Quantum affine wreath algebras

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \uparrow \downarrow \end{array}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \uparrow \downarrow \end{array}
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \uparrow \downarrow \end{array}
\end{array}
\end{array}
- \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \uparrow \downarrow \end{array}
\end{array}
\end{array}
= z
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \end{array}
\end{array}
\end{array}
\end{align*}
\]

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Goal: Unify and generalize existing algebras by defining families of Hecke-like algebras depending on Frobenius algebras.

Overview:
1. Strict monoidal categories and string diagrams
2. Warm up: symmetric groups, degenerate affine Hecke algebras
3. Frobenius algebras
4. Affine wreath product algebras
5. Quantum affine wreath algebras
A strict monoidal category is a category $C$ equipped with
- a bifunctor (the tensor product) $\otimes : C \times C \to C$, and
- a unit object $1$,

such that
- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all objects $A, B, C$,
- $1 \otimes A = A = A \otimes 1$ for all objects $A$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.
A $k$-linear monoidal category is a strict monoidal category such that

- each morphism space is a $k$-module,
- composition of morphisms is $k$-bilinear,
- tensor product of morphisms is $k$-bilinear.

The interchange law

The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

$$
\begin{array}{c}
A_1 \otimes B_1 & \xrightarrow{1 \otimes g} & A_1 \otimes B_2 \\
\downarrow f \otimes 1 & & \downarrow f \otimes 1 \\
A_2 \otimes B_1 & \xrightarrow{1 \otimes g} & A_2 \otimes B_2
\end{array}
$$
String diagrams

Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f : A \to B$ by:

$$\begin{array}{c}
B \\
\downarrow \quad f \\
A
\end{array}$$

The identity map $1_A : A \to A$ is a string with no label:

$$\begin{array}{c}
A \\
\downarrow \\
A
\end{array}$$

We sometimes omit the object labels when they are clear or unimportant.
String diagrams

Composition is **vertical stacking** and tensor product is **horizontal juxtaposition**:

\[
\begin{align*}
\bullet f & \quad = \quad \bullet fg \\
\bullet g & \quad = \quad \bullet g \\
\bullet f & \quad \otimes \quad \bullet g \quad = \quad \bullet f \quad \otimes \quad \bullet g
\end{align*}
\]

The **interchange law** then becomes:

\[
\begin{align*}
\bullet f & \quad = \quad \bullet g \\
\bullet g & \quad = \quad \bullet f \\
\bullet f & \quad = \quad \bullet g \\
\bullet g & \quad = \quad \bullet f
\end{align*}
\]

A morphism \( f : A_1 \otimes A_2 \to B_1 \otimes B_2 \) can be depicted:

\[
\begin{tikzpicture}
  \node (A1) at (0,0) [circle,draw] {\( A_1 \)};
  \node (A2) at (1,0) [circle,draw] {\( A_2 \)};
  \node (B1) at (0,1) [circle,draw] {\( B_1 \)};
  \node (B2) at (1,1) [circle,draw] {\( B_2 \)};
  \draw (A1) -- (B1);
  \draw (A2) -- (B2);
  \draw (A1) -- (B2);
  \draw (A2) -- (B1);
  \node at (0.5,0.5) [circle,draw] {\( f \)};
\end{tikzpicture}
\]
Presentations of strict monoidal categories

One can give presentations of some strict \( \mathbb{k} \)-linear monoidal categories, just as for monoids, groups, algebras, etc.

**Objects:** If the objects are generated by some collection \( A_i, i \in I \), then we have all possible tensor products of these objects:

\[
1, \ A_i, \ A_i \otimes A_j \otimes A_k \otimes A_\ell, \ etc.
\]

**Morphisms:** If the morphisms are generated by some collection \( f_j, j \in J \), then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

\[
1_{A_i}, \ f_j \otimes (f_i f_k) \otimes (f_\ell), \ etc.
\]

We then often impose some relations on these morphism spaces.

**String diagrams:** We can build complex diagrams out of our simple generating diagrams.
The symmetric group category

Define a strict $\mathbb{k}$-linear monoidal category $Sym$ with one generating object $\uparrow$ and denote $1_{\uparrow} = \uparrow$.

We have one generating morphism

$$\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{array} \quad : \quad \uparrow \otimes \uparrow \rightarrow \uparrow \otimes \uparrow .$$

We impose the relations:

$$
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{array} ,
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{array} .
\end{array}
$$

Then

$$\text{End}_{Sym}(\uparrow \otimes^n) = \mathbb{k}S_n$$

is the group algebra of the symmetric group on $n$ letters.
The symmetric group category

This monoidal presentation of $\mathbb{k}S_n$ is very efficient! We only needed

- one generating morphism, and
- two relations,

to get all the symmetric groups.

Note that the “distant braid relation”

$$s_is_j = s_js_i, \quad |i - j| > 1$$

for simple transpositions follows for free from the interchange law:

\[
\begin{array}{cccc}
\uparrow & \uparrow & \cdots & \uparrow \\
\downarrow & \downarrow & \cdots & \downarrow \\
\end{array}
\]

= \[
\begin{array}{cccc}
\uparrow & \uparrow & \cdots & \uparrow \\
\uparrow & \uparrow & \cdots & \uparrow \\
\end{array}
\]
Degenerate affine Hecke algebras

The degenerate affine Hecke algebra $H_n$ of type $A$ is

$$k[x_1, \ldots, x_n] \otimes kS_n$$

as a $k$-module.

The factors $k[x_1, \ldots, x_n]$ and $kS_n$ are subalgebras, and

$$s_ix_j = x_js_i, \quad j \neq i, i + 1,$$
$$s_ix_i = x_{i+1}s_i - 1.$$
Obtain $\mathcal{H}$ from $\text{Sym}$ by adjoining one additional morphism (a dot)

\[
\begin{array}{c}
\bullet \\
\uparrow
\end{array} : 
\begin{array}{c}
\uparrow \\
\rightarrow \\
\uparrow
\end{array}
\]

and one additional relation:

\[
\begin{array}{c}
\bullet \\
\rightarrow \quad \quad \quad \quad \quad \bullet \\
\rightarrow
\end{array} - 
\begin{array}{c}
\bullet \\
\rightarrow \quad \quad \quad \quad \quad \bullet \\
\rightarrow
\end{array} = 
\begin{array}{c}
\uparrow \\
\rightarrow \\
\uparrow
\end{array} .
\]

Then

\[
\text{End}_{\mathcal{H}}(\uparrow \otimes n) = H_n
\]
Frobenius algebras

Definition (symmetric Frobenius algebra)

An associative algebra $A$ together with a linear trace map

$$\text{tr}: A \to k, \quad \text{tr}(ab) = \text{tr}(ba),$$

such that $\ker \text{tr}$ contains no nonzero left ideals.

Example ($k$)

$k$ with $\text{tr} = \text{id}_k$.

Example ($k[x]/(x^k)$)

$k[x]/(x^k)$ with $\text{tr}(x^\ell) = \delta_{\ell,k-1}$.

Example (Matrix algebra)

Matrix algebras with the usual trace.
Frobenius algebras: Examples

Example (Group algebra)
If $G$ is a finite group, then the group algebra $kG$ is a Frobenius algebra with
$$\text{tr}(g) = \delta_{g,1_G}, \quad g \in G.$$  

Example (Hopf algebras)
Every f.d. Hopf algebra is a Frobenius algebra.

From now on: $A$ is a symmetric Frobenius algebra with trace $\text{tr}$.

Remark: Can actually work more generally, with graded Frobenius superalgebras (not necessarily symmetric).
Wreath product algebras

The symmetric group $S_n$ acts on $A^\otimes n$ by permutations:

$$\pi(a_n \otimes \cdots \otimes a_1) = a_{\pi^{-1}(n)} \otimes \cdots \otimes a_{\pi^{-1}(1)},$$

Wreath product algebra

The wreath product algebra is

$$\text{Wr}_n(A) = A^\otimes n \otimes k S_n$$

as $k$-modules. Multiplication is determined by

$$(a \otimes \pi)(b \otimes \sigma) = a_{\pi}(b) \otimes \pi \sigma.$$
Example ($A = \mathbb{k}$)
\[ \text{Wr}_n(\mathbb{k}) \cong \mathbb{k}S_n \]

Example ($A = \text{Cl}$)
\[ \text{Wr}_n(\text{Cl}) \] is the Sergeev algebra, which plays an important role in the projective representation theory of the symmetric group.

Example ($A = \mathbb{k}G$, $G = \mathbb{Z}/2\mathbb{Z}$)
\[ \text{Wr}_n(\mathbb{k}G) \] is the group algebra of the hyperoctahedral group, the Weyl group of type $B$.

Example ($A = \mathbb{k}G$, $G = \mathbb{Z}/r\mathbb{Z}$)
\[ \text{Wr}_n(\mathbb{k}G) \] is the group algebra of the complex reflection group $G(r, 1, n)$. 
The wreath product category

Define $\mathcal{W}r(A)$ by adjoining to $Sym$ morphisms (tokens)

$$\uparrow a : \uparrow \to \uparrow, \quad a \in A,$$

subject to the relations ($\alpha, \beta \in \mathbb{k}, \ a, b \in A$)

$$\uparrow 1 = \uparrow, \quad \uparrow \alpha a + \beta b = \alpha \uparrow a + \beta \uparrow b, \quad \uparrow a = \uparrow ab,$$

(so $A \mapsto \text{End}_{\mathcal{W}r(A)}(\uparrow)$, $a \mapsto \uparrow a$ is an algebra homomorphism) and

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\longrightarrow \end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\longrightarrow \end{array}
\end{array}
\end{array},
\quad a \in A.
$$

Then

$$\text{End}_{\mathcal{W}r(A)}(\uparrow \otimes^n) = \mathcal{W}r_n(A).$$
Teleporters

Fix a basis $B$ of $A$. The dual basis is

$$B^\vee = \{ b^\vee \mid b \in B \} \text{ defined by } \text{tr} (b^\vee c) = \delta_{b,c}, \quad b, c \in B.$$

Exercise 1: $\sum_{b \in B} b \otimes b^\vee \in A \otimes A$ is independent of the basis $B$.

Exercise 2: For all $a \in A$, we have

$$\sum_{b \in B} ab \otimes b^\vee = \sum_{b \in B} b \otimes b^\vee a, \quad \sum_{b \in B} ba \otimes b^\vee = \sum_{b \in B} b \otimes ab^\vee.$$

Define the teleporter

$$\uparrow \downarrow \uparrow \downarrow := \sum_{b \in B} b \uparrow \downarrow b^\vee \quad .$$

Then tokens “teleport” across teleporters:

$$\begin{align*}
\uparrow \downarrow \uparrow \downarrow a &= a \uparrow \downarrow \uparrow \downarrow , \\
a \uparrow \downarrow \uparrow \downarrow &= \uparrow \downarrow \uparrow \downarrow a .
\end{align*}$$
Define $\mathcal{W}_{r,\text{aff}}(A)$ by adjoining to $\mathcal{W}_r(A)$ the morphism (dot) \[
abla: \downarrow \rightarrow \uparrow \]

and relations
\[
\begin{align*}
\begin{array}{ccc}
\uparrow & - & \uparrow \downarrow \uparrow = \downarrow \uparrow \downarrow , \\
\mathcal{A} & = & \mathcal{A} \uparrow \\
\mathcal{A} & = & \mathcal{A} \uparrow \\
\mathcal{A} & = & \mathcal{A} \uparrow 
\end{array}
\end{align*}
\]

We define the **affine wreath product algebra** to be \[
\mathcal{W}_{r,\text{aff}}(A) := \text{End}_{\mathcal{W}_{r,\text{aff}}(A)}(\uparrow \otimes n).\]
Affine wreath product algebras

Example ($A = k$)

\[ \uparrow \uparrow \uparrow = \uparrow \uparrow \uparrow \] and $\text{Wr}_n^{\text{aff}}(k)$ is the degenerate affine Hecke algebra.

Example ($A = Cl$, Clifford algebra)

$\text{Wr}_n^{\text{aff}}(Cl)$ is the affine Sergeev algebra, aka the degenerate affine Hecke–Clifford algebra.

Example ($A = kG$)

$\text{Wr}_n^{\text{aff}}(kG)$ is the wreath Hecke algebra (Wan–Wang).

Example (Affine zigzag algebras)

When $A$ is a certain zigzag algebra, $\text{Wr}_n^{\text{aff}}(A)$ is related to imaginary strata for quiver Hecke algebras (Kleshchev–Muth).
Hecke algebras

Fix $z \in k$. Let $\mathcal{H}(z)$ be the strict $k$-linear monoidal category with one generating object $\uparrow$, generating morphisms

\[
\begin{array}{c}
\xymatrix{ 
\uparrow & \uparrow \\
\uparrow & \uparrow & \uparrow \\
\end{array}
, \quad
\begin{array}{c}
\xymatrix{ 
\uparrow & \uparrow & \\
\uparrow & \uparrow & \\
\uparrow & \uparrow & \\
\end{array}

\end{array}
\]

and relations

\[
\begin{array}{c}
\xymatrix{ 
\uparrow & \uparrow \ar[r] & \uparrow \\
\uparrow & \uparrow & \uparrow \\
\end{array}
, \quad
\begin{array}{c}
\xymatrix{ 
\uparrow & \uparrow \ar[r] & \uparrow \\
\uparrow & \uparrow & \uparrow \\
\end{array}
, \quad
\begin{array}{c}
\xymatrix{ 
\uparrow & \uparrow \ar[r] & \uparrow \\
\uparrow & \uparrow & \uparrow \\
\end{array}
, \\
\begin{array}{c}
\xymatrix{ 
\uparrow & \uparrow \ar[r] & \uparrow \\
\uparrow & \uparrow & \uparrow \\
\end{array}
- \begin{array}{c}
\xymatrix{ 
\uparrow & \uparrow \ar[r] & \uparrow \\
\uparrow & \uparrow & \uparrow \\
\end{array}
= z \uparrow \uparrow \\
\end{array}
\]

(skein relation).

Then

\[
H_n(z) := \text{End}_{\mathcal{H}(z)}(\uparrow \otimes n)
\]

is the Iwahori–Hecke algebra of type $A_{n-1}$ (often $z = q - q^{-1}$).
Affine Hecke algebras

Define $\mathcal{H}^{\text{aff}}(z)$ by adjoining to $\mathcal{H}(z)$ the invertible morphism

$$\begin{array}{c}
\uparrow : \uparrow \rightarrow \uparrow \\
\end{array}$$

and relations

$$\begin{array}{cc}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
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\end{array}
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\begin{array}{c}
\begin{array}{c}
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\begin{array}{c}
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= \begin{array}{c}
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, \\
\begin{array}{c}
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\begin{array}{c}
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\begin{array}{c}
\bullet
\end{array}
\end{array}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\end{array}
\end{array}.
\end{array}$$

Then

$$H_n^{\text{aff}}(z) := \text{End}_{\mathcal{H}^{\text{aff}}}(\uparrow \otimes^n)$$

is the affine Hecke algebra of type $A_{n-1}$ (often $z = q - q^{-1}$).
Define $\mathcal{H}(A, z)$ by adjoining to $\mathcal{H}(z)$ morphisms

$$\begin{array}{c}
\begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture} \quad a : \uparrow \rightarrow \uparrow, \quad a \in A,
\end{array}$$

subject to the relations ($\alpha, \beta \in \mathbb{k}, a, b \in A$)

$$\begin{array}{c}
\begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture} \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture} = \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture},
\begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture} \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture} = \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture},
\alpha a + \beta b = \alpha \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture} + \beta \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture},
\begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture} \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture} = \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture},
\begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture} = \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture},
\begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture} = \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[->] (0,0) -- (0.5,0);
\end{tikzpicture}.
\end{array}$$

We call

$$H_n(A, z) := \text{End}_{\mathcal{H}(A, z)}(\uparrow \otimes n)$$

a Frobenius Hecke algebra.
Frobenius Hecke algebras

Example \((A = \mathbb{k})\)

\(H_n(\mathbb{k}, z)\) is an Iwahori–Hecke algebra.

Example \((A = \mathbb{k}G, G\) a cyclic group\)

\(H_n(\mathbb{k}G, z)\) is a Yokonuma–Hecke algebra.

Other choices of \(A\) yield new algebras.
Define $\mathcal{W}_r^{\text{aff}}(A, z)$ by adjoining to $\mathcal{H}(A, z)$ the invertible morphism

$$
\uparrow: \uparrow \rightarrow \uparrow
$$

and relations

$$
\begin{align*}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},
scale=0.5]
\draw[thick,->] (0,0) -- (1,0);
\draw[thick,->] (0,1) -- (1,1);
\end{tikzpicture}
\end{align*}
= \begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},
scale=0.5]
\draw[thick,->] (0,0) -- (1,0);
\draw[thick,->] (0,1) -- (1,1);
\end{tikzpicture},
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},
scale=0.5]
\draw[thick,->] (0,0) -- (1,0);
\draw[thick,->] (0,1) -- (1,1);
\draw[thick,->] (0,0) -- (0,1);
\draw[thick,->] (0,1) -- (1,0);
\end{tikzpicture}
= \begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},
scale=0.5]
\draw[thick,->] (0,0) -- (1,0);
\draw[thick,->] (0,1) -- (1,1);
\draw[thick,->] (0,0) -- (0,1);
\draw[thick,->] (0,1) -- (1,0);
\end{tikzpicture},
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},
scale=0.5]
\draw[thick,->] (0,0) -- (1,0);
\draw[thick,->] (0,1) -- (1,1);
\draw[thick,->] (0,0) -- (0,1);
\draw[thick,->] (0,1) -- (1,0);
\draw[thick,->] (0,0) -- (0,1);\end{tikzpicture}
\cdot a
= \begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},
scale=0.5]
\draw[thick,->] (0,0) -- (1,0);
\draw[thick,->] (0,1) -- (1,1);
\draw[thick,->] (0,0) -- (0,1);
\draw[thick,->] (0,1) -- (1,0);
\draw[thick,->] (0,0) -- (0,1);\end{tikzpicture}.
\end{align*}
$$

We call $W_r^{\text{aff}}(n, A, z) := \text{End}_{\mathcal{W}_r^{\text{aff}}(A, z)}(\uparrow \otimes n)$ a quantum affine wreath algebra.

One could also call it a affine Frobenius Hecke algebra.
Example ($A = k$)

$W_{r_{\text{aff}}} (k, z)$ is an affine Hecke algebra.

Example ($A = kG$, $G$ a cyclic group)

$W_{r_{\text{aff}}} (kG, z)$ is an affine Yokonuma–Hecke algebra.

Other choices of $A$ yield new algebras.

Example ($A = \text{zigzag algebra}$)

$W_{r_{\text{aff}}} (A, z)$ is a quantum analogue of affine zigzag algebras.
Jucys–Murphy elements

The Jucys–Murphy elements in $\mathbb{C}S_n$ are

$$J_1 = 0, \quad J_i = (1i) + (2i) + \cdots + (i-1i), \quad i = 2, \ldots, n.$$  

Useful facts

- $J_n$ commutes with elements of $\mathbb{C}S_{n-1}$.
- the $J_i$ generate a commutative subalgebra of $\mathbb{C}S_n$.
- The basis elements of Young’s seminormal representation are eigenvectors for the $J_i$.

Theorem (Jucys)

The center of $\mathbb{C}S_n$ is generated by symmetric polynomials in the $J_i$.

Jucys–Murphy elements play a central role in the Okounkov–Vershik approach to the representation theory of symmetric groups.
Recall the degenerate affine Hecke algebra:

\[ H_n = \mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}S_n \]

with relations

\[ s_i x_i = x_{i+1} s_i - 1, \quad s_i x_j = x_j s_i, \quad j \neq i, i + 1. \]

Clearly we have an injection

\[ \mathbb{C}S_n \hookrightarrow H_n. \]

We also have a surjection

\[ H_n \twoheadrightarrow \mathbb{C}S_n, \quad x_i \mapsto J_i. \]

This is an example of a cyclotomic quotient. Map is uniquely determined by \( x_1 \mapsto 0 \). In general, we can quotient by any polynomial in \( x_1 \).
For $1 \leq i < j \leq n$, define

$$t_{i,j} = \begin{array}{c}
\uparrow \quad \uparrow \quad \uparrow \\
\quad j \quad \quad \quad \quad \quad i
\end{array}$$

In the wreath product algebra, we define the Jucys–Murphy elements

$$J_1 = 0, \quad J_i = t_{1,i}(1i) + t_{2,i}(2i) + \cdots + t_{i-1,i}(i-1i), \quad 1 \leq i \leq n.$$ 

We have a surjection

$$\text{Wr}_n^{\text{aff}}(A) \twoheadrightarrow \text{Wr}_n(A), \quad x_i \mapsto J_i,$$

where $x_i$ is a dot on the $i$-th strand.

**Quantum version:** Can also define Jucys–Murphy elements in $\text{Wr}_n(A, z)$ generalizing usual Jucys–Murphy elements for the Hecke algebra.
Other structure theory results

One can prove many general structure theory results in a uniform way:

- **Demazure operators** (aka divided difference operators)
- **Basis theorem:**

\[
W_{r_n}^{\text{aff}}(A) \cong A^{\otimes n} \otimes \mathbb{k}[x_1, \ldots, x_n] \otimes \mathbb{k}S_n
\]

\[
W_{r_n}^{\text{aff}}(A, z) \cong A^{\otimes n} \otimes \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \otimes H_n(z)
\]

as \(\mathbb{k}\)-modules.

- **Center:**

\[
Z(W_{r_n}^{\text{aff}}(A)) = (Z(A)^{\otimes n} \otimes \mathbb{k}[x_1, \ldots, x_n])^{S_n}
\]

\[
Z(W_{r_n}^{\text{aff}}(A, z)) = (Z(A)^{\otimes n} \otimes \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])^{S_n}
\]

- **Mackey theorem**
- **Cyclotomic quotients** (basis theorem, Mackey theorem, etc.)
Fix $k \in \mathbb{Z}$. The Heisenberg category (Khovanov, Mackaay–S., Brundan) is defined by adjoining to the degenerate affine Hecke category $\mathcal{H}$ an object $\downarrow$ and morphisms and relations so that

- $\uparrow$ is right dual to $\downarrow$,
- we have an isomorphism

\[ \uparrow \otimes \downarrow \cong \downarrow \otimes \uparrow \oplus 1 \oplus k \quad (\text{when } k \geq 0), \]
\[ \uparrow \otimes \downarrow \oplus 1 \oplus (-k) \cong \downarrow \otimes \uparrow \quad (\text{when } k \leq 0) \]

(the inversion relation).

Acts on modules for degenerate cyclotomic Hecke algebras, categorifies the Heisenberg algebra.
We can now repeat this with our (quantum) affine wreath categories!

We get:

- quantum Heisenberg category (Licata–S., Brundan–S.–Webster)
- Frobenius Heisenberg category (Rosso-S., S.)
- quantum Frobenius Heisenberg category (Brundan–S.–Webster, work in progress)

These act on modules for the corresponding cyclotomic quotients.

Can also define an odd quantum Frobenius Heisenberg category...