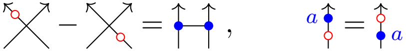
Affine oriented Frobenius Brauer categories



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Slides: alistairsavage.ca/talks

Preprint: arXiv:2101.04582 (joint with Alexandra McSween)

Outline

Goal: Generalize and unify Brauer algebra/category techniques to the setting of general linear Lie superalgebras over Frobenius superalgebras.

Overview:

1. Monodial supercategories
2. Oriented Brauer category
3. Frobenius superalgebras A
4. $\mathfrak{gl}_{m|n}(A)$
5. (Affine) oriented Frobenius Brauer category
6. Natural actions
7. Further directions

Supercategories

Let $\mathcal{SV}ec$ be the category of vector superspaces with parity-preserving morphisms.

A supercategory is a category enriched in $\mathcal{SV}ec$. Thus,

- morphism spaces are superspaces, and
- composition is parity preserving.

Example

If B is a superalgebra, then the category of B-supermodules, with all (i.e. not necessarily parity-preserving) homomorphisms, is a supercategory.

Example

If $\mathfrak{h}$ is a Lie superalgebra, then the category of $\mathfrak{h}$-supermodules, with all (i.e. not necessarily parity-preserving) homomorphisms, is a supercategory.

A superfunctor between supercategories induces a parity-preserving linear map between morphism spaces.

Strict monoidal supercategories

A strict monoidal supercategory is a supercategory $\mathcal{C}$ equipped with

- a super bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all objects A, B, C,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$ for all objects A,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ for all morphisms f, g, h,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$ for all morphisms f.

Remark: Non-strict monoidal supercategories

In a (not necessarily strict) monoidal supercategory, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

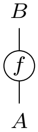
The axioms of a monoidal supercategory imply the super interchange law: If f, g are homogeneous morphisms, then

$$f \otimes g = (f \otimes 1) \circ (1 \otimes g) = (-1)^{\bar{f}\bar{g}} (1 \otimes g) \circ (f \otimes 1).$$

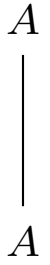
String diagrams

Fix a strict monoidal supercategory $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:



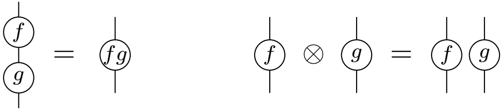
The identity map $1_A\colon A \to A$ is a string with no label:



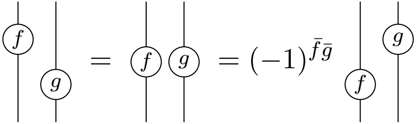
We sometimes omit the object labels when they are clear or unimportant.

String diagrams

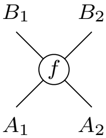
Composition is vertical stacking and tensor product is horizontal juxtaposition:



The super interchange law then becomes:



A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



Symmetric monoidal supercategories

A symmetric monoidal supercategory is a monoidal supercategory $\mathcal{C}$ such that, for each pair of objects X, Y in $\mathcal{C}$, there is an isomorphism

$$s_{XY} \colon X \otimes Y \xrightarrow{\cong} Y \otimes X$$

that is

- natural in both X and Y,
- $s_{YX} s_{XY} = 1_{X \otimes Y}$ for all objects X, Y,
- the s_{XY} satisfy the braid relation,
- the s_{XY} satisfy certain (associativity and unit) coherence conditions.

Examples

- Vector superspaces over a given field
- Supermodules over a cocommutative Hopf superalgebra (e.g. group algebra of a group)

In both cases, $s_{XY}(x \otimes y) = (-1)^{\bar{x}\bar{y}} y \otimes x$.

A universal symmetric monoidal supercategory

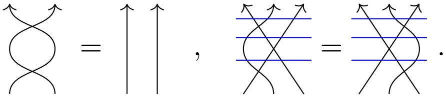
Define a strict monoidal supercategory $\mathcal{S}$ with one generating object $\uparrow$ and denote

$$1_{\uparrow} = \uparrow$$

We have one generating morphism

$$: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow .$$

We impose the relations:



Then

$$\mathrm{End}_{\mathcal{S}}(\uparrow^{\otimes n}) = \Bbbk S_n$$

is the group algebra of the symmetric group on n letters.

A universal symmetric monoidal category

Universal property: If $\mathcal{C}$ is any symmetric monoidal supercategory and X is an object of $\mathcal{C}$, then there is a unique monoidal superfunctor

$\mathcal{S} \to \mathcal{C}, \quad \uparrow \mapsto X,$  $\mapsto s_{XX}.$

Immediate consequence: We have an S_n-action on $\mathrm{End}_{\mathcal{C}}(X^{\otimes n})$ coming from the induced map

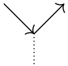
$$\Bbbk S_n = \mathrm{End}_{\mathcal{S}}(\uparrow^{\otimes n}) \to \mathrm{End}_{\mathcal{C}}(X^{\otimes n}).$$

Duals

Suppose a strict monoidal supercategory $\mathcal{C}$ has two objects $\uparrow$ and $\downarrow$, with

$$1_\uparrow = \Big\uparrow \quad , \qquad 1_\downarrow = \Big\downarrow .$$

A morphism $\mathbb{1} \to \downarrow \otimes \uparrow$ would have string diagram

 , where $= 1_{\mathbb{1}}$.

We typically omit the dotted line and draw:

 $: \mathbb{1} \to \downarrow \otimes \uparrow .$

Similarly, we can have

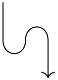
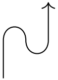
 $: \uparrow \otimes \downarrow \to \mathbb{1}.$

Duals

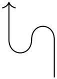
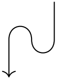
We say that $\downarrow$ is right dual to $\uparrow$ (and $\uparrow$ is left dual to $\downarrow$) if there exist morphisms

 $: \mathbb{1} \to \downarrow \otimes \uparrow$ and  $: \uparrow \otimes \downarrow \to \mathbb{1}$

such that

 $= \downarrow$ and  $= \uparrow$.

Similarly, $\downarrow$ is left dual to $\uparrow$ if we have

 $: \mathbb{1} \to \uparrow \otimes \downarrow$,  $: \downarrow \otimes \uparrow \to \mathbb{1}$ with  $= \uparrow$,  $= \downarrow$.

Remark

If $\downarrow$ is right dual to $\uparrow$ in a symmetric monoidal supercategory, then it is also left dual, with

 $:=$  and  $:=$ .

Duals: example

Let $\Bbbk$ be a field and consider the category $\mathrm{Vect}_{\Bbbk}$ of f.d. $\Bbbk$-vector spaces.

Unit object: $\Bbbk$

Fix a f.d. $\Bbbk$-vector space V.

Claim: The dual vector space V^* is both left and right dual to V, in the sense mentioned above.

Proof: Fix a basis B of V. Let $\{\delta_v : v \in B\}$ be the dual basis of V^*. Viewing V and $\uparrow$ and V^* as $\downarrow$, we define

 $: \Bbbk \to V^* \otimes V, \quad 1 \mapsto \sum_{v \in B} \delta_v \otimes v,$

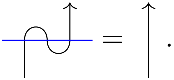
 $: V \otimes V^* \to \Bbbk, \quad v \otimes f \mapsto f(v),$

 $: \Bbbk \to V \otimes V^*, \quad 1 \mapsto \sum_{v \in B} v \otimes \delta_v,$

 $: V^* \otimes V \to \Bbbk, \quad f \otimes v \mapsto f(v).$

Duals: example

Let's check the relation



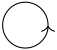
The left-hand side is the composition

$$V \cong V \otimes \Bbbk \xrightarrow{1_V \otimes \cup} V \otimes V^* \otimes V \xrightarrow{\cap \otimes 1_V} \Bbbk \otimes V \cong V,$$

$$w \mapsto w \otimes 1 \mapsto \sum_{v \in B} w \otimes \delta_v \otimes v \mapsto \sum_{v \in B} \delta_v(w) \otimes v \mapsto \sum_{v \in B} \delta_v(w) v = w.$$

The verification of the other relations is analogous.

Note:



$$: \Bbbk \to \Bbbk, \quad 1 \mapsto \sum_{v \in B} \delta_v(v) = \dim V.$$

So the "bubble" corresponds to the dimension of the object V.

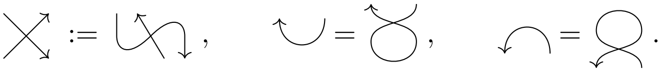
The oriented Brauer category

The oriented Brauer category $\mathcal{OB}$ is the free $\Bbbk$-linear symmetric monoidal category on a pair of dual objects.

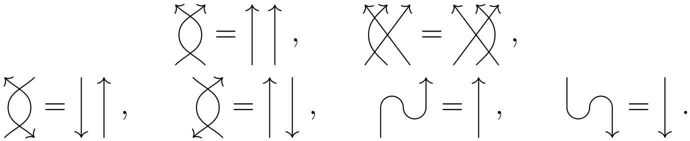
Two generating objects: $\uparrow$ and $\downarrow$

Four generating morphisms:

Define



Relations:



The oriented Brauer category

Universal property: Any $\Bbbk$-linear symmetric monoidal category $\mathcal{C}$ with dual objects V, W admits a $\Bbbk$-linear monoidal functor

$$\mathcal{OB} \to \mathcal{C}, \quad \uparrow \mapsto V, \quad \downarrow \mapsto W.$$

Example (Vector spaces)

For any f.d. $\Bbbk$-vector space V, we have a $\Bbbk$-linear monoidal functor

$$\mathcal{OB} \to \mathrm{Vect}_{\Bbbk}, \quad \uparrow \mapsto V, \quad \downarrow \mapsto V^*.$$

Example ($\mathfrak{gl}_m(\Bbbk)$-mod)

Let $V = \Bbbk^m$ be the natural $\mathfrak{gl}_m(\Bbbk)$-module. We have a $\Bbbk$-linear monoidal functor

$$\mathcal{OB} \to \mathfrak{gl}_m(\Bbbk)\text{-mod}, \quad \uparrow \mapsto V, \quad \downarrow \mapsto V^*.$$

Schur–Weyl duality: This functor is full.

Frobenius superalgebras

Fix a ground field $\Bbbk$.

A symmetric Frobenius superalgebra is an associative superalgebra with trace map

$$\operatorname{tr}_A \colon A \to \Bbbk, \qquad \operatorname{tr}_A(ab) = (-1)^{\bar{a}\bar{b}} \operatorname{tr}_A(ba), \quad a, b \in A,$$

such that $\ker(\operatorname{tr}_A)$ contains no nonzero left ideal.

Choose a homogeneous basis $\mathbf{B}_A$. Define the dual basis $\{b^\vee : b \in \mathbf{B}_A\}$ by

$$\operatorname{tr}_A(b^\vee c) = \delta_{b,c}, \qquad b, c \in \mathbf{B}_A.$$

Remark

A (not necessarily symmetric) Frobenius superalgebra has a Nakayama automorphism $\psi \colon A \to A$ with

$$\operatorname{tr}_A(ab) = (-1)^{\bar{a}\bar{b}} \operatorname{tr}_A(b\psi(a)).$$

Examples of Frobenius superalgebras

Ground field $\mathbb{k}$

$\mathbb{k}$ with $\mathrm{tr}_{\mathbb{k}} = \mathrm{id}_{\mathbb{k}}$.

Matrix algebra

Algebra of matrices $\mathrm{Mat}_n(\mathbb{k})$ with the usual trace.

Matrix superalgebra

Superalgebra of super matrices $\mathrm{Mat}_{m|n}(\mathbb{k})$ with the super trace.

Truncated polynomial algebra

$A = \mathbb{k}[x]/(x^k)$ with $\mathrm{tr}_A(x^l) = \delta_{l,k-1}$.

If k is even, then we can declare x to be odd.

Examples of Frobenius superalgebras

Group algebra

If G is a finite group, then the group algebra $\Bbbk G$ is a symmetric Frobenius algeba with

$$\mathrm{tr}_{\Bbbk G}(g) = \delta_{g,1_G}.$$

Note: If we instead fix $h \in G$ and choose

$$\mathrm{tr}_{\Bbbk G}(g) = \delta_{g,h},$$

then we get a Frobenius algebra with Nakayama automorphism

$$\psi(g) = h^{-1}gh.$$

Hopf algebras

Every f.d. Hopf superalgebra is a Frobenius superalgebra.

Clifford superalgebra

Let Cl denote the two-dimensional Clifford superalgebra generated by c with

$$\bar{c} = 1, \qquad c^2 = 1.$$

Up to scalar multiples, there are two choices of trace map.

Even trace

If we define

$$\mathrm{tr}_{\mathrm{Cl}}(1) = 1, \quad \mathrm{tr}_{\mathrm{Cl}}(c) = 0,$$

then Cl is a Frobenius superalgebra with Nakayama automorphism

$$1 \mapsto 1, \quad c \mapsto -c. \qquad \text{(not symmetric!)}$$

Odd trace

If we define

$$\mathrm{tr}_{\mathrm{Cl}}(1) = 0, \qquad \mathrm{tr}_{\mathrm{Cl}}(c) = 1,$$

then Cl is a symmetric Frobenius superalgebra.

Lie superalgebras $\mathfrak{gl}_{m|n}(A)$

Fix a symmetric Frobenius superalgebra A with even trace map.

Fix $m, n \in \mathbb{N}$. For $1 \le i \le m+n$, define

$$p(i) = \begin{cases} \bar{0} & \text{if } 1 \le i \le m, \\ \bar{1} & \text{if } m+1 \le i \le m+n, \end{cases} \in \mathbb{Z}/2\mathbb{Z}.$$

Let $\mathrm{Mat}_{m|n}(A)$ be the associative superalgebra of $(m+n) \times (m+n)$ matrices with entries in A. Parity is defined by

$$\overline{a_{(i,j)}} = \bar{a} + p(i) + p(j), \qquad \left(\begin{array}{c|c} \bar{a} & \bar{a}+\bar{1} \\ \hline \bar{a}+\bar{1} & \bar{a} \end{array} \right)$$

where $a_{(i,j)}$ is the matrix with $a \in A$ in the (i,j) position and 0 in all other positions.

Let $\mathfrak{g} = \mathfrak{gl}_{m|n}(A)$ be the Lie superalgebra associated to $\mathrm{Mat}_{m|n}(A)$, with bracket

$$[M, N] = MN - (-1)^{\bar{M}\bar{N}} NM.$$

Examples

General linear Lie superalgebras

$\mathfrak{gl}_{m|n}(\Bbbk)$ is the usual general linear Lie superalgebra over $\Bbbk$.

Truncated current superalgebras

$\mathfrak{gl}_{m|n}(\Bbbk[t]/(t^l))$ is a truncated current superalgebra. When $n = 0$, this is called a Takiff algebra.

Queer Lie superalgebras

Recall that Cl is the two-dimensional Clifford superalgebra generated by an odd element c with $c^2 = 1$. (Need odd trace map here.)

Then $\mathfrak{gl}_{m|n}(\mathrm{Cl}) \cong \mathfrak{q}_{m+n}(\Bbbk)$, the queer Lie superalgebra.

E.g. $\mathfrak{gl}_m(\mathrm{Cl}) \cong \mathfrak{q}_m(\Bbbk)$.

Natural modules

Let $A^{m|n}$ be the $\Bbbk$-superspace equal to A^{m+n} as a $\Bbbk$-module, with parity given by

$$\overline{ae_i} = \bar{a} + p(i), \qquad a \in A,\ 1 \le i \le m+n, \qquad (\underbrace{\bar{a}}_{m \text{ positions}}, \underbrace{\bar{a}+\bar{1}}_{n \text{ positions}})$$

where $e_i \in A^{m|n}$ has a 1 in the i-th entry and a 0 in all other entries.

Define

$$V_+ = A^{m|n}, \qquad \text{written as row matrices,}$$
$$V_- = A^{m|n}, \qquad \text{written as column matrices.}$$

Then V_+ is naturally a right $\mathfrak{g}$-supermodule, and V_- is a right $\mathfrak{g}$-supermodule with action

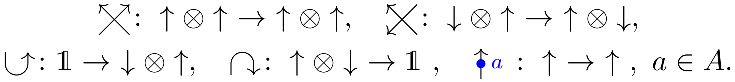
$$v \cdot M = -(-1)^{\bar{v}\bar{M}} Mv, \qquad v \in V_-,\ M \in \mathfrak{g}.$$

We have a natural nondegenerate $\Bbbk$-bilinear form

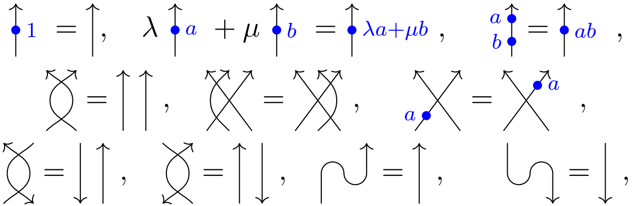
$$B \colon V_- \otimes V_+ \to \Bbbk, \quad B(v \otimes w) = (-1)^{\bar{v}\bar{w}} \operatorname{tr}_A(wv).$$

Oriented Frobenius Brauer category

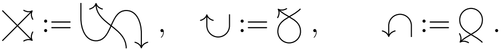
Let $\mathcal{OB}(A)$ be the strict monoidal supercategory generated by objects $\uparrow$ and $\downarrow$ and morphisms



subject to the relations ($a, b \in A$, $\lambda, \mu \in \Bbbk$)



where we define



The parity of  is $\bar{a}$, and all the other generating morphisms are even.

Oriented Frobenius Brauer category

We have

$$\mathrm{End}_{\mathcal{OB}(A)}(\uparrow^{\otimes n}) \cong A^{\otimes n} \rtimes S_n,$$

which are wreath product algebras.

Case $A = \Bbbk$

$\mathcal{OB}(\Bbbk)$ is the oriented Brauer category.

Endomorphism algebras are oriented Brauer algebras, which are isomorphic to walled Brauer algebras.

Case $A = \mathrm{Cl}$

$\mathcal{OB}(\mathrm{Cl})$ is the oriented Brauer–Clifford supercategory of Brundan–Comes–Kujawa.

Theorem (McSween–S.)

There is a monoidal superfunctor $\psi\colon \mathcal{OB}(A) \to$ smod-$\mathfrak{g}$ given on objects by

$$\uparrow \mapsto V_+, \qquad \downarrow \mapsto V_-$$

and on morphisms by

$$\psi(\ )\colon V_+ \otimes V_+ \to V_+ \otimes V_+, \qquad v \otimes w \mapsto (-1)^{\bar{v}\bar{w}} w \otimes v,$$

$$\psi(\ )\colon V_- \otimes V_+ \to V_+ \otimes V_-, \qquad v \otimes w \mapsto (-1)^{\bar{v}\bar{w}} w \otimes v,$$

$$\psi(\ a\ )\colon V_+ \mapsto V_+, \qquad v \mapsto av,$$

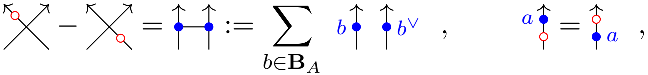
$$\psi(\ )\colon V_+ \otimes V_- \to \Bbbk, \qquad v \otimes w \mapsto (-1)^{\bar{v}\bar{w}} B(w \otimes v),$$

$$\psi(\ )\colon \Bbbk \mapsto V_- \otimes V_+, \qquad 1 \mapsto \sum_{v \in \mathbf{B}_{V_+}} (-1)^{\bar{v}} v^\vee \otimes v.$$

Note: $\psi\left(\ \right) = \psi\left(\ \right) = \mathrm{sdim}(V_+) 1_{\mathbb{1}}.$

Affine oriented Frobenius Brauer category

Let $\mathcal{AOB}(A)$ be the strict monoidal supercategory obtained from $\mathcal{OB}(A)$ by adjoining an even generator : $\uparrow \to \uparrow$, subject to the relations



$a \in A$.

Remarks

1. $\mathrm{End}_{\mathcal{AOB}(A)}(\uparrow^{\otimes n})$ are affine wreath product algebras.
2. $\mathcal{AOB}(\Bbbk)$ is the affine oriented Brauer category.
3. $\mathcal{AOB}(\mathrm{Cl})$ is the affine oriented Brauer–Clifford category.
4. $\mathcal{AOB}(A) = \mathcal{H}eis_0(A)$ is the Frobenius Heisenberg category at central charge zero.

Theorem (McSween–S.)

Let $\mathcal{E}nd\,(\text{smod-}\mathfrak{g})$ denote the supercategory with:

- Objects: superfunctors smod-$\mathfrak{g} \to$ smod-$\mathfrak{g}$,
- Morphisms: supernatural transformations between superfunctors.

There is a monoidal superfunctor

$$\Psi \colon \mathcal{AOB}(A) \to \mathcal{E}nd\,(\text{smod-}\mathfrak{g})$$

given on objects by

$$\uparrow\, \mapsto V_+ \otimes -, \qquad \downarrow\, \mapsto V_- \otimes -$$

and on morphisms by

$$\Psi(f) = \psi(f) \otimes -, \quad f \in \{\nwarrow\!\!\!\!\nearrow, \swarrow\!\!\!\!\nearrow, \uparrow\!\!\bullet\, a, \circlearrowleft, \cap\!\!\downarrow : a \in A\},$$

and $\Psi(\uparrow\!\!\circ) \colon V_+ \otimes - \to V_+ \otimes -$ is the functor with components

$$\Psi(\uparrow\!\!\circ)_W \colon V_+ \otimes W \to V_+ \otimes W, \quad v \otimes w \mapsto (v \otimes w) \sum_{M \in \mathbf{B}_\mathfrak{g}} M \otimes M^\vee.$$

Special cases

$A = \Bbbk$

When $A = \Bbbk$, previous two theorems specialize to known results on the

- oriented Brauer category (Turaev, Koike, and others) and the
- affine oriented Brauer category (Brundan–Comes–Nash–Reynolds).

$A = \mathrm{Cl}$

When A is the two-dimensional Clifford algebra, we recover actions of the

- oriented Brauer–Clifford category and the
- affine oriented Brauer–Clifford category

on supermodules for the queer Lie superalgebra (Brundan–Comes–Kujawa).

Other cases seem to be new.

Application: central elements

The center of a monoidal category $\mathcal{C}$ is

$$Z(\mathcal{C}) := \mathrm{End}_{\mathcal{C}}(\mathbb{1}).$$

We have a natural identification

$$\rho\colon Z(U(\mathfrak{g})) \xrightarrow{\cong} Z(\mathcal{E}nd(\mathsf{smod}\text{-}\mathfrak{g})), \quad u \mapsto \rho_u,$$

where ρ_u is the supernatural transformation with components

$$(\rho_u)_W\colon W \to W, \quad w \mapsto (-1)^{\bar{u}\bar{w}} wu, \qquad W \in \mathsf{smod}\text{-}\mathfrak{g}.$$

The center of $\mathcal{AOB}(A)$ consists of closed diagrams.

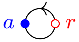
Application: central elements

So we have maps

$$\underbrace{Z(\mathcal{AOB}(A))}_{\text{closed diagrams}} \xrightarrow{\Psi} Z(\mathcal{E}nd(\mathsf{smod}\text{-}\mathfrak{g})) \xrightarrow{\rho^{-1}} Z(U(\mathfrak{g})).$$

In particular, we have canonical central elements

$$\rho^{-1} \circ \Psi \Bigg($$



$$\Bigg) \in Z(U(\mathfrak{g})), \quad a \in A,\ r \in \mathbb{N}.$$

When $A = \Bbbk$ these are the known elements

$$\sum_{\substack{1 \le i_1, \dots, i_r \le n \\ i_1 = i_r}} e_{i_{r-1}, i_r} \cdots e_{i_2, i_3} e_{i_1, i_2}.$$

When $A = \mathrm{Cl}$ these central elements were computed by Brundan–Comes–Kujawa.

Further directions

Interpolating categories

Idempotent completion of $\mathcal{OB}(A)$ is a candidate for an interpolating supercategory for smod-$\mathfrak{gl}_{m|n}(A)$, generalizing Deligne's interpolating categories (case $n = 0$, $A = \Bbbk$).

Frobenius Schur algebras

Define Schur superalgebras depending on Frobenius superalgebra A such that, $A = \Bbbk$ recovers usual Schur algebras.

Cyclotomic quotients

When $A = \Bbbk$ and $A = \mathrm{Cl}$, cyclotomic quotients of $\mathcal{AOB}(A)$ have been studied by Brundan–Comes–Nash–Reynolds and Brundan–Comes–Kujawa. These results can likely be extended to general A.

Further directions

Type BCD (work in progress with Samchuck–Schnarch)

In types BCD (i.e. orthosymplectic), $\mathcal{OB}(A)$ is replaced by a Frobenius Brauer category (no longer oriented).

Functor to supercategory of $\mathfrak{osp}_{m|2n}(A)$-modules.

Quantum analogues

Quantum Frobenius Heisenberg categories (Brundan–S.–Webster) are quantum analogues of Frobenius Heisenberg categories.

Central charge zero special case yields a natural quantum affine oriented Frobenius Brauer category. When $A = \Bbbk$, this is the affine HOMFLY-PT skein category.

Should have functor to modules for (yet-to-be-defined) quantized enveloping algebras of $\mathfrak{gl}_{m|n}(A)$ and yield Frobenius analogues of the HOMFLY-PT link invariant.

Summary

Main idea

Many of the tools that have been used to study the representation theory of $\mathfrak{gl}_n$, $\mathfrak{gl}_{m|n}$, and $\mathfrak{q}_n$ can be unified and generalized to the setting of $\mathfrak{gl}_{m|n}(A)$:

- oriented/walled Brauer algebras,
- (affine) oriented Brauer categories,
- Schur–Weyl duality (?).