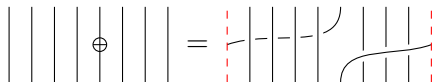


# Affinization of monoidal categories

(Monoidal categories on cylinders)



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Preprint: [arXiv:2010.13598](https://arxiv.org/abs/2010.13598) (joint with Youssef Mousaaid)

# Outline

**Goal:** Guided by intuition of monoidal categories on cylinders, formalize the notion of **affinization** of a monoidal category.

## Overview:

- 1 String diagrams for strict monoidal categories
- 2 Examples
- 3 Definition of affinization
- 4 Dot presentation
- 5 Examples
- 6 Horizontal and vertical trace

# Strict monoidal categories

A **strict monoidal category** is a category  $\mathcal{C}$  equipped with

- a bifunctor (the **tensor product**)  $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , and
- a **unit object**  $\mathbb{1}$ ,

such that, for objects  $A, B, C$  and morphisms  $f, g, h$ ,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ ,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$ ,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ ,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$ .

## Remark: Non-strict monoidal categories

In a (not necessarily strict) **monoidal category**, the equalities above are replaced by isomorphism, and we impose some **coherence conditions**.

Every monoidal category is monoidally equivalent to a strict one.

## $\mathbb{k}$ -linear monoidal categories

Fix a commutative ground ring  $\mathbb{k}$ .

A **strict  $\mathbb{k}$ -linear monoidal category** is a strict monoidal category such that

- each morphism space is a  $\mathbb{k}$ -module,
- composition of morphisms is  $\mathbb{k}$ -bilinear,
- tensor product of morphisms is  $\mathbb{k}$ -bilinear.

### The interchange law

The axioms of a strict monoidal category imply the **interchange law**: For  $A_1 \xrightarrow{f} A_2$  and  $B_1 \xrightarrow{g} B_2$ , the following diagram commutes:

$$\begin{array}{ccc} A_1 \otimes B_1 & \xrightarrow{1 \otimes g} & A_1 \otimes B_2 \\ f \otimes 1 \downarrow & \searrow f \otimes g & \downarrow f \otimes 1 \\ A_2 \otimes B_1 & \xrightarrow{1 \otimes g} & A_2 \otimes B_2 \end{array}$$

# Strict monoidal categories

## Example (Monoids)

A (strict) monoidal category with one object is simply a commutative monoid. More precisely, the endomorphisms of  $\mathbb{1}$  form a commutative monoid.

Conversely, every commutative monoid gives rise to a one-object monoidal category.

## Example (Associative algebras)

A (strict)  $\mathbb{k}$ -linear monoidal category with one object is simply a commutative associative unital  $\mathbb{k}$ -algebra.

# String diagrams

Fix a strict monoidal category  $\mathcal{C}$ .

We will denote a morphism  $f: A \rightarrow B$  by:



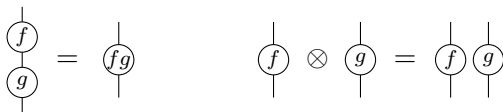
The **identity map**  $1_A: A \rightarrow A$  is a string with no label:



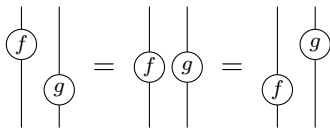
We sometimes omit the object labels when they are clear or unimportant.

# String diagrams

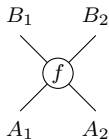
Composition is **vertical stacking** and tensor product is **horizontal juxtaposition**:



The **interchange law** then becomes:



A morphism  $f: A_1 \otimes A_2 \rightarrow B_1 \otimes B_2$  can be depicted:



## Example: monoidally generated symmetric groups

Define a strict monoidal category  $Sym$  with one generating object  $\uparrow$  and denote

$$1_{\uparrow} = \uparrow$$

We have one generating morphism

$$\begin{array}{c} \nearrow \\ \times \\ \searrow \end{array} : \uparrow \otimes \uparrow \rightarrow \uparrow \otimes \uparrow .$$

We impose the relations:

$$\begin{array}{c} \nearrow \\ \text{ } \\ \searrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array} , \quad \begin{array}{c} \nearrow \\ \times \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \times \\ \searrow \end{array} .$$

Then

$$\text{End}_{Sym}(\uparrow^{\otimes n}) = S_n$$

is the **symmetric group** on  $n$  letters.



## Example: monoidally generated symmetric groups

This monoidal presentation of  $S_n$  is very efficient! We only needed

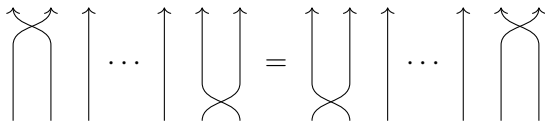
- one generating morphism, and
- two relations,

to get **all** the symmetric groups.

Note that the “distant braid relation”

$$s_i s_j = s_j s_i, \quad |i - j| > 1$$

for simple transpositions follows for free from the interchange law:



**Note:** If we define  $Sym$  to be  $\mathbb{k}$ -linear, then  $\text{End}_{Sym}(\uparrow^{\otimes n}) = \mathbb{k}S_n$ .

# Braidings

A strict monoidal category  $\mathcal{C}$  is **braided** if it is equipped with isomorphisms

$$\beta_{X,Y} = \begin{array}{c} \diagup \quad \diagdown \\ X \quad Y \end{array} : X \otimes Y \rightarrow Y \otimes X, \quad \beta_{X,Y}^{-1} = \begin{array}{c} \diagdown \quad \diagup \\ Y \quad X \end{array},$$

for all  $X, Y \in \text{Ob}(\mathcal{C})$ , satisfying

$$\begin{array}{c} \diagup \quad \diagdown \\ X \quad Y \otimes Z \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ X \quad Y \quad Z \end{array}, \quad \begin{array}{c} \diagdown \quad \diagup \\ X \otimes Y \quad Z \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ X \quad Y \quad Z \end{array},$$

and

$$\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array}, \quad \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} \circ f, \quad \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} \circ f,$$

for all morphisms  $f$  in  $\mathcal{C}$  (and any compatible labelling of strands).

# Braid groups

The category *Braid* of braids over the disc is isomorphic to the strict monoidal category generated by a single object  $\uparrow$ , and morphisms

$$\begin{array}{c} \nearrow \\ \searrow \end{array}, \begin{array}{c} \nwarrow \\ \swarrow \end{array} : \uparrow \otimes \uparrow \rightarrow \uparrow \otimes \uparrow,$$

subject to the relations

$$\begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array} = \begin{array}{c} \nwarrow \\ \swarrow \\ \nwarrow \\ \swarrow \end{array}, \quad \begin{array}{c} \nearrow \\ \nwarrow \\ \swarrow \\ \searrow \end{array} = \begin{array}{c} \nwarrow \\ \swarrow \\ \nearrow \\ \searrow \end{array}.$$

**Universal property:** *Braid* is the free braided monoidal category generated by a single object.

Note that

$$\text{End}_{\text{Braid}}(\uparrow^{\otimes n}) = \text{braid group of type } A_{n-1}.$$

# Hecke algebras

Let  $\mathcal{Braid}_{\mathbb{k}}$  be the  $\mathbb{k}$ -linearization of  $\mathcal{Braid}$ , i.e. the strict  $\mathbb{k}$ -linear monoidal category with the same generators and relations.

Fix  $z \in \mathbb{k}^{\times}$ , and let  $\mathcal{H}$  be obtained from  $\mathcal{Braid}_{\mathbb{k}}$  by imposing the **Conway skein relation**

$$\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} - \begin{array}{c} \nwarrow \nearrow \\ \nearrow \searrow \end{array} = z \begin{array}{c} \uparrow \\ \uparrow \end{array}.$$

Then

$$\text{End}_{\mathcal{H}}(\uparrow^{\otimes n}) = \text{Iwahori-Hecke algebra of type } A_{n-1}.$$

(Often one sets  $z = q - q^{-1}$  for some  $q \in \mathbb{k}^{\times}$ .)

# Duals

Suppose a strict monoidal category  $\mathcal{C}$  has two objects  $\uparrow$  and  $\downarrow$ , with

$$1_{\uparrow} = \uparrow \quad , \quad 1_{\downarrow} = \downarrow .$$

A morphism  $\mathbb{1} \rightarrow \downarrow \otimes \uparrow$  would have string diagram

$$\begin{array}{c} \swarrow \quad \searrow \\ \downarrow \text{ (dotted)} \end{array} \quad , \quad \text{where} \quad \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} = 1_{\mathbb{1}} .$$

We typically omit the dotted line and draw:

$$\cup : \mathbb{1} \rightarrow \downarrow \otimes \uparrow .$$

Similarly, we can have

$$\cap : \uparrow \otimes \downarrow \rightarrow \mathbb{1} .$$

# Duals

We say that  $\downarrow$  is **right dual** to  $\uparrow$  (and  $\uparrow$  is **left dual** to  $\downarrow$ ) if there exist morphisms

$$\cup : \mathbb{1} \rightarrow \downarrow \otimes \uparrow \quad \text{and} \quad \cap : \uparrow \otimes \downarrow \rightarrow \mathbb{1}.$$

such that

$$\begin{array}{c} \downarrow \\ \cup \\ \downarrow \end{array} = \downarrow \quad \text{and} \quad \begin{array}{c} \uparrow \\ \cap \\ \uparrow \end{array} = \uparrow.$$

Objects  $\uparrow$  and  $\downarrow$  are both left and right dual to each other if we also have

$$\cup : \mathbb{1} \rightarrow \uparrow \otimes \downarrow \quad \text{and} \quad \cap : \downarrow \otimes \uparrow \rightarrow \mathbb{1}$$

such that

$$\begin{array}{c} \uparrow \\ \cup \\ \uparrow \end{array} = \uparrow \quad \text{and} \quad \begin{array}{c} \downarrow \\ \cap \\ \downarrow \end{array} = \downarrow.$$

We say  $\mathcal{C}$  is **rigid** if all objects have left and right duals.

## Duals: example

Let  $\mathbb{k}$  be a field and consider the category  $\text{Vect}_{\mathbb{k}}$  of f.d.  $\mathbb{k}$ -vector spaces.

**Unit object:**  $\mathbb{k}$

Fix a f.d.  $\mathbb{k}$ -vector space  $V$ .

**Claim:** The dual vector space  $V^*$  is both left and right dual to  $V$ , in the sense mentioned above.

**Proof:** Fix a basis  $B$  of  $V$ . Let  $\{\delta_v : v \in B\}$  be the dual basis of  $V^*$ . Viewing  $V$  and  $\uparrow$  and  $V^*$  as  $\downarrow$ , we define

$$\cup : \mathbb{k} \rightarrow V^* \otimes V,$$

$$\cap : V \otimes V^* \rightarrow \mathbb{k},$$

$$\cup : \mathbb{k} \rightarrow V \otimes V^*,$$

$$\cap : V^* \otimes V \rightarrow \mathbb{k},$$

$$1 \mapsto \sum_{v \in B} \delta_v \otimes v,$$

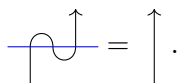
$$v \otimes f \mapsto f(v),$$

$$1 \mapsto \sum_{v \in B} v \otimes \delta_v,$$

$$f \otimes v \mapsto f(v).$$

## Duals: example

Let's check the relation


$$\text{wavy line with arrow} = \text{straight arrow}.$$

The left-hand side is the composition

$$V \cong V \otimes \mathbb{k} \xrightarrow{1_V \otimes \cup} V \otimes V^* \otimes V \xrightarrow{\cap \otimes 1_V} \mathbb{k} \otimes V \cong V,$$
$$w \mapsto w \otimes 1 \mapsto \sum_{v \in B} w \otimes \delta_v \otimes v \mapsto \sum_{v \in B} \delta_v(w) \otimes v \mapsto \sum_{v \in V} \delta_v(w) v = w.$$

The verification of the other relations is analogous.



# Self-dual objects

An object  $I$  is **self-dual** if there exist morphisms

$$\cup : \mathbb{1} \rightarrow I \otimes I. \quad \text{and} \quad \cap : I \otimes I \rightarrow \mathbb{1}.$$

such that

$$\cup \cap = \text{id} = \cap \cup.$$

## Example

In the category of **finite-dimensional inner product spaces** (over  $\mathbb{R}$  or  $\mathbb{C}$ ), all objects  $V$  are self-dual.

$$\begin{aligned} \cap : V \otimes V &\rightarrow \mathbb{k}, & v \otimes w &\mapsto \langle v, w \rangle, \\ \cup : \mathbb{k} &\rightarrow V \otimes V, & 1 &\mapsto \sum_{v \in B} v \otimes v, \end{aligned}$$

where  $B$  is an orthonormal basis for  $V$ .

## Pivotal categories

A strict monoidal category  $\mathcal{C}$  is a **strict pivotal category** if every object  $X$  has a right dual  $X^\vee$  and the following conditions are satisfied:

- 1 For all objects  $X$  and  $Y$  in  $\mathcal{C}$ ,

$$(X^\vee)^\vee = X, \quad (X \otimes Y)^\vee = Y^\vee \otimes X^\vee, \quad \mathbf{1}^\vee = \mathbf{1},$$

$X \otimes Y = XY$  and  $X \otimes Y = XY$ .

- 2 For every morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$ ,

$X \xrightarrow{f} Y = X \xrightarrow{f} Y : Y^\vee \rightarrow X^\vee$ .

**Intuition:** In a strict pivotal category, morphisms are invariant under isotopy.

# Framed tangles

The category  $\mathcal{FT}$  of **framed tangles** over the disc is generated by a single object  $I$  and morphisms

$$\times, \times, \cup, \cap,$$

subject to the relations

$$\begin{aligned} \cup = \cap, \quad \cap = \cup, \quad \cup = \cup, \\ \cup = \cup, \quad \cap = \cap, \quad \cup = \cup. \end{aligned}$$

**Universal property:**  $\mathcal{FT}$  is the free ribbon category generated by a self-dual object.

## Kauffman skein category

Fix  $z, t \in \mathbb{k}^\times$ . The **Kauffman skein category**  $\mathcal{KS}(z, t)$  is the category obtained from  $\mathcal{FT}_{\mathbb{k}}$  by imposing the relations

$$\begin{array}{c} \diagdown \\ \diagup \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} = z \left| \begin{array}{c} | \\ | \end{array} \right. + z \begin{array}{c} \cup \\ \cap \end{array} \quad (\text{Kauffman skein relation})$$

and

$$\begin{array}{c} \circlearrowleft \end{array} = t \left| \begin{array}{c} | \\ | \end{array} \right., \quad \bigcirc = \left( 1 - \frac{t + t^{-1}}{z} \right) \mathbb{1}_{\mathbb{1}}.$$

Then

$$\begin{aligned} \text{End}_{\mathcal{KS}(z,t)}(\mathbb{1}^{\otimes n}) &\cong \text{Kauffman tangle algebra} \\ &\cong \text{Birman–Murakami–Wenzl (BMW) algebra.} \end{aligned}$$

Viewing a link as an element of  $\text{End}_{\mathcal{KS}(z,t)}(\mathbb{1})$  gives rise to its **Kauffman polynomial**.

# Affinization

## Definition

The **affinization** of a strict monoidal category  $\mathcal{C}$  is the category  $\text{Aff}(\mathcal{C})$  obtained from  $\mathcal{C}$  by adjoining invertible morphisms

$$\xi_{X,Y}: X \otimes Y \rightarrow Y \otimes X, \quad X, Y \in \text{Ob}(\mathcal{C}),$$

subject to the relations

$$\xi_{X,Y \otimes Z} = \xi_{Z \otimes X, Y} \circ \xi_{X \otimes Y, Z}, \quad \xi_{X_2, Y_2} \circ (g \otimes f) = (f \otimes g) \circ \xi_{X_1, Y_1}.$$

for all  $X, Y, Z \in \text{Ob}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(Y_1, Y_2)$ , and  $g \in \text{Hom}_{\mathcal{C}}(X_1, X_2)$ .

Picture morphisms in  $\text{Aff}(\mathcal{C})$  as  $\mathcal{C}$ -diagrams on the cylinder, with

$$\xi_{X,Y} = \begin{array}{c} \text{[Cylinder with two strands crossing]} \\ X \quad Y \end{array}, \quad \xi_{X,Y}^{-1} = \begin{array}{c} \text{[Cylinder with two strands crossing]} \\ Y \quad X \end{array}. \quad \text{“coils”}$$

# Affinization

Cutting open the cylinder, we draw

$$\xi_{X,Y} = \begin{array}{c} \text{Cylinder with } X \text{ and } Y \text{ strands} \\ \text{---} \\ \begin{array}{c} Y \quad X \\ \diagup \quad \diagdown \\ X \quad Y \end{array} \end{array}, \quad \xi_{X,Y}^{-1} = \begin{array}{c} \text{Cylinder with } X \text{ and } Y \text{ strands} \\ \text{---} \\ \begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ Y \quad X \end{array} \end{array}.$$

Then the defining relations become

$$\begin{array}{c} Y \otimes Z \quad X \\ \diagup \quad \diagdown \\ X \quad Y \otimes Z \end{array} = \begin{array}{c} YZ \quad X \\ \diagup \quad \diagdown \\ X \quad YZ \end{array}, \quad \begin{array}{c} Y_2 \quad X_2 \\ \diagup \quad \diagdown \\ X_1 \quad Y_1 \end{array} = \begin{array}{c} Y_2 \quad X_2 \\ \oplus \quad \oplus \\ X_1 \quad Y_1 \end{array}.$$

If  $\mathcal{C}$  is **braided**, then  $\text{Aff}(\mathcal{C})$  is **strict monoidal**, with tensor product given by nesting cylinders. For example,

$$\begin{array}{c} Y \quad X \\ \diagup \quad \diagdown \\ X \quad Y \end{array} \otimes \begin{array}{c} | \\ | \\ Z \end{array} = \begin{array}{c} Y \quad X \\ \diagup \quad \diagdown \\ X \quad YZ \end{array}, \quad \begin{array}{c} | \\ | \\ X \end{array} \otimes \begin{array}{c} Z \quad Y \\ \diagup \quad \diagdown \\ Y \quad Z \end{array} = \begin{array}{c} Z \quad Y \\ \diagup \quad \diagdown \\ XY \quad Z \end{array}.$$

# Dot presentation

Suppose  $\mathcal{C}$  is **braided**, so that  $\text{Aff}(\mathcal{C})$  is strict monoidal. Define

$$\begin{array}{c} \oplus \\ X \end{array} := \xi_{\mathbb{1}, X} = \begin{array}{c} X \\ \text{[Cylinder with dot]} \\ X \end{array} = \begin{array}{c} X \\ \text{[Crossing]} \\ X \end{array}, \quad \begin{array}{c} \ominus \\ X \end{array} := \xi_{\mathbb{1}, X}^{-1} = \begin{array}{c} X \\ \text{[Cylinder with dot]} \\ X \end{array} = \begin{array}{c} X \\ \text{[Crossing]} \\ X \end{array}.$$

Thus  $\begin{array}{c} | \\ | \\ | \\ | \\ \oplus \\ | \\ | \\ | \\ | \end{array} = \begin{array}{c} \text{[Crossing]} \\ | \\ | \\ | \\ | \end{array}, \quad \begin{array}{c} | \\ | \\ | \\ | \\ \ominus \\ | \\ | \\ | \\ | \end{array} = \begin{array}{c} \text{[Crossing]} \\ | \\ | \\ | \\ | \end{array}.$

## Theorem

The **affinization**  $\text{Aff}(\mathcal{C})$  is obtained from  $\mathcal{C}$  by adjoining morphisms

$$\begin{array}{c} \oplus \\ X \end{array}, \quad \begin{array}{c} \ominus \\ X \end{array} : X \rightarrow X, \quad X \in \text{Ob}(\mathcal{C}), \quad \text{subject to relations}$$

$$\begin{array}{c} \oplus \\ \oplus \end{array} = \begin{array}{c} \oplus \\ \oplus \end{array}, \quad \begin{array}{c} \oplus \\ X \otimes Y \end{array} = \begin{array}{c} \oplus \\ X \end{array} \begin{array}{c} \oplus \\ Y \end{array}, \quad \begin{array}{c} \oplus \\ f \end{array} = \begin{array}{c} f \\ \oplus \end{array}, \quad \begin{array}{c} \oplus \\ \oplus \end{array} = \begin{array}{c} \oplus \\ \oplus \end{array} = \begin{array}{c} | \end{array}.$$

# Pivotal structures

Suppose  $\mathcal{C}$  is braided and **rigid/pivotal**. Then  $\text{Aff}(\mathcal{C})$  is also rigid/pivotal.

## Self-dual case (unoriented)

If  $X$  is self-dual, for strands labelled  $X$  we have

$$\oplus \cap = \cap \ominus, \quad \ominus \cap = \cap \oplus, \quad \oplus \cup = \cup \ominus, \quad \ominus \cup = \cup \oplus.$$

## Non-self-dual case (oriented)

Define the **invertible dots**

$$\begin{array}{c} \uparrow \\ \circ \\ | \\ X \end{array} := \begin{array}{c} \uparrow \\ \oplus \\ | \\ X \end{array}, \quad \begin{array}{c} \downarrow \\ \circ \\ | \\ X \end{array} := \begin{array}{c} \downarrow \\ \ominus \\ | \\ X \end{array}.$$

Then we have

$$\begin{array}{c} \uparrow \\ \circ \\ | \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \ominus \\ | \\ \uparrow \end{array}, \quad \begin{array}{c} \downarrow \\ \circ \\ | \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \oplus \\ | \\ \downarrow \end{array}, \quad \begin{array}{c} \downarrow \\ \circ \\ | \\ \downarrow \end{array} = \begin{array}{c} \uparrow \\ \ominus \\ | \\ \uparrow \end{array}, \quad \begin{array}{c} \uparrow \\ \circ \\ | \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \oplus \\ | \\ \downarrow \end{array}.$$



## Examples: Towers of algebras

Suppose  $\mathcal{C}$  is a strict linear monoidal category such that

- objects are generated by a single object  $X$ ,
- $\text{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m}) = 0$  when  $m \neq n$ .

The collection  $\mathcal{C}(n) := \text{End}_{\mathcal{C}}(X^{\otimes n})$ ,  $n \in \mathbb{N}$ , is a **tower of algebras**.




### Braids

Category *Braid* of braids over the disc has

generators , , relations  =  = ,  = .

### Affine braids

The **affinization**  $\text{Aff}(\textit{Braid})$  is the category of braids over the annulus. Dot presentation obtained from *Braid* by adding

invertible generator , relation  = .

# Examples: Towers of algebras

## Hecke algebras

The category  $\mathcal{H}$  corresponding to the tower of **Hecke algebras of type  $A$**  is the  $\mathbb{k}$ -linearization of *Braid* modulo the Conway skein relation

$$\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} - \begin{array}{c} \nwarrow \nearrow \\ \nearrow \searrow \end{array} = (q - q^{-1}) \begin{array}{c} \uparrow \uparrow \end{array}, \quad q \in \mathbb{k}^\times.$$

## Affine Hecke algebras

The **affinization**  $\text{Aff}(\mathcal{H})$  corresponds to the tower of **affine Hecke algebras of type  $A$** . Obtained from  $\mathcal{H}$  by adding

invertible generator  $\hat{\phi}$ , relation  $\begin{array}{c} \nearrow \searrow \\ \circ \quad \nearrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \nearrow \quad \circ \end{array}.$

## Examples: Tangles

The category  $\mathcal{FT}$  of **framed tangles** over the disc is generated by a single object  $I$  and morphisms

$$\times, \succ, \cup, \cap,$$

subject to the relations

$$\begin{aligned} \cup &= | = \cap, & \cap &= \cap', & \cup &= \cup', \\ d_1 &= \rho, & \cup &= || = \cup, & \times &= \times'. \end{aligned}$$

The **affinization**  $\text{Aff}(\mathcal{FT})$  is the category of framed tangles over the **annulus**.

$\text{Aff}(\mathcal{FT})$  is obtained from  $\mathcal{FT}$  adding morphisms  $\oplus, \ominus$  and relations

$$\begin{aligned} \times &= \times^{\oplus}, & \cap &= \cap^{\ominus}, & \oplus &= \oplus = |. \end{aligned}$$

Similar results hold for non-framed tangles and oriented (framed and non-framed) tangles.

## Examples: Skein categories

### Kauffman skein category

The affinization of the **Kauffman skein category** over the disc is the Kauffman skein category over the annulus.

Dot presentation matches the **affine Kauffman skein category** of Gao–Rui–Song.

### Temperley–Lieb category

The affinization of the **Temperley–Lieb category** is the **affine Temperley–Lieb category**.

## Actions

Suppose  $\mathcal{C}, \mathcal{M}$  are braided strict pivotal categories and  $F: \mathcal{C} \rightarrow \mathcal{M}$  is a monoidal functor.

Then  $\mathcal{C}$  acts on  $\mathcal{M}$  via the action

$$X \cdot M = F(X) \otimes M, \quad f \cdot g = F(f) \otimes g.$$

### Theorem

The above can be extended to an action of  $\text{Aff}(\mathcal{C})$  on  $\mathcal{M}$  with

$$\xi_X \cdot g = \text{diagram}, \quad \xi_X^{-1} \cdot g = \text{diagram}.$$

**Note:** More generally, have action whenever  $\mathcal{C}$  is a strict monoidal category and  $\mathcal{M}$  is a **balanced** strict monoidal category.

## Horizontal trace

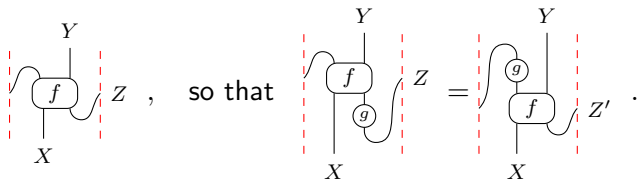
Suppose  $\mathcal{C}$  is a strict monoidal category. Fix  $X, Y \in \text{Ob}(\mathcal{C})$  and consider pairs

$$(Z, f), \quad Z \in \text{Ob}(\mathcal{C}), \quad f: X \otimes Z \rightarrow Z \otimes Y.$$

Define an equivalence relation on such pairs by

$$(Z, f \circ (1_X \otimes g)) \sim (Z', (g \otimes 1_Y) \circ f), \\ g \in \text{Hom}_{\mathcal{C}}(Z, Z'), \quad f \in \text{Hom}_{\mathcal{C}}(X \otimes Z', Z \otimes Y).$$

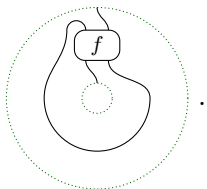
Picture the equivalence class  $[Z, f]$  as the cylindrical diagram



**Important:** The cups and caps above have no precise meaning!

## Horizontal trace

Also sometimes see  $[Z, f]$  as the annular diagram



**Important (again):** The cups and caps above have no precise meaning!

Can define a natural composition, turning the horizontal trace  $\text{Tr}_h(\mathcal{C})$  into a category.

If  $\mathcal{C}$  is **braided**, then the horizontal trace is a **monoidal category**.

# Horizontal trace

## Theorem

If  $\mathcal{C}$  is a **rigid** strict monoidal category then  $\text{Aff}(\mathcal{C})$  is **isomorphic** to the horizontal trace  $\text{Tr}_h(\mathcal{C})$  of  $\mathcal{C}$ .

In general (i.e. when  $\mathcal{C}$  is not rigid) the affinization and the horizontal trace are **different**.

It is the **affinization**, not the horizontal trace, that naturally corresponds to  $\mathcal{C}$ -diagrams on the cylinder.

## Example

Recall that  $\text{Aff}(\mathit{Braid})$  is the category of **annular braids**.

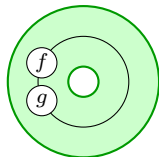
However, the strings in  $\text{Tr}_h(\mathit{Braid})$  can only wrap in one direction around the cylinder, and one also has closed components wrapping around the cylinder.



## Vertical trace

The **vertical trace** of a  $\mathbb{k}$ -linear category  $\mathcal{C}$  is the  $\mathbb{k}$ -module

$$\mathrm{Tr}_v(\mathcal{C}) := \left( \bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(X) \right) / \mathrm{Span}_{\mathbb{k}}\{f \circ g - g \circ f\},$$



where  $f$  and  $g$  range over  $f: X \rightarrow Y$ ,  $g: Y \rightarrow X$  in  $\mathcal{C}$ .

### Toroidal diagrams

If  $\mathcal{C}$  is strict pivotal,  $\mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C}))$  corresponds to  $\mathcal{C}$ -diagrams on the **torus**.

If  $\mathcal{C} \rightarrow \mathcal{M}$  is a monoidal functor, then the  $\mathrm{Tr}_v(\mathrm{Aff}(\mathcal{C}))$  **acts** on  $\mathrm{Tr}_v(\mathcal{M})$ .  
(Nesting of annulus inside torus.)

If  $\mathcal{C}$  is right or left rigid, then  $\mathrm{Tr}_v(\mathcal{C}) \cong Z(\mathrm{Aff}(\mathcal{C})) := \mathrm{End}_{\mathrm{Aff}(\mathcal{C})}(\mathbb{1})$ .

## Final example: HOMFLYPT skein category

The category  $\mathcal{FOT}$  of **framed oriented tangles** over the disc is isomorphic to the strict monoidal category generated by objects  $\uparrow, \downarrow$ , and morphisms

$$\nearrow, \nwarrow, \searrow, \swarrow, \downarrow, \uparrow, \downarrow, \uparrow, \cup, \cap, \cup, \cap,$$

subject to the relations

$$\begin{aligned} \cup &= | = \cap, & \cap &= \cap', & \cup &= \cup', \\ \downarrow &= \downarrow, & \downarrow &= || = \downarrow, & \downarrow &= \downarrow, \end{aligned}$$

for all orientations of the strands.

## Final example: HOMFLYPT skein category

Work over  $\mathbb{k} = \mathbb{Z}[z, z^{-1}, t, t^{-1}]$ . The **HOMFLYPT skein category**  $\mathcal{OS}(z, t)$  over the disc is the category obtained from  $\mathcal{FOT}_{\mathbb{k}}$  by imposing the relations

$$\begin{array}{c} \nearrow \nearrow \\ \nwarrow \nwarrow \end{array} - \begin{array}{c} \nwarrow \nwarrow \\ \nearrow \nearrow \end{array} = z \begin{array}{c} \uparrow \\ \uparrow \end{array} \quad (\text{Conway skein relation})$$

and

$$\begin{array}{c} \uparrow \\ \circlearrowleft \end{array} = t \begin{array}{c} \uparrow \\ \uparrow \end{array}, \quad \begin{array}{c} \circlearrowright \end{array} = \frac{t - t^{-1}}{z} \mathbf{1}_{\mathbb{1}}.$$

Given an oriented link diagram  $L$ , we have

$$t^{-\text{writhe}(L)} L = H(L) \frac{t - t^{-1}}{z} \mathbf{1}_{\mathbb{1}},$$

where

- $\text{writhe}(L)$  is the **writhe number** ( $\#$  pos crossings  $-$   $\#$  neg crossings),
- $H(L)$  is the **HOMFLYPT** polynomial of  $L$ .

## Final example: HOMFLYPT skein category

The **affinization**  $\text{Aff}(\mathcal{OS}(z, t))$  is isomorphic to the **HOMFLPYT skein category** over the annulus.

Dot presentation matches the **affine oriented skein category** of Brundan.

**Affinization**  $\text{Aff}(\mathcal{OS}(z, t))$  is obtained from  $\mathcal{OS}(z, t)$  by adjoining morphisms

$$\uparrow\circlearrowleft: \uparrow \rightarrow \uparrow, \quad \downarrow\circlearrowright: \downarrow \rightarrow \downarrow,$$

subject to the relations

$$\begin{array}{c} \nearrow \\ \circlearrowleft \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \circlearrowleft \\ \nearrow \end{array}, \quad \begin{array}{c} \circlearrowleft \\ \downarrow \end{array} = \begin{array}{c} \cap \\ \downarrow \end{array}, \quad \begin{array}{c} \circlearrowright \\ \downarrow \end{array} = \begin{array}{c} \cup \\ \downarrow \end{array}, \quad \uparrow\circlearrowleft \text{ is invertible.}$$

The **vertical trace** of the affinization

$$\text{Tr}_v(\text{Aff}(\mathcal{OS}(z, t)))$$

is the **HOMFLYPT skein algebra of the torus**. It acts on  $\text{Tr}_v(\mathcal{OS}(z, t))$ , which is the **skein of the solid torus**. (See work of Morton–Samuelson.)