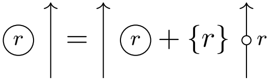
Categorification of the elliptic Hall algebra



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Slides: alistairsavage.ca/talks
Preprint: arXiv:2104.07530 (joint with Youssef Mousaaid)

Outline

Main result: The trace of the quantum Heisenberg category at central charge k is isomorphic to the central charge k reduction of a central extension of the elliptic Hall algebra.

Overview:

1. Elliptic Hall algebra
2. String diagrams for monoidal categories
3. Trace of a category
4. Quantum Heisenberg category
5. Main result
6. Application 1: action on symmetric functions
7. Application 2: action on cocenters of cyclotomic Hecke algebras

The elliptic Hall algebra: Definition

We work over the ground ring

$$\Bbbk = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}, \{d\}^{-1} : d \geq 1] \quad \text{where } \{d\} := q^d - q^{-d},\ d \neq 0.$$

Let $\mathfrak{EH}$ be the Lie algebra over $\Bbbk$ with basis

$$w_{\mathbf{x}}, \quad \mathbf{x} \in \mathbb{Z}^2 \setminus \{(0,0)\},$$

and Lie bracket

$$[w_{\mathbf{x}}, w_{\mathbf{y}}] = \{d\} w_{\mathbf{x}+\mathbf{y}}, \quad \text{where } d = \det \begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix}.$$

The elliptic Hall algebra associated to an elliptic curve X over a finite field is the Drinfeld double of the Hall algebra of the category of coherent sheaves over X.

Burban–Schiffmann defined a generic elliptic Hall algebra EH depending on two formal parameters σ, $\bar{\sigma}$, which specializes to the EHA for any X.

Morton–Samuelson: $\mathrm{EH} := U(\mathfrak{EH})$ is this generic Hall algebra at $\bar{\sigma} = \sigma^{-1}$.

Universal central extension

Easy fact: $\mathfrak{EH}$ is perfect, i.e. $[\mathfrak{EH}, \mathfrak{EH}] = \mathfrak{EH}$.

Consequence: $\mathfrak{EH}$ has a universal central extension.

Define $\widetilde{\mathfrak{EH}} = \mathfrak{EH} \oplus \Bbbk^2$ as $\Bbbk$-modules, with Lie bracket

$$[w_{\mathbf{x}}, w_{\mathbf{y}}] = \{d\} w_{\mathbf{x}+\mathbf{y}} + \delta_{\mathbf{x},-\mathbf{y}} \mathbf{x}, \quad \text{where } d = \det \begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix},$$

and elements of $\Bbbk^2$ are central.

Proposition (Mousaaid–S. 2021)

If q is not integral over $\mathbb{Z}$, then $\widetilde{\mathfrak{EH}}$ is the universal central extension of $\mathfrak{EH}$.

Remark

For every $\mathbf{x} \in \mathbb{Z}^2 \setminus \{(0,0)\}$,

$$\mathrm{Span}_{\Bbbk}\{w_{i\mathbf{x}} : i \in \mathbb{Z} \setminus \{0\}\}$$

is an infinite rank Heisenberg algebra with central element $\mathbf{x}$.

Central reductions

Motivation

In action on simple modules, centers act via central characters (Schur's lemma). So action factors through central reduction.

For a $\mathbb{Z}$-linear map

$$\lambda \colon \mathbb{Z}^2 \to \mathbb{Z},$$

define the corresponding central reduction

$$\mathrm{EH}_\lambda = U(\widetilde{\mathfrak{EH}})/\langle \mathbf{x} - \lambda(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^2 \rangle.$$

For $k \in \mathbb{Z}$, let

$$\lambda_k \colon \mathbb{Z}^2 \to \mathbb{Z}, \qquad (r, n) \mapsto kn,$$

and define $\mathrm{EH}_k := \mathrm{EH}_{\lambda_k}$.

Central reductions

Thus EH_k is the associative $\Bbbk$-algebra generated by

$$w_{\mathbf{x}}, \quad \mathbf{x} \in \mathbb{Z}^2 \setminus \{(0,0)\},$$

and relations

$$[w_{\mathbf{x}}, w_{\mathbf{y}}] = \{d\} w_{\mathbf{x}+\mathbf{y}} + kn\delta_{\mathbf{x},-\mathbf{y}}, \quad \text{where } d = \det \begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix}, \ \mathbf{x} = (r,n).$$

We call EH_k the elliptic Hall algebra at central charge k.

Proposition

Every central reduction is isomorphic to EH_k for some $k \in \mathbb{Z}$.

Remark

$\mathrm{EH}_0 \cong U(\mathfrak{E}\mathfrak{H})$ is isomorphic to the elliptic Hall algebra of Burban and Schiffmann, specialized at $\bar{\sigma} = q^2 = \sigma^{-1}$.

Importance of the elliptic Hall algebra

Other appearances

- generalized quantum affine algebra (Ding–Iohara)
- (q,γ)-analogue of the $W_{1+\infty}$ algebra (Miki)
- shuffle algebra (Feigin–Tsymbaliuk, Neguț)
- spherical $\mathfrak{gl}_\infty$ double affine Hecke algebra (Schiffmann–Vasserot, Feigin–Feigin–Jimbo–Miwa–Mukhin)
- quantum continuous $\mathfrak{gl}_\infty$ (Feigin–Feigin–Jimbo–Miwa–Mukhin)

Geometry

The EHA is intimately related to the equivariant K-theory of the Hilbert scheme of points on $\mathbb{A}^2$.

Goal

Relate EHA to categories that have appeared in representation theory and theory of link invariants.

Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.

$\Bbbk$-linear monoidal categories

Fix a commutative ground ring $\Bbbk$.

A strict $\Bbbk$-linear monoidal category is a strict monoidal category such that

- each morphism space is a $\Bbbk$-module,
- composition of morphisms is $\Bbbk$-bilinear,
- tensor product of morphisms is $\Bbbk$-bilinear.

The interchange law

The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

$$\begin{tikzcd} A_1 \otimes B_1 \arrow[r, "1\otimes g"] \arrow[d, "f\otimes 1"'] \arrow[dr, "f\otimes g"] & A_1 \otimes B_2 \arrow[d, "f\otimes 1"] \\ A_2 \otimes B_1 \arrow[r, "1\otimes g"'] & A_2 \otimes B_2 \end{tikzcd}$$

Strict monoidal categories

Example (Monoids)

A (strict) monoidal category with one object is simply a commutative monoid. More precisely, the endomorphisms of $\mathbb{1}$ form a commutative monoid (with operation $\otimes = \circ$).

Conversely, every commutative monoid gives rise to a one-object monoidal category.

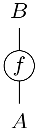
Example (Associative algebras)

A (strict) $\Bbbk$-linear monoidal category with one object is simply a commutative associative unital $\Bbbk$-algebra.

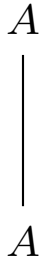
String diagrams

Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f : A \to B$ by:



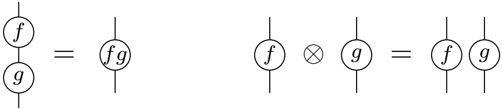
The identity map $1_A : A \to A$ is a string with no label:



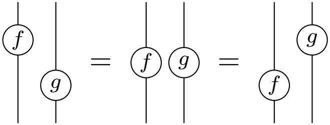
We sometimes omit the object labels when they are clear or unimportant.

String diagrams

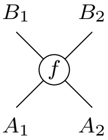
Composition is vertical stacking and tensor product is horizontal juxtaposition:



The interchange law then becomes:



A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:

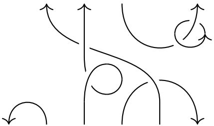


Framed oriented tangles over the disc

Let $\mathcal{FOT}(D)$ be the category of framed oriented tangles over the disc D.

Objects: Finite sequences of $\uparrow$ and $\downarrow$.

Morphisms: Framed oriented tangles in $D \times [0,1]$, up to ambient isotopy, whose orientation matches the $\uparrow$ and $\downarrow$ at the endpoints.

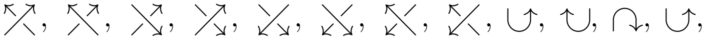
 $\in \mathrm{Hom}_{\mathcal{FOT}(D)}((\downarrow,\uparrow,\uparrow,\uparrow,\uparrow,\downarrow),(\uparrow,\uparrow,\downarrow,\uparrow)),$

Composition: Vertical stacking.

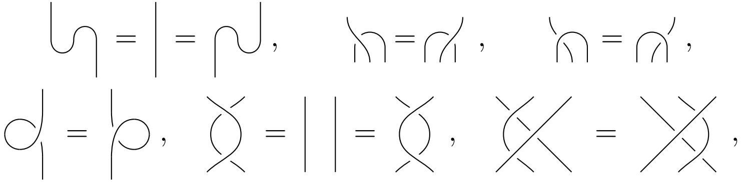
Tensor product: Horizontal juxtaposition.

Framed oriented tangles over the disc

The category $\mathcal{FOT}(D)$ of framed oriented tangles over the disc is isomorphic to the strict monoidal category generated by objects $\uparrow$, $\downarrow$, and morphisms



subject to the relations

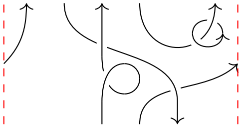


for all orientations of the strands.

Framed oriented tangles over the annulus

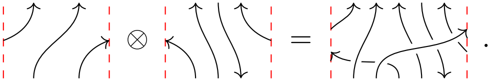
Replacing the disc D by the annulus A, we get the category $\mathcal{FOT}(A)$ of framed oriented tangles over the annulus.

Now tangles can wrap around:

 $\in \mathrm{Hom}_{\mathcal{FOT}(A)}(\uparrow \otimes \uparrow \otimes \downarrow, \uparrow \otimes \downarrow \otimes \uparrow \otimes \downarrow \otimes \uparrow)$,

Composition: Vertical stacking.

Tensor product: Nesting of cylinders:



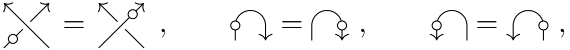
Affinization of monoidal categories (Mousaaid–S. 2020)

Given a monoidal category $\mathcal{C}$, there is a formal procedure of affinization that produces the category $\mathrm{Aff}(\mathcal{C})$ of $\mathcal{C}$-diagrams on the cylinder/annulus.

The affinization $\mathrm{Aff}(\mathcal{FOT}(D))$ is obtained from $\mathcal{FOT}(D)$ by adjoining morphisms

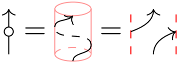
$$\overset{\uparrow}{\circ}\colon \uparrow \to \uparrow, \qquad \underset{\downarrow}{\circ}\colon \downarrow \to \downarrow,$$

subject to the relations



$\overset{\uparrow}{\circ}$ is invertible.

$\mathrm{Aff}(\mathcal{FOT}(D))$ is isomorphic to $\mathcal{FOT}(A)$.



HOMFLYPT skein category over the disc

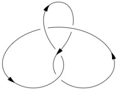
Work over $\Bbbk = \mathbb{Z}[z, z^{-1}, t, t^{-1}]$. The HOMFLYPT skein category $\mathcal{OS}$ over the disc is the category obtained from $\mathcal{FOT}_{\Bbbk}(D)$ by imposing the relations

 $-$  $= z \uparrow\uparrow$ (Conway skein relation)

and

 $= t \uparrow$,  $= \dfrac{t - t^{-1}}{z} 1_{\mathbb{1}}.$

Given an oriented link diagram L, we have

$L =$  , $L = t^{\mathrm{writhe}(L)} H(L)$  ,

where

- writhe(L) is the writhe number (# pos crossings $-$ # neg crossings),
- $H(L)$ is the HOMFLYPT polynomial of L.

Affinization: Imposing the same relations on $\mathcal{FOT}_{\Bbbk}(A)$ gives the HOMFLYPT skein category over the annulus $\mathrm{Aff}(\mathcal{OS})$.

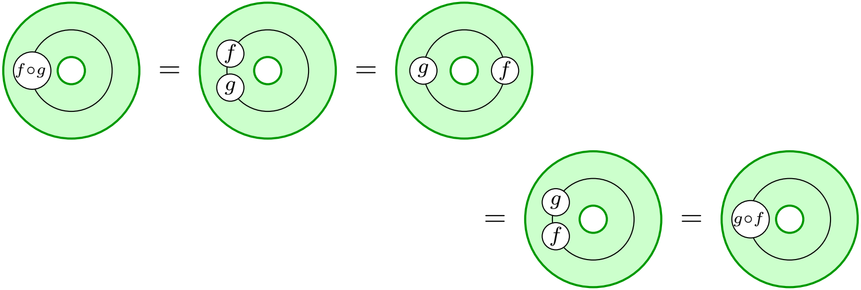
Trace of a category

The trace, or zeroth Hoschschild homology, of a $\Bbbk$-linear category $\mathcal{C}$ is the $\Bbbk$-module

$$\operatorname{Tr}(\mathcal{C}) := \left(\bigoplus_{X \in \operatorname{Ob}(\mathcal{C})} \operatorname{End}_{\mathcal{C}}(X) \right) / \operatorname{Span}_{\Bbbk}\{f \circ g - g \circ f\},$$

where f and g range over $f \colon X \to Y$, $g \colon Y \to X$ in $\mathcal{C}$.

If $\mathcal{C}$ is pivotal, $\operatorname{Tr}(\mathcal{C})$ corresponds to $\mathcal{C}$-diagrams on the annulus:



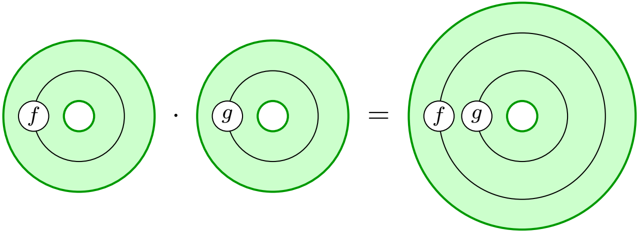
Trace of a category

If $\mathcal{C}$ is also monoidal, then $\mathrm{Tr}(\mathcal{C})$ is an associative $\Bbbk$-algebra:

$$[f] \cdot [g] = [f \otimes g],$$

where $[f]$ denotes the image of $f \in \mathrm{End}_{\mathcal{C}}(X)$ in $\mathrm{Tr}(\mathcal{C})$.

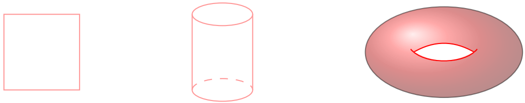
This corresponds to nesting of annuli:



Diagrams on the torus

Suppose $\mathcal{C}$ is a strict pivotal braided monoidal category.

Category:	$\mathcal{C}$	$\mathrm{Aff}(\mathcal{C})$	$\mathrm{Tr}(\mathrm{Aff}(\mathcal{C}))$
Diagrams:	plane	cylinder	torus



$\mathrm{Tr}(\mathrm{Aff}(\mathcal{C}))$ is an algebra, with multiplication given by nesting tori.

Theorem (Morton–Samuelson 2017)

Skein algebra of the torus, $\mathrm{Tr}(\mathrm{Aff}(\mathcal{OS}))$, is isomorphic to EH.

General central charge

Goal: Want to extend the above isomorphism to general central charge k.

$$\begin{tikzcd}
\text{skein of torus} \arrow[r, "\cong", "\text{[MS]}"'] & \mathrm{EH} \\
{} \arrow[u, "\cong"] \arrow[r, "\cong"] & {} \arrow[u, "\cong"'] \\
{} \arrow[u, rightsquigarrow, "k=0"] \arrow[r, "\cong", "\text{main result}"'] & {} \arrow[u, rightsquigarrow, "k=0"']
\end{tikzcd}$$

Missing ingredient: Central charge k version of the HOMFLYPT skein category over the annulus $\mathrm{Aff}(\mathcal{OS})$.

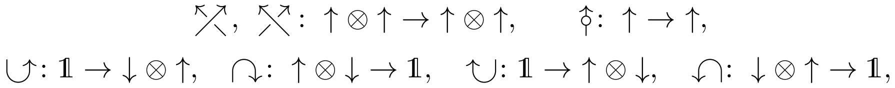
Solution: Quantum Heisenberg category $\mathcal{H}eis_k$ with

$$\mathcal{H}eis_0 \cong \mathrm{Aff}(\mathcal{OS}).$$

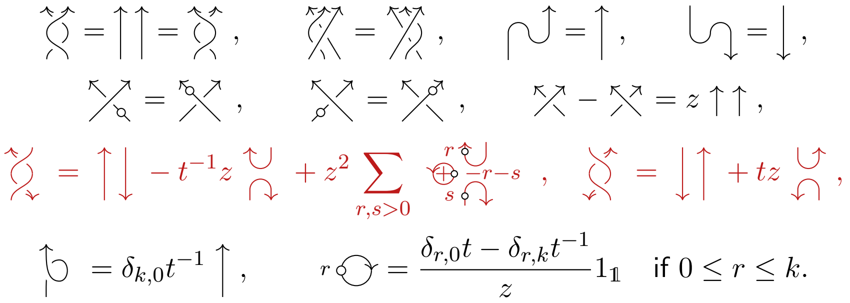
The quantum Heisenberg category

Assume $k \geq 0$ for simplicity.

The quantum Heisenberg category $\mathcal{H}eis_k$ is the strict $\Bbbk$-linear monoidal category generated by objects $\uparrow$, $\downarrow$ and morphisms

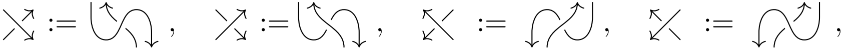


subject to relations that the dot is invertible and:

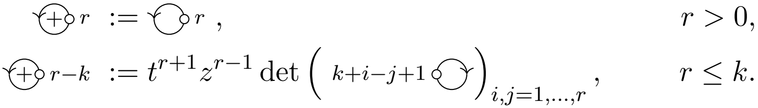


The quantum Heisenberg category

In the relations we used:



and



Special case: $\mathcal{H}eis_0 \cong \mathrm{Aff}(\mathcal{OS})$

Heisenberg categorification

The relations ensure that

$$\uparrow \otimes \downarrow \cong \downarrow \otimes \uparrow \oplus \mathbb{1}^{\oplus k},$$

which is a categorification of the relation in the central charge k reduction of the universal enveloping algebra of the infinite rank Heisenberg algebra.

Main Theorem (Mousaaid–S. 2021)

For $k \in \mathbb{Z}$, there is a unique isomorphism of algebras

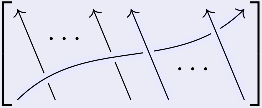
$$\varphi_k \colon \mathrm{EH}_k \xrightarrow{\cong} \mathrm{Tr}(\mathcal{H}eis_k) \qquad \text{such that}$$

$$w_{r,1} \mapsto \left[\uparrow\!\circ\, r\right], \quad w_{r,-1} \mapsto \left[\downarrow\!\circ\, r\right], \quad r \in \mathbb{Z}.$$

Remarks

1. Uniqueness follows from fact that $w_{r,\pm 1}$, $r \in \mathbb{Z}$, generate EH_k.
2. Can give explicit description of image of every $w_{r,n}$. E.g., for $n > 0$,

$$w_{0,n} \mapsto \frac{z}{\{n\}} \sum_{i=0}^{n-1}$$



,

where the up-right strand passes over i strands and under $n - i - 1$ strands.

Relation to the Heisenberg algebra

The Heisenberg algebra

The central charge k Heisenberg algebra Heis_k has generators p_n, $n \in \mathbb{Z}$, $n \neq 0$ and relations

$$[p_n, p_m] = \delta_{n,-m} kn.$$

Recall

EH_k contains many copies of the infinite rank Heisenberg algebra.

For example, we have embedding

$$\mathrm{Heis}_k \hookrightarrow \mathrm{EH}_k, \quad p_n \mapsto w_{0,n}, \quad n \in \mathbb{Z},\ n \neq 0.$$

Relation to the Heisenberg algebra

Conjecture: We have an algebra isomorphism $K_0(\mathcal{H}eis_k) \cong \mathrm{Heis}_k$.

For the degenerate Heisenberg category, this is a theorem (Brundan–S.–Webster).

The Chern character map is an algebra homomorphism

$$K_0(\mathcal{C}) \to \mathrm{Tr}(\mathcal{C}), \qquad [X] \mapsto [1_X].$$

Assuming the above conjecture, this corresponds to embedding

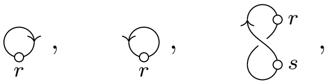
$$\mathrm{Heis}_k \cong K_0(\mathcal{H}eis_k) \to \mathrm{Tr}(\mathcal{H}eis_k) \cong \mathrm{EH}_k.$$

Center of a monoidal category

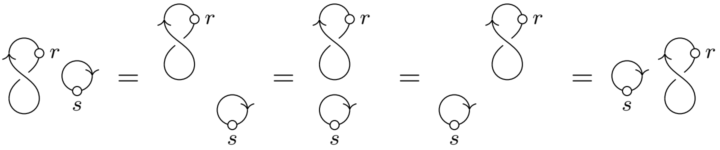
The center of a monoidal category $\mathcal{C}$ is

$$\mathrm{End}_{\mathcal{C}}(\mathbb{1}).$$

In terms of string diagrams, this corresponds to closed diagrams:



$\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is a commutative algebra, with operation $\otimes = \circ$:



Proposition (Brundan–S.–Webster 2020)

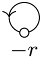
$\mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1}) \cong \mathrm{Sym}^{\otimes 2}$, where Sym is the ring of symmetric functions.

For central charge $k = 0$, above isomorphism is given by $(r > 0)$

$\mapsto -t^{-1}z^{-1}h_r \otimes 1,$

$\mapsto tz^{-1}1 \otimes h_r,$

$\mapsto (-1)^r tz^{-1}e_r \otimes 1,$



$\mapsto (-1)^{r-1}t^{-1}z^{-1}e_r \otimes 1.$

For general k, we need to tweak this a bit. E.g.

$\mapsto (-1)^r tz^{-1}e_r \otimes 1,$

$\mapsto (-1)^{r-1}t^{-1}z^{-1}e_r \otimes 1.$

where

$:=$


$-$


$, \qquad r \in \mathbb{Z}.$

Action of the trace on the center

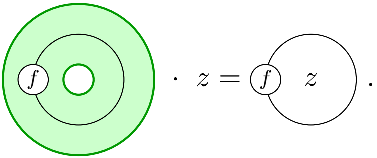
Suppose $\mathcal{C}$ is a strict pivotal $\Bbbk$-linear category.

The trace $\mathrm{Tr}(\mathcal{C})$ acts naturally on the center $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$.

If

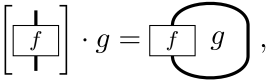
- $f \in \mathrm{End}_{\mathcal{C}}(X)$,
- $z \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is a closed diagram,

then the action of $[f] \in \mathrm{Tr}(\mathcal{C})$ on z is



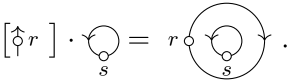
Application: action on symmetric functions

The trace $\mathrm{Tr}(\mathcal{H}eis_k)$ acts naturally on the center $\mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1})$ by



$$f \in \mathrm{End}_{\mathcal{H}eis_k}(X),\ X \in \mathcal{H}eis_k,\ g \in \mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1}).$$

For example,



Immediate consequence

We obtain a natural action of EH_k on $\mathrm{Sym}^{\otimes 2}$.

Note: The action on $\mathrm{Sym}^{\otimes 2}$ depends on the parameter t even though the definition of EH_k does not.

Affine Hecke algebras

Let

- $\Bbbk$ be a field of characteristic zero, $q, t \in \Bbbk^\times$, q not a root of unity,
- H_n be the Iwahori–Hecke algebra (type A) of rank n,
- $\mathrm{AH}_n = \mathrm{H}_n \otimes_\Bbbk \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the affine Hecke algebra.

There is an injective algebra homomorphism

$$\mathrm{AH}_n \hookrightarrow \mathrm{End}_{\mathcal{H}eis_k}(\uparrow^{\otimes n}),$$

given by

$$x_i \mapsto \underbrace{\uparrow \otimes \cdots \otimes \uparrow}_{n-i \text{ factors}} \otimes \overset{\circ}{\uparrow} \otimes \underbrace{\uparrow \otimes \cdots \otimes \uparrow}_{i-1 \text{ factors}},$$

$$\tau_i \mapsto \underbrace{\uparrow \otimes \cdots \otimes \uparrow}_{n-i-1 \text{ factors}} \otimes \nwarrow\!\!\!\!\!\nearrow \otimes \underbrace{\uparrow \otimes \cdots \otimes \uparrow}_{i-1 \text{ factors}}.$$

Note: This extends to an isomorphism $\mathrm{AH}_n \otimes \mathrm{Sym}^{\otimes 2} \cong \mathrm{End}_{\mathcal{H}eis_k}(\uparrow^{\otimes n})$.

Cyclotomic Hecke algebras

Fix a polynomial

$$f(u) = f_0 u^l + f_1 u^{l-1} + \cdots + f_l \in \Bbbk[u], \quad f_0 = 1, \quad f_l = t^2.$$

The corresponding cyclotomic Hecke algebra is

$$\mathrm{H}_n^f = \mathrm{AH}_n / (f(x_1)).$$

Its cocenter is

$$C(\mathrm{H}_n^f) = \mathrm{H}_n^f / \operatorname{span}_{\Bbbk}\{ab - ba : a, b \in \mathrm{H}_n^f\}.$$

Note: This is a vector space (not an algebra).

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Example

If

$$f(u) = u + t^2,$$

then we have quotient map

$$\mathrm{AH}_n \twoheadrightarrow \mathrm{H}_n^f \cong \mathrm{H}_n.$$

Here

$$x_i \mapsto i\text{-th Jucys–Murphy element}.$$

So one can think of cyclotomic quotients H_n^f as higher level analogues of the finite Iwahori–Hecke algebras.

Action on cocenters of cyclotomic Hecke algebras

Proposition (Brundan–S.–Webster 2020)

The H_n^f are endomorphism algebras in a generalized cyclotomic quotient of $\mathcal{Heis}_{-l}$.

Conclusion

We have a natural action of $\mathrm{EH}_{-l} \cong \mathrm{Tr}(\mathcal{Heis}_{-l})$ on $\bigoplus_{n\geq 0} C(\mathrm{H}_n^f)$.

Remarks

1. When $l = 1$, above action is closely related to one defined by Schiffmann–Vasserot in terms of the K-theory of the Hilbert scheme of points on $\mathbb{A}^2$.
2. For $l > 1$, we expect that above action is related to the moduli space of framed torsion-free sheaves on $\mathbb{P}^2$.