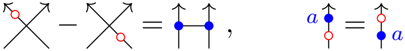
Affine oriented Frobenius Brauer categories



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Preprint: arXiv:2101.04582 (joint with Alexandra McSween)

Outline

Goal: Generalize and unify Brauer algebra/category techniques to the setting of general linear Lie superalgebras over Frobenius superalgebras.

Overview:

1. Frobenius superalgebras A
2. $\mathfrak{gl}_{m|n}(A)$
3. (Affine) oriented Frobenius Brauer category
4. Natural actions

Frobenius superalgebras

Fix a ground field $\Bbbk$.

A symmetric Frobenius superalgebra is an associative superalgebra with parity-preserving trace map

$$\mathrm{tr}_A \colon A \to \Bbbk, \qquad \mathrm{tr}_A(ab) = (-1)^{\bar{a}\bar{b}}\,\mathrm{tr}_A(ba), \quad a, b \in A,$$

such that $\ker(\mathrm{tr}_A)$ contains no nonzero left ideal.

Choose a homogeneous basis $\mathbf{B}_A$. Define the dual basis $\{b^\vee : b \in \mathbf{B}_A\}$ by

$$\mathrm{tr}_A(b^\vee c) = \delta_{b,c}, \qquad b, c \in \mathbf{B}_A.$$

Lie superalgebras $\mathfrak{gl}_{m|n}(A)$

Fix $m, n \in \mathbb{N}$. For $1 \le i \le m+n$, define

$$p(i) = \begin{cases} \bar{0} & \text{if } 1 \le i \le m, \\ \bar{1} & \text{if } m+1 \le i \le m+n, \end{cases} \in \mathbb{Z}/2\mathbb{Z}.$$

Let $\mathrm{Mat}_{m|n}(A)$ be the associative superalgebra of $(m+n) \times (m+n)$ matrices with entries in A. Parity is defined by

$$\overline{a_{(i,j)}} = \bar{a} + p(i) + p(j), \qquad \left(\begin{array}{c|c} \bar{a} & \bar{a}+\bar{1} \\ \hline \bar{a}+\bar{1} & \bar{a} \end{array}\right)$$

where $a_{(i,j)}$ is the matrix with $a \in A$ in the (i,j) position and 0 in all other positions.

Let $\mathfrak{g} = \mathfrak{gl}_{m|n}(A)$ be the Lie superalgebra associated to $\mathrm{Mat}_{m|n}(A)$, with bracket

$$[M, N] = MN - (-1)^{\bar{M}\bar{N}} NM.$$

Examples

Example (General linear Lie superalgebras)

$\mathfrak{gl}_{m|n}(\Bbbk)$ is the usual general linear Lie superalgebra over $\Bbbk$.

Example (Truncated current superalgebras)

$\mathfrak{gl}_{m|n}(\Bbbk[t]/(t^l))$ is a truncated current superalgebra. When $n = 0$, this is called a Takiff algebra.

Example (Queer Lie superalgebras)

Let Cl be the two-dimensional Clifford superalgebra generated by an odd element c with $c^2 = 1$. (Actually need odd trace map here.)

Then $\mathfrak{gl}_{m|n}(\mathrm{Cl}) \cong \mathfrak{q}_{m+n}(\Bbbk)$, the queer Lie superalgebra.

E.g. $\mathfrak{gl}_m(\mathrm{Cl}) \cong \mathfrak{q}_m(\Bbbk)$.

Natural modules

Let $A^{m|n}$ be the $\Bbbk$-superspace equal to A^{m+n} as a $\Bbbk$-module, with parity given by

$$\overline{ae_i} = \bar{a} + p(i), \qquad a \in A,\ 1 \le i \le m+n, \qquad (\underbrace{\bar{a}}_{m \text{ positions}}, \underbrace{\bar{a}+\bar{1}}_{n \text{ positions}})$$

where $e_i \in A^{m|n}$ has a 1 in the i-th entry and a 0 in all other entries.

Define

$$V_+ = A^{m|n}, \qquad \text{written as row matrices,}$$
$$V_- = A^{m|n}, \qquad \text{written as column matrices.}$$

Then V_+ is naturally a right $\mathfrak{g}$-supermodule, and V_- is a right $\mathfrak{g}$-supermodule with action

$$v \cdot M = -(-1)^{\bar{v}\bar{M}} M v, \qquad v \in V_-,\ M \in \mathfrak{g}.$$

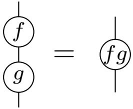
We have a natural nondegenerate $\Bbbk$-bilinear form

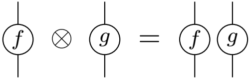
$$B \colon V_- \otimes V_+ \to \Bbbk, \quad B(v \otimes w) = (-1)^{\bar{v}\bar{w}} \operatorname{tr}_A(wv).$$

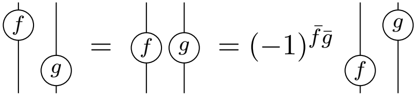
String diagrams for strict monoidal supercategories

We will denote a morphism $f\colon A \to B$ by: 

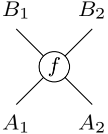
Composition is vertical stacking and tensor product is horizontal juxtaposition:





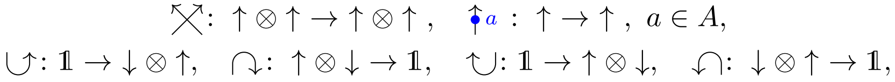
The super interchange law is: 

A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:

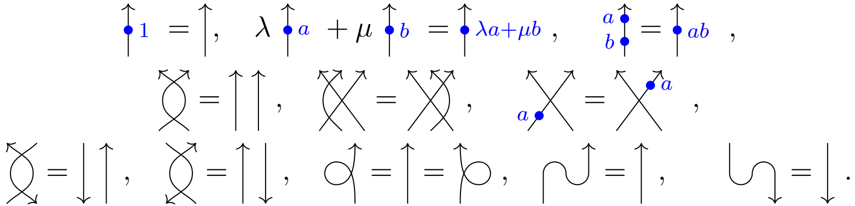


Oriented Frobenius Brauer category

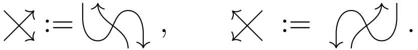
Let $\mathcal{OB}(A)$ be the strict monoidal supercategory generated by objects $\uparrow$ and $\downarrow$ and morphisms



subject to the relations ($a, b \in A$, $\lambda, \mu \in \Bbbk$)



In the above, the left and right crossings are defined by



The parity of $\uparrow_a$ is $\bar{a}$, and all the other generating morphisms are even.

Oriented Frobenius Brauer category

We have

$$\mathrm{End}_{\mathcal{OB}(A)}(\uparrow^{\otimes n}) \cong A^{\otimes n} \rtimes S_n.$$

Case $A = \Bbbk$

$\mathcal{OB}(\Bbbk)$ is the oriented Brauer category.

Endomorphism algebras are oriented Brauer algebras, which are isomorphic to walled Brauer algebras.

Case $A = \mathrm{Cl}$

$\mathcal{OB}(\mathrm{Cl})$ is the oriented Brauer–Clifford supercategory of Brundan–Comes–Kujawa.

Theorem (McSween–S.)

There is a monoidal superfunctor $\psi \colon \mathcal{OB}(A) \to$ smod-$\mathfrak{g}$ given on objects by

$$\uparrow \mapsto V_+, \qquad \downarrow \mapsto V_-$$

and on morphisms by

$\psi($$) \colon V_+ \otimes V_+ \to V_+ \otimes V_+, \qquad v \otimes w \mapsto (-1)^{\bar{v}\bar{w}} w \otimes v,$

$\psi($$) \colon V_+ \mapsto V_+, \qquad v \mapsto av,$

$\psi($$) \colon V_- \otimes V_+ \to \Bbbk, \qquad v \otimes w \mapsto B(v \otimes w)$

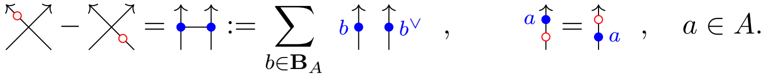
$\psi($$) \colon \Bbbk \mapsto V_+ \otimes V_-, \qquad 1 \mapsto \sum_{v \in \mathbf{B}_{V_+}} v \otimes v^\vee,$

$\psi($$) \colon V_+ \otimes V_- \to \Bbbk, \qquad v \otimes w \mapsto (-1)^{\bar{v}\bar{w}} B(w \otimes v),$

$\psi($$) \colon \Bbbk \mapsto V_- \otimes V_+, \qquad 1 \mapsto \sum_{v \in \mathbf{B}_{V_+}} (-1)^{\bar{v}} v^\vee \otimes v.$

Affine oriented Frobenius Brauer category

Let $\mathcal{AOB}(A)$ be the strict monoidal supercategory obtained from $\mathcal{OB}(A)$ by adjoining an even generator $\mathring{\uparrow}$: $\uparrow \to \uparrow$, subject to the relations



Remarks

1. $\mathrm{End}_{\mathcal{AOB}(A)}(\uparrow^{\otimes n})$ are affine wreath product algebras.
2. $\mathcal{AOB}(\Bbbk)$ is the affine oriented Brauer category.
3. $\mathcal{AOB}(\mathrm{Cl})$ is the affine oriented Brauer–Clifford category.
4. $\mathcal{AOB}(A) = \mathcal{H}eis_0(A)$ is the Frobenius Heisenberg category at central charge zero.

Theorem (McSween–S.)

There is a monoidal superfunctor $\Psi \colon \mathcal{AOB}(A) \to \mathcal{E}nd_{\Bbbk}(\text{smod-}\mathfrak{g})$ given on objects by

$$\uparrow \mapsto V_+ \otimes -, \qquad \downarrow \mapsto V_- \otimes -$$

and on morphisms by

$$\Psi(f) = \psi(f) \otimes -, \quad f \in \{ \nwarrow\!\!\!\!\!\nearrow, \overset{\bullet}{\uparrow}\, a, \circlearrowright, \circlearrowleft, \curvearrowright, \curvearrowleft : a \in A\},$$

and $\Psi(\overset{\circ}{\uparrow}) \colon V_+ \otimes - \to V_+ \otimes -$ is the functor with components

$$\Psi(\overset{\circ}{\uparrow})_W \colon V_+ \otimes W \to V_+ \otimes W, \quad v \otimes w \mapsto (v \otimes w) \sum_{M \in \mathbf{B}_{\mathfrak{g}}} M \otimes M^{\vee},$$

for $W \in \text{smod-}\mathfrak{g}$.

Summary

Main idea

Many of the tools that have been used to student the representation theory of $\mathfrak{gl}_n$, $\mathfrak{gl}_{m|n}$, and $\mathfrak{q}_n$ can be unified and generalized to the setting of $\mathfrak{gl}_{m|n}(A)$:

- Brauer algebras,
- walled Brauer algebras,
- (affine) oriented Brauer categories,
- Schur–Weyl duality (?).

Other directions

There also exist:

- quantum versions,
- type BCD versions.