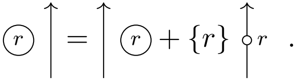
Categorification of the elliptic Hall algebra



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Slides: alistairsavage.ca/talks
Preprint: arXiv:2104.07530 (joint with Youssef Mousaaid)

Outline

Main result: The trace of the quantum Heisenberg category at central charge k is isomorphic to the central charge k reduction of a central extension of the elliptic Hall algebra.

Overview:

1. Elliptic Hall algebra
2. Quantum Heisenberg category
3. Main result
4. Application: actions

The elliptic Hall algebra: Definition

We work over the ground ring

$$\Bbbk = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}, \{d\}^{-1} : d \geq 1] \quad \text{where } \{d\} := q^d - q^{-d},\ d \neq 0.$$

Fix a central charge $k \in \mathbb{Z}$.

Let EH_k be the associative $\Bbbk$-algebra generated by

$$w_{\mathbf{x}}, \quad \mathbf{x} \in \mathbb{Z}^2 \setminus \{(0,0)\},$$

and relations

$$[w_{\mathbf{x}}, w_{\mathbf{y}}] = \{d\} w_{\mathbf{x}+\mathbf{y}} + kn\delta_{\mathbf{x},-\mathbf{y}}, \quad \text{where } d = \det \begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix},\ \mathbf{x} = (r,n).$$

We call EH_k the elliptic Hall algebra at central charge k.

The elliptic Hall algebra of Burban and Schiffmann

The elliptic Hall algebra associated to an elliptic curve X over a finite field is the Drinfeld double of the Hall algebra of the category of coherent sheaves over X.

Burban–Schiffmann defined a generic elliptic Hall algebra EH depending on two formal parameters σ, $\bar{\sigma}$, which specializes to the EHA for any X.

Morton–Samuelson: EH at $\bar{\sigma} = \sigma^{-1}$ is isomorphic to the skein algebra of the torus.

The algebra EH_k

1. EH is the universal enveloping algebra of a Lie algebra $\mathfrak{EH}$.
2. $\mathfrak{EH}$ has a universal central extension $\widetilde{\mathfrak{EH}}$.
3. Every central reduction of $U(\widetilde{\mathfrak{EH}})$ is isomorphic to EH_k for some $k \in \mathbb{Z}$.

The elliptic Hall algebra: Motivation

Other appearances

- generalized quantum affine algebra (Ding–Iohara)
- (q, γ)-analogue of the $W_{1+\infty}$ algebra (Miki)
- shuffle algebra (Feigin–Tsymbaliuk, Neguț)
- spherical $\mathfrak{gl}_\infty$ double affine Hecke algebra (Schiffmann–Vasserot, Feigin–Feigin–Jimbo–Miwa–Mukhin)
- quantum continuous $\mathfrak{gl}_\infty$ (Feigin–Feigin–Jimbo–Miwa–Mukhin)

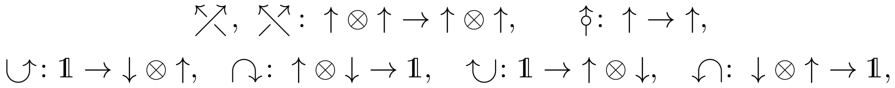
Geometry

The EHA is intimately related to the equivariant K-theory of the Hilbert scheme of points on $\mathbb{A}^2$.

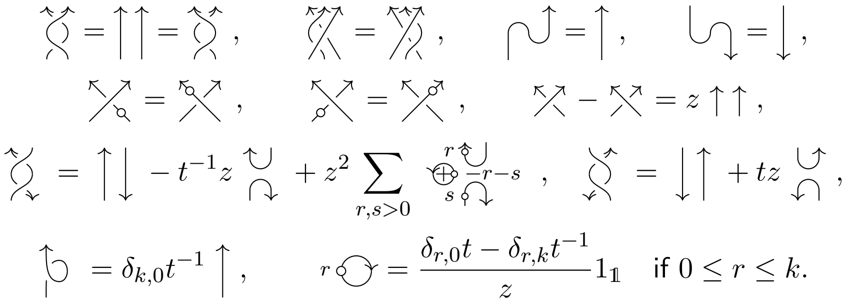
The quantum Heisenberg category

Assume $k \geq 0$ for simplicity.

The quantum Heisenberg category $\mathcal{H}eis_k$ is the strict $\Bbbk$-linear monoidal category generated by objects $\uparrow$, $\downarrow$ and morphisms

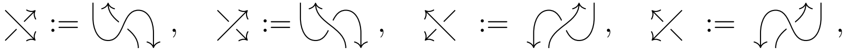


subject to relations that the dot is invertible and:

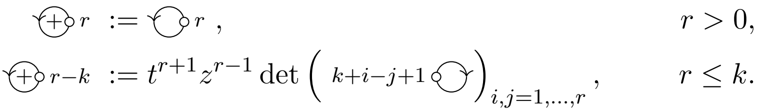


The quantum Heisenberg category

In the relations we used:



and



Heisenberg categorification

The relations ensure that

$$\uparrow \otimes \downarrow \cong \downarrow \otimes \uparrow \oplus \mathbb{1}^{\oplus k},$$

which is a categorification of the relation in the central charge k reduction of the universal enveloping algebra of the infinite rank Heisenberg algebra.

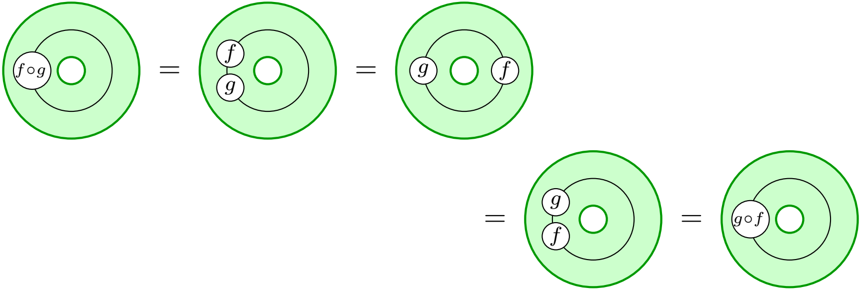
Trace of a category

The trace, or zeroth Hoschschild homology of a $\mathbb{k}$-linear category $\mathcal{C}$ is the $\mathbb{k}$-module

$$\operatorname{Tr}(\mathcal{C}) := \left(\bigoplus_{X \in \operatorname{Ob}(\mathcal{C})} \operatorname{End}_{\mathcal{C}}(X) \right) / \operatorname{Span}_{\mathbb{k}}\{f \circ g - g \circ f\},$$

where f and g range over $f \colon X \to Y$, $g \colon Y \to X$ in $\mathcal{C}$.

If $\mathcal{C}$ is pivotal, $\operatorname{Tr}(\mathcal{C})$ corresponds to $\mathcal{C}$-diagrams on the annulus:



Categorification of the elliptic Hall algebra

Main Theorem (Mousaaid–S. 2021)

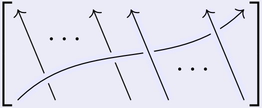
For $k \in \mathbb{Z}$, there is a unique isomorphism of algebras

$$\varphi_k \colon \mathrm{EH}_k \xrightarrow{\cong} \mathrm{Tr}(\mathcal{H}eis_k) \qquad \text{such that}$$
$$w_{r,1} \mapsto \left[\ \uparrow\!\circ\, r\ \right], \quad w_{r,-1} \mapsto \left[\ \downarrow\!\circ\, r\ \right], \quad r \in \mathbb{Z}.$$

Remarks

1. Uniqueness follows from fact that $w_{r,\pm 1}$, $r \in \mathbb{Z}$, generate EH_k.
2. Can give explicit description of image of every $w_{r,n}$. E.g., for $n > 0$,

$$w_{0,n} \mapsto \frac{z}{\{n\}} \sum_{i=0}^{n-1}$$



,

where the up-right strand passes over i strands and under $n - i - 1$ strands.

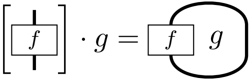
Application: action on symmetric functions

The center $\mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1})$ of $\mathcal{H}eis_k$ corresponds to closed diagrams.

Proposition (Brundan–S.–Webster 2020)

$\mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1}) \cong \mathrm{Sym}^{\otimes 2}$, where Sym is the ring of symmetric functions.

The trace $\mathrm{Tr}(\mathcal{H}eis_k)$ acts naturally on the center $\mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1})$ by



$$, \quad f \in \mathrm{End}_{\mathcal{H}eis_k}(X),\ X \in \mathcal{H}eis_k,\ g \in \mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1}).$$

Conclusion

We obtain a natural action of EH_k on $\mathrm{Sym}^{\otimes 2}$.

Note: The action on $\mathrm{Sym}^{\otimes 2}$ depends on the parameter t even though the definition of EH_k does not.

Cocenters of cyclotomic Hecke algebras

Let

- $\Bbbk$ be a field of characteristic zero, $q, t \in \Bbbk^\times$, q not a root of unity,
- H_n be the Iwahori–Hecke algebra (type A) of rank n,
- $\mathrm{AH}_n = \mathrm{H}_n \otimes_\Bbbk \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the affine Hecke algebra.

Fix a polynomial

$$f(u) = f_0 u^l + f_1 u^{l-1} + \cdots + f_l \in \Bbbk[u], \quad f_0 = 1, \quad f_1 = t^2.$$

Consider the cyclotomic Hecke algebra

$$\mathrm{H}_n^f = \mathrm{AH}_n / (f(x_1)).$$

Its cocenter is

$$C(\mathrm{H}_n^f) = \mathrm{H}_n^f / \operatorname{span}_\Bbbk \{ab - ba : a, b \in \mathrm{H}_n^f\}.$$

Action on cocenters of cyclotomic Hecke algebras

Proposition (Brundan–S.–Webster 2020)

The H_n^f are endomorphism algebras in a generalized cyclotomic quotient of $\mathcal{H}eis_{-l}$.

Conclusion

We have a natural action of $\mathrm{EH}_{-l} \cong \mathrm{Tr}(\mathcal{H}eis_{-l})$ on $\bigoplus_{n \geq 0} C(\mathrm{H}_n^f)$.

Remarks

1. When $l = 1$, above action is closely related to one defined by Schiffmann–Vasserot in terms of the K-theory of the Hilbert scheme of points on $\mathbb{A}^2$.
2. For $l > 1$, we expect that above action is related to the moduli space of framed torsion-free sheaves on $\mathbb{P}^2$.