

$$\mathrm{B}_r(\delta) \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(V^{\otimes r}),$$

where $\mathrm{B}_r(\delta)$ is the Brauer algebra.

Exceptional types

Less is known in exceptional types.

We will focus on the Lie algebra $\mathfrak{f}$ of type F_4.

If V is the natural $\mathfrak{f}$-module, we want

$$?? \twoheadrightarrow \mathrm{End}_{\mathfrak{f}}(V^{\otimes r})$$

Bigger picture

In fact, things become much more natural if we describe more.

General idea

Instead of only considering endomorphisms of $V^{\otimes r}$, we consider all morphisms $V^{\otimes r} \to V^{\otimes s}$ (and include dual V^* in type A).

Want a "nice" monoidal category $\mathcal{C}$, together with a full and essentially surjective monoidal functor

$$\mathcal{C} \to \mathfrak{g}\text{-mod}.$$

Desired features

1. $\mathcal{C}$ should be easy to describe, e.g. want a presentation in terms of generators and relations.
2. Should recover above centralizer statements when restricting to appropriate morphism spaces.

Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.

Linear monoidal categories

For simplicity, we work over the ground field $\mathbb{C}$.

A strict linear monoidal category is a strict monoidal category such that

- each morphism space is a $\mathbb{C}$-module,
- composition of morphisms is $\mathbb{C}$-bilinear,
- tensor product of morphisms is $\mathbb{C}$-bilinear.

The interchange law

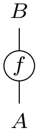
The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

$$\begin{tikzcd}
A_1 \otimes B_1 \arrow[r, "1\otimes g"] \arrow[d, "f\otimes 1"'] \arrow[dr, "f\otimes g"] & A_1 \otimes B_2 \arrow[d, "f\otimes 1"] \\
A_2 \otimes B_1 \arrow[r, "1\otimes g"'] & A_2 \otimes B_2
\end{tikzcd}$$

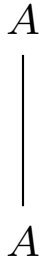
String diagrams

Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:



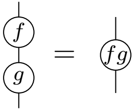
The identity map $1_A\colon A \to A$ is a string with no label:

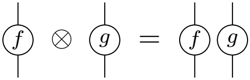


We sometimes omit the object labels when they are clear or unimportant.

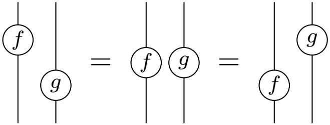
String diagrams

Composition is vertical stacking and tensor product is horizontal juxtaposition:

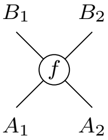




The interchange law then becomes:



A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



Presentations of strict monoidal categories

One can give presentations of some strict $\mathbb{C}$-linear monoidal categories, just as for monoids, groups, algebras, etc.

Objects: If the objects are generated by some collection A_i, $i \in I$, then we have all possible tensor products of these objects:

$$\mathbb{1}, \quad A_i, \quad A_i \otimes A_j \otimes A_k \otimes A_\ell, \quad \text{etc.}$$

Morphisms: If the morphisms are generated by some collection f_j, $j \in J$, then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

$$1_{A_i}, \quad f_j \otimes (f_i f_k) \otimes (f_\ell), \quad \text{etc.}$$

We then often impose some relations on these morphism spaces.

String diagrams: We can build complex diagrams out of our simple generating diagrams.

Example: monoidally generated symmetric groups

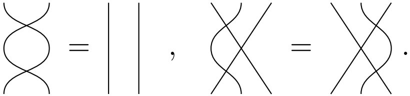
Define a strict monoidal category $\mathcal{Sym}$ with one generating object I and denote

$$1_{\mathsf{I}} = \Big|$$

We have one generating morphism

 $: \mathsf{I} \otimes \mathsf{I} \to \mathsf{I} \otimes \mathsf{I}.$

We impose the relations:



Then

$$\mathrm{End}_{\mathcal{Sym}}(\mathsf{I}^{\otimes n}) = S_n$$

is the symmetric group on n letters.

Example: monoidally generated symmetric groups

This monoidal presentation of S_n is very efficient! We only needed

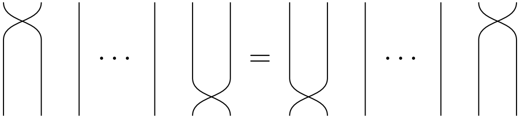
- one generating morphism, and
- two relations,

to get all the symmetric groups.

Note that the "distant braid relation"

$$s_i s_j = s_j s_i, \qquad |i-j| > 1$$

for simple transpositions follows for free from the interchange law:



Note: If we define $\mathcal{Sym}$ to be linear, then $\mathrm{End}_{\mathcal{Sym}}(\uparrow^{\otimes n}) = \mathbb{C}S_n$.

Oriented Brauer category

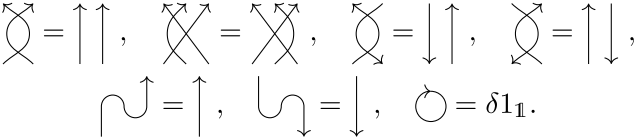
For $\delta \in \mathbb{C}$, the oriented Brauer category $\mathcal{OB}(\delta)$ is the linear monoidal category defined as follows.

Two generating objects: $\uparrow$ and $\downarrow$

Four generating morphisms: 

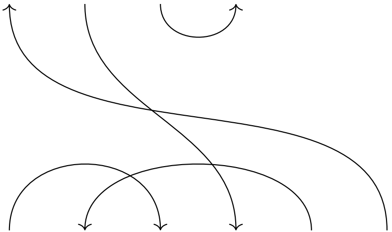
Define 

Relations:



Oriented Brauer category

An arbitrary morphism in $\mathcal{OB}(\delta)$ is a $\mathbb{C}$-linear combination of oriented Brauer diagrams. E.g.



$$: \uparrow \otimes \downarrow^{\otimes 3} \otimes \uparrow^{\otimes 2} \to \uparrow \otimes \downarrow^{\otimes 2} \otimes \uparrow .$$

Composition: vertical "gluing", replace closed components by a factor of δ.

Tensor product: horizontal concatenation.

Oriented Brauer category

Universal property

Any linear symmetric monoidal category $\mathcal{C}$ with dual objects V, V^* of dimension δ admits a linear monoidal functor

$$\mathcal{OB}(\delta) \to \mathcal{C}, \quad \uparrow \mapsto V, \quad \downarrow \mapsto V^*.$$

Corollary

We have a linear monoidal functor (V = natural module)

$$\mathcal{OB}(\delta) \to \mathfrak{gl}_\delta\text{-mod}, \quad \uparrow \mapsto V, \quad \downarrow \mapsto V^*.$$

 $\mapsto \left(V^{\otimes 2} \to V^{\otimes 2}, \quad v \otimes w \mapsto w \otimes v\right),$

 $\mapsto (V^* \otimes V \to V \otimes V^*, \quad f \otimes v \mapsto v \otimes f),$

 $\mapsto (V \otimes V^* \to \mathbb{C}, \quad v \otimes f \mapsto f(v)),$

 $\mapsto \left(\mathbb{C} \to V^* \otimes V, \quad 1 \mapsto \sum_{v \in \mathbf{B}_V} v^* \otimes v\right),$

where $\mathbf{B}_V$ is a basis for V and v^*, $v \in \mathbf{B}_V$, is the dual basis.

Oriented Brauer category

In fact, this functor is full. In particular, we recover the surjection

$$\mathrm{OB}_{r,s}(\delta) \cong \mathrm{End}_{\mathcal{OB}(\delta)}(\uparrow^{\otimes r} \otimes \downarrow^{\otimes s}) \twoheadrightarrow \mathrm{End}_{\mathfrak{gl}_\delta}\left(V^{\otimes r} \otimes (V^*)^{\otimes s}\right),$$

where $\mathrm{OB}_{r,s}(\delta)$ is the walled Brauer algebra (or oriented Brauer algebra).

Fact: Every f.d. $\mathfrak{gl}_\delta$-module is a summand of $V^{\otimes r} \otimes (V^*)^{\otimes s}$ for some r, s.

Philosophy

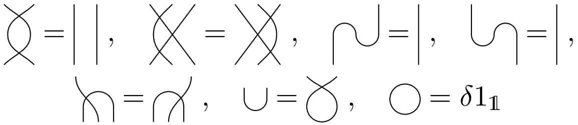
All homomorphisms between f.d. $\mathfrak{gl}_\delta$-modules are built from the "flip" symmetry and the natural pairing of V and V^*.

Fix $\delta \in \mathbb{C}$. The Brauer category $\mathcal{B}(\delta)$ is the linear monoidal category defined as follows.

One generating object: |

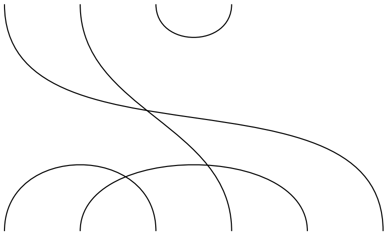
Three generating morphisms: $\cup$, $\cap$, $\times$

Relations:



Brauer category

An arbitrary morphism in $\mathcal{B}(\delta)$ is a linear combination of Brauer diagrams. E.g.



$: \mathsf{I}^{\otimes 6} \to \mathsf{I}^{\otimes 4}.$

Composition: vertical "gluing", replace closed components by a factor of δ.

Tensor product: horizontal concatenation.

Brauer category

Universal property

Any linear symmetric monoidal category $\mathcal{C}$ with self-dual object V of dimension δ admits a linear monoidal functor

$$\mathcal{B}(\delta) \to \mathcal{C}, \quad | \mapsto V.$$

Corollary

We have a linear monoidal functor ($V =$ natural module)

$$\mathcal{B}(\delta) \to \mathfrak{so}_\delta\text{-mod}, \qquad | \mapsto V.$$

$$\begin{aligned}
\times &\mapsto \left(V^{\otimes 2} \to V^{\otimes 2}, \quad v \otimes w \mapsto w \otimes v\right), \\
\cap &\mapsto \left(V^{\otimes 2} \to \mathbb{C}, \quad v \otimes w \mapsto \langle v, w \rangle\right), \\
\cup &\mapsto \left(\mathbb{C} \to V^{\otimes 2}, \quad 1 \mapsto \textstyle\sum_{v \in \mathbf{B}_V} v \otimes v^\vee\right),
\end{aligned}$$

where $\langle\ ,\ \rangle$ is the bilinear form, and $^\vee$ denotes the dual basis.

Brauer category

In fact, this functor is full. In particular, we recover the surjection

$$\mathrm{B}_r(\delta) \cong \mathrm{End}_{\mathcal{B}(\delta)}(\mathsf{I}^{\otimes r}) \twoheadrightarrow \mathrm{End}_{\mathfrak{so}_\delta}(V^{\otimes r}),$$

where $\mathrm{B}_r(\delta)$ is the Brauer algebra

Fact: Every f.d. $\mathfrak{so}_\delta$-module is a summand of $V^{\otimes r}$ for some r.

Philosophy

All homomorphisms between f.d. $\mathfrak{so}_\delta$-modules are built from the "flip" symmetry and the bilinear pairing.

Similar statements are true in the symplectic case.

Goal

We want analogous results in type F_4.

Thus, we want a linear monoidal category $\mathcal{F}$ and a functor

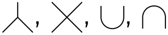
$$\Phi \colon \mathcal{F} \to \mathfrak{f}\text{-mod}$$

such that

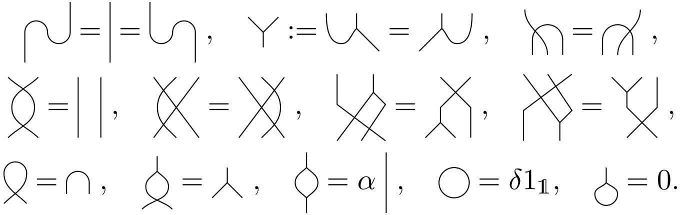
- $\mathcal{F}$ has a simple presentation,
- Φ is full (surjective on morphisms)
- Φ is essentially surjective (surjective on isomorphism classes) after adding in summands.

A preliminary category

For $\alpha, \delta \in \mathbb{C}$, $\mathcal{T} = \mathcal{T}_{\alpha,\delta}$ is the linear monoidal category defined as follows.

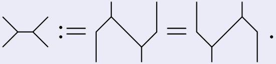
One generating objects: I   Four generating morphisms: 

Relations:

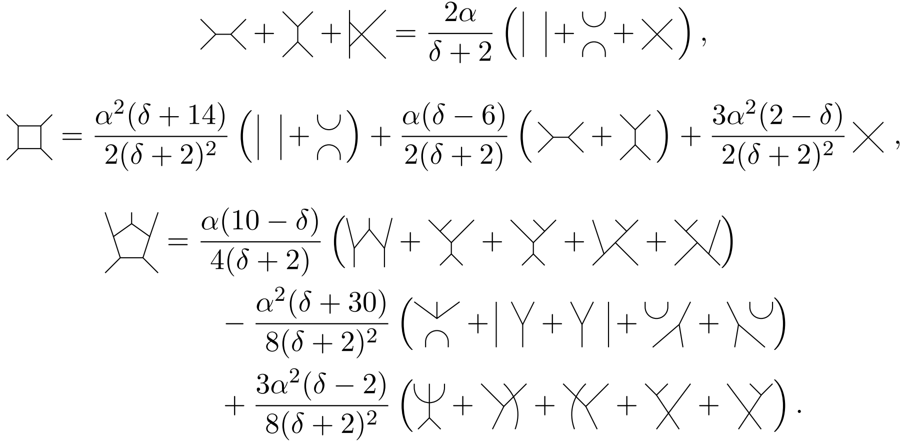


Notes

- Can scale  by $\alpha^{-1/2}$ to reduce to case $\alpha = 1$.
- Category is strict pivotal (isotopy invariance), so we can define



Assume $\delta \neq -2$. Then $\mathcal{F} = \mathcal{F}_{\alpha,\delta}$ is obtained from $\mathcal{T}_{\alpha,\delta}$ by imposing three additional relations:



The Albert algebra

Let

$$A_{\mathbb{R}} = \left\{ \begin{pmatrix} \lambda_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \lambda_2 & x_1 \\ x_2 & \bar{x}_1 & \lambda_3 \end{pmatrix} : \lambda_i \in \mathbb{R},\ x_i \in \mathbb{O} \right\}$$

denote the set of 3×3 self-adjoint matrices over the octonions $\mathbb{O}$, equipped with the bilinear operation

$$a \circ b := \frac{1}{2}(ab + ba), \quad a, b \in A_{\mathbb{R}},$$

where the juxtaposition ab denotes usual matrix multiplication.

We have a trace map

$$\mathrm{tr}\colon A_{\mathbb{R}} \to \mathbb{R}, \qquad \mathrm{tr} \begin{pmatrix} \lambda_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \lambda_2 & x_1 \\ x_2 & \bar{x}_1 & \lambda_3 \end{pmatrix} = \lambda_1 + \lambda_2 + \lambda_3.$$

$A := \mathbb{C} \otimes_{\mathbb{R}} A_{\mathbb{R}}$ is the unique simple exceptional Jordan algebra.

The Lie algebra of type F_4

The group G of algebra automorphisms of $A_{\mathbb{R}}$ is the compact connected real Lie group of type F_4.

Let $\mathfrak{f}$ be the complexification of the Lie algebra of G.

The symmetric bilinear form

$$B\colon A \otimes A \to \mathbb{C}, \quad B(a \otimes b) := \mathrm{tr}(a \circ b)$$

is nondegenerate and G-invariant (hence $\mathfrak{f}$-invariant).

Thus, we have a decomposition of $\mathfrak{f}$-modules

$$A = \mathbb{C}1_A \oplus V, \quad V = \ker(\mathrm{tr}).$$

V is the natural $\mathfrak{f}$-module, and

$$\dim V = 26.$$

The functor

Fix a basis $\mathbf{B}_V$ of V, with dual basis $\{b^\vee : b \in B_V\}$, and let

$$\pi\colon A = \mathbb{C}1_A \oplus V \to V,$$

be the natural projection.

Theorem (Gandhi–S.–Zaynullin 2021)

There is a unique monoidal functor

$$\Phi\colon \mathcal{T}_{7/3,26} \to \mathfrak{f}\text{-mod}$$

given on objects by $\mathsf{I} \mapsto V$ and on morphisms by

$$\begin{aligned}
\Phi(\curlywedge)&\colon V \otimes V \to V, && a \otimes b \mapsto \pi(a \circ b),\\
\Phi(\times)&\colon V \otimes V \to V \otimes V, && a \otimes b \mapsto b \otimes a,\\
\Phi(\cup)&\colon \mathbb{C} \to V \otimes V, && 1 \mapsto \sum_{b \in \mathbf{B}_V} b \otimes b^\vee,\\
\Phi(\cap)&\colon V \otimes V \to \mathbb{C}, && a \otimes b \mapsto B(a \otimes b) = \mathrm{tr}(a \circ b).
\end{aligned}$$

The functor

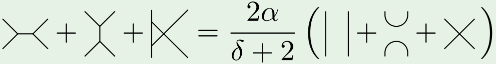
Theorem (Gandhi–S.–Zaynullin 2021)

1. The functor Φ is full (surjective on morphisms).
2. The functor Φ factors through $\mathcal{F}_{7/3,26}$, giving

$$\Phi\colon \mathcal{F}_{7/3,26} \to \mathfrak{f}\text{-mod}.$$

Remark

The relation



corresponds to the Cayley–Hamilton equation for traceless 3×3 octonionic matrices.

Idempotent completion

Additive envelope

The additive envelope $\mathrm{Add}(\mathcal{F})$ is a linear monoidal category with:

Objects: Formal finite direct sums $\bigoplus_{i=1}^{n} X_i$, $X_i \in \mathcal{F}$.

Morphisms: Morphisms

$$\bigoplus_{i=1}^{n} X_i \to \bigoplus_{j=1}^{m} Y_j$$

are $m \times n$ matrices, where the (j,i)-entry is a morphism $f_{ij} \colon X_i \to Y_j$ in $\mathcal{F}$. Composition is matrix multiplication.

Karoubi envelope

The additive Karoubi envelope $\mathrm{Kar}(\mathcal{F})$ is a linear monoidal category with:

Objects: Pairs (X, e) with $X \in \mathrm{Add}(\mathcal{F})$ and $e \colon X \to X$ an idempotent.

Morphisms: $\mathrm{Hom}_{\mathrm{Kar}(\mathcal{F})}\big((X,e),(X',e')\big) = e' \, \mathrm{Hom}_{\mathrm{Add}(\mathcal{F})}(X, X')e$.

Idempotent completion

Since the category $\mathfrak{f}$-mod is idempotent complete (i.e. it contains the images of idempotents), we have an induced functor

$$\mathrm{Kar}(\Phi)\colon\ \mathrm{Kar}(\mathcal{F}_{7/3,26}) \to \mathfrak{f}\text{-mod}.$$

Proposition (Gandhi–S.–Zaynullin 2021)

The functor $\mathrm{Kar}(\Phi)$ is full and essentially surjective (i.e. every finite-dimensional $\mathfrak{f}$-module is isomorphic to an object in the image of $\mathrm{Kar}(\Phi)$).

Corollary (Centralizer property)

We have a surjective algebra isomorphism

$$\mathrm{End}_{\mathcal{F}_{7/3,26}}(\mathsf{I}^{\otimes r}) \twoheadrightarrow \mathrm{End}_{\mathfrak{f}}(V^{\otimes r}), \quad r \in \mathbb{N}.$$

The upshot

Since

$$\mathrm{Kar}(\Phi)\colon\ \mathrm{Kar}(\mathcal{F}_{7/3,26}) \to \mathfrak{f}\text{-mod}$$

is full and essentially surjective,

$$\mathfrak{f}\text{-mod is a quotient of } \mathrm{Kar}(\mathcal{F}_{7/3,26}).$$

Philosophy

All homomorphisms between f.d. $\mathfrak{f}$-modules are built from the “flip” symmetry, the bilinear form, and a trilinear form (or the multiplication in the Albert algebra).

Question

Is the functor $\mathrm{Kar}(\Phi)$ faithful, and hence an equivalence of categories, or are there further relations?

Summary

- $\mathcal{F}$ is a type F_4 analogue of the unoriented and oriented Brauer categories.

- $\mathrm{End}_{\mathcal{F}}(\mathsf{I}^{\otimes r})$ is a type F_4 analogue of the oriented (walled) and unoriented Brauer algebras.

- We now have diagrammatic tools to study representation theory in type F_4.

Further directions

Quantum analogue

There are quantum diagrammatic categories:

- Type A: framed HOMFLYPT skein category
- Types BCD: Kauffman skein category
- Type G_2: Kuperberg (1994)

There should exist a quantum analogue of the category $\mathcal{F}$.

Webs

- Webs given a diagrammatic description of the category generated by the fundamental modules.
- First described in rank 2 by Kuperberg (1994, 1996).
- Arbitrary type A by Cautis–Kamnizter–Morrison (2014) and type C by Bodish–Elias–Rose–Tatham (2021).
- $\mathcal{F}$ is a starting point for developing F_4 webs.