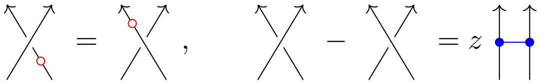
Quantum affine wreath algebras



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Preprint: arXiv:1902.00143 (joint with Daniele Rosso)

Outline

Goal: Unify and generalize existing algebras by defining families of Hecke-like algebras depending on Frobenius algebras.

Overview:

1. Strict monoidal categories and string diagrams
2. Warm up: symmetric groups, degenerate affine Hecke algebras
3. Frobenius algebras
4. Affine wreath product algebras
5. Quantum affine wreath algebras

Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.

$\Bbbk$-linear monoidal categories

Fix a commutative ground ring $\Bbbk$.

A strict $\Bbbk$-linear monoidal category is a strict monoidal category such that

- each morphism space is a $\Bbbk$-module,
- composition of morphisms is $\Bbbk$-bilinear,
- tensor product of morphisms is $\Bbbk$-bilinear.

The interchange law

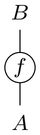
The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

$$\begin{tikzcd}
A_1 \otimes B_1 \arrow[r, "1\otimes g"] \arrow[d, "f\otimes 1"'] \arrow[dr, "f\otimes g"] & A_1 \otimes B_2 \arrow[d, "f\otimes 1"] \\
A_2 \otimes B_1 \arrow[r, "1\otimes g"'] & A_2 \otimes B_2
\end{tikzcd}$$

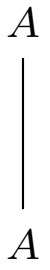
String diagrams

Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:



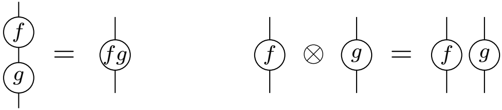
The identity map $1_A\colon A \to A$ is a string with no label:



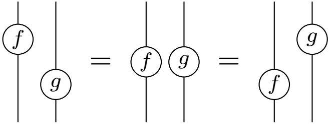
We sometimes omit the object labels when they are clear or unimportant.

String diagrams

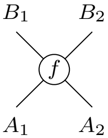
Composition is vertical stacking and tensor product is horizontal juxtaposition:



The interchange law then becomes:



A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



Presentations of strict monoidal categories

One can give presentations of some strict $\Bbbk$-linear monoidal categories, just as for monoids, groups, algebras, etc.

Objects: If the objects are generated by some collection A_i, $i \in I$, then we have all possible tensor products of these objects:

$$\mathbb{1}, \quad A_i, \quad A_i \otimes A_j \otimes A_k \otimes A_\ell, \quad \text{etc.}$$

Morphisms: If the morphisms are generated by some collection f_j, $j \in J$, then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

$$1_{A_i}, \quad f_j \otimes (f_i f_k) \otimes (f_\ell), \quad \text{etc.}$$

We then often impose some relations on these morphism spaces.

String diagrams: We can build complex diagrams out of our simple generating diagrams.

The symmetric group category

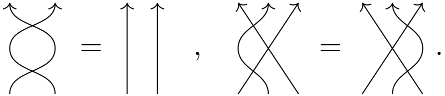
Define a strict $\Bbbk$-linear monoidal category $\mathcal{S}ym$ with one generating object $\uparrow$ and denote

$$1_\uparrow = \uparrow$$

We have one generating morphism

$$: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow .$$

We impose the relations:



Then

$$\mathrm{End}_{\mathcal{S}ym}(\uparrow^{\otimes n}) = \Bbbk S_n$$

is the group algebra of the symmetric group on n letters.

The symmetric group category

This monoidal presentation of $\Bbbk S_n$ is very efficient! We only needed

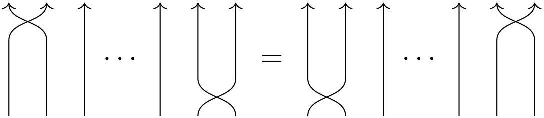
- one generating morphism, and
- two relations,

to get all the symmetric groups.

Note that the "distant braid relation"

$$s_i s_j = s_j s_i, \qquad |i-j| > 1$$

for simple transpositions follows for free from the interchange law:



Degenerate affine Hecke algebras

The degenerate affine Hecke algebra H_n of type A is

$$\Bbbk[x_1, \ldots, x_n] \otimes \Bbbk S_n$$

as a $\Bbbk$-module.

The factors $\Bbbk[x_1, \ldots, x_n]$ and $\Bbbk S_n$ are subalgebras, and

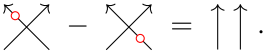
$$\begin{aligned} s_i x_j &= x_j s_i, && j \neq i, i+1, \\ s_i x_i &= x_{i+1} s_i - 1. \end{aligned}$$

String diagrams

Obtain $\mathcal{H}$ from $\mathcal{S}ym$ by adjoining one additional morphism (a dot)

$$: \uparrow \to \uparrow$$

and one additional relation:



Then

$$\mathrm{End}_{\mathcal{H}}(\uparrow^{\otimes n}) = H_n$$

Frobenius algebras

Definition (symmetric Frobenius algebra)

An associative algebra A together with a linear trace map

$$\operatorname{tr}\colon A \to \Bbbk, \qquad \operatorname{tr}(ab) = \operatorname{tr}(ba),$$

such that $\ker \operatorname{tr}$ contains no nonzero left ideals.

Example ($\Bbbk$)

$\Bbbk$ with $\operatorname{tr} = \mathrm{id}_{\Bbbk}$.

Example ($\Bbbk[x]/(x^k)$)

$\Bbbk[x]/(x^k)$ with $\operatorname{tr}(x^\ell) = \delta_{\ell,k-1}$.

Example (Matrix algebra)

Matrix algebras with the usual trace.

Frobenius algebras: Examples

Example (Group algebra)

If G is a finite group, then the group algebra $\Bbbk G$ is a Frobenius algebra with

$$\mathrm{tr}(g) = \delta_{g,1_G}, \quad g \in G.$$

Example (Hopf algebras)

Every f.d. Hopf algebra is a Frobenius algebra.

From now on: A is a symmetric Frobenius algebra with trace tr.

Remark: Can actually work more generally, with graded Frobenius superalgebras (not necessarily symmetric).

Wreath product algebras

The symmetric group S_n acts on $A^{\otimes n}$ by permutations:

$$\pi(a_n \otimes \cdots \otimes a_1) = a_{\pi^{-1}(n)} \otimes \cdots \otimes a_{\pi^{-1}(1)},$$

Wreath product algebra

The wreath product algebra is

$$\mathrm{Wr}_n(A) = A^{\otimes n} \otimes \Bbbk S_n$$

as $\Bbbk$-modules. Multiplication is determined by

$$(\mathbf{a} \otimes \pi)(\mathbf{b} \otimes \sigma) = \mathbf{a}\pi(\mathbf{b}) \otimes \pi\sigma.$$

Wreath product algebras: Examples

Example ($A = \Bbbk$)

$\mathrm{Wr}_n(\Bbbk) \cong \Bbbk S_n$

Example ($A = \mathrm{Cl}$)

$\mathrm{Wr}_n(\mathrm{Cl})$ is the Sergeev algebra, which plays an important role in the projective representation theory of the symmetric group.

Example ($A = \Bbbk G$, $G = \mathbb{Z}/2\mathbb{Z}$)

$\mathrm{Wr}_n(\Bbbk G)$ is the group algebra of the hyperoctahedral group, the Weyl group of type B.

Example ($A = \Bbbk G$, $G = \mathbb{Z}/r\mathbb{Z}$)

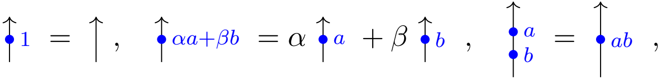
$\mathrm{Wr}_n(\Bbbk G)$ is the group algebra of the complex reflection group $G(r, 1, n)$.

The wreath product category

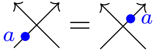
Define $\mathcal{W}r(A)$ by adjoining to $\mathcal{S}ym$ morphisms (tokens)

$$: \uparrow \to \uparrow, \quad a \in A,$$

subject to the relations ($\alpha, \beta \in \Bbbk,\ a, b \in A$)



(so $A \mapsto \mathrm{End}_{\mathcal{W}r(A)}(\uparrow)$, $a \mapsto$  is an algebra homomorphism) and



$$, \quad a \in A.$$

Then

$$\mathrm{End}_{\mathcal{W}r(A)}(\uparrow^{\otimes n}) = \mathrm{Wr}_n(A).$$

Teleporters

Fix a basis B of A. The dual basis is

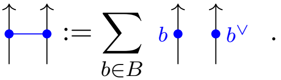
$$B^\vee = \{b^\vee \mid b \in B\} \quad \text{defined by} \quad \operatorname{tr}\left(b^\vee c\right) = \delta_{b,c}, \qquad b, c \in B.$$

Exercise 1: $\sum_{b\in B} b \otimes b^\vee \in A \otimes A$ is independent of the basis B.

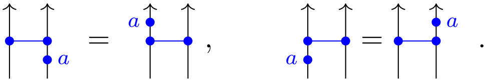
Exercise 2: For all $a \in A$, we have

$$\sum_{b\in B} ab \otimes b^\vee = \sum_{b\in B} b \otimes b^\vee a, \qquad \sum_{b\in B} ba \otimes b^\vee = \sum_{b\in B} b \otimes ab^\vee.$$

Define the teleporter



Then tokens "teleport" across teleporters:

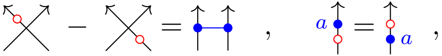


Affine wreath product algebras

Define $\mathcal{W}r^{\mathsf{aff}}(A)$ by adjoining to $\mathcal{W}r(A)$ the morphism (dot)

 $: \uparrow \to \uparrow$

and relations



$, \quad a \in A.$

We define the affine wreath product algebra to be

$$\mathrm{Wr}_n^{\mathsf{aff}}(A) := \mathrm{End}_{\mathcal{W}r^{\mathsf{aff}}(A)}(\uparrow^{\otimes n}).$$

Affine wreath product algebras

Example ($A = \Bbbk$)

 $= \uparrow \ \uparrow$ and $\mathrm{Wr}_n^{\mathsf{aff}}(\Bbbk)$ is the degenerate affine Hecke algebra.

Example ($A = \mathrm{Cl}$, Clifford algebra)

$\mathrm{Wr}_n^{\mathsf{aff}}(\mathrm{Cl})$ is the affine Sergeev algebra, aka the degenerate affine Hecke–Clifford algebra.

Example ($A = \Bbbk G$)

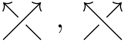
$\mathrm{Wr}_n^{\mathsf{aff}}(\Bbbk G)$ is the wreath Hecke algebra (Wan–Wang).

Example (Affine zigzag algebras)

When A is a certain zigzag algebra, $\mathrm{Wr}_n^{\mathsf{aff}}(A)$ is related to imaginary strata for quiver Hecke algebras (Kleshchev–Muth).

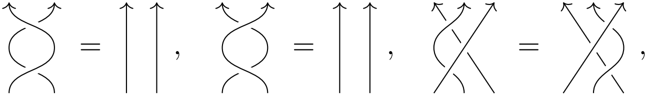
Hecke algebras

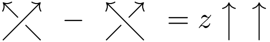
Fix $z \in \Bbbk$. Let $\mathcal{H}(z)$ be the strict $\Bbbk$-linear monoidal category with one generating object $\uparrow$, generating morphisms



$$: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow$$

and relations





(skein relation).

Then

$$H_n(z) := \mathrm{End}_{\mathcal{H}(z)}(\uparrow^{\otimes n})$$

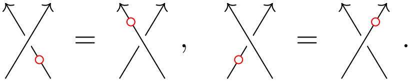
is the Iwahori–Hecke algebra of type A_{n-1} (often $z = q - q^{-1}$).

Affine Hecke algebras

Define $\mathcal{H}^{\mathsf{aff}}(z)$ by adjoining to $\mathcal{H}(z)$ the invertible morphism

 $\colon \uparrow \to \uparrow$

and relations



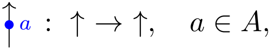
Then

$$H_n^{\mathsf{aff}}(z) := \operatorname{End}_{\mathcal{H}^{\mathsf{aff}}}(\uparrow^{\otimes n})$$

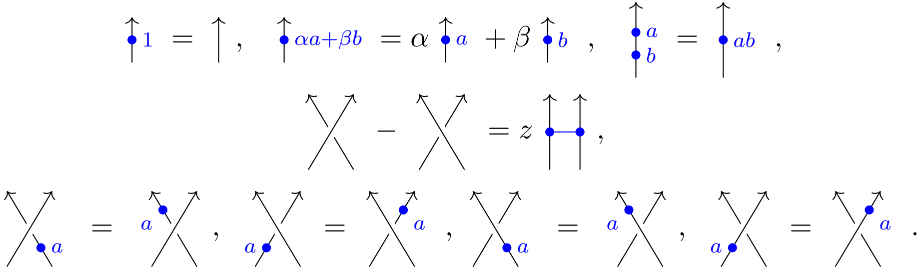
is the affine Hecke algebra of type A_{n-1} (often $z = q - q^{-1}$).

Frobenius Hecke algebras

Define $\mathcal{H}(A, z)$ by adjoining to $\mathcal{H}(z)$ morphisms



subject to the relations ($\alpha, \beta \in \Bbbk$, $a, b \in A$)



We call

$$H_n(A, z) := \mathrm{End}_{\mathcal{H}(A,z)}(\uparrow^{\otimes n})$$

a Frobenius Hecke algebra.

Frobenius Hecke algebras

Example ($A = \mathbb{k}$)

$H_n(\mathbb{k}, z)$ is an Iwahori–Hecke algebra.

Example ($A = \mathbb{k}G$, G a cyclic group)

$H_n(\mathbb{k}G, z)$ is a Yokonuma–Hecke algebra.

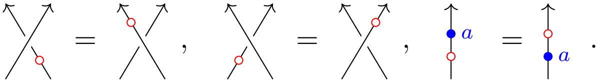
Other choices of A yield new algebras.

Quantum affine wreath algebras

Define $\mathcal{W}r^{\mathsf{aff}}(A, z)$ by adjoining to $\mathcal{H}(A, z)$ the invertible morphism

$: \uparrow \to \uparrow$

and relations



We call

$$\mathrm{Wr}_n^{\mathsf{aff}}(A, z) := \mathrm{End}_{\mathcal{W}r^{\mathsf{aff}}(A,z)}(\uparrow^{\otimes n})$$

a quantum affine wreath algebra.

One could also call it a affine Frobenius Hecke algebra.

Quantum affine wreath algebras

Example ($A = \Bbbk$)

$\mathrm{Wr}_n^{\mathsf{aff}}(\Bbbk, z)$ is an affine Hecke algebra.

Example ($A = \Bbbk G$, G a cyclic group)

$\mathrm{Wr}_n^{\mathsf{aff}}(\Bbbk G, z)$ is an affine Yokonuma–Hecke algebra.

Other choices of A yield new algebras.

Example ($A =$ zigzag algebra)

$\mathrm{Wr}_n^{\mathsf{aff}}(A, z)$ is a quantum analogue of affine zigzag algebras.

Jucys–Murphy elements

The Jucys–Murphy elements in $\mathbb{C}S_n$ are

$$J_1 = 0, \quad J_i = (1\,i) + (2\,i) + \cdots + (i-1\,i), \quad i = 2, \ldots, n.$$

Useful facts

- J_n commutes with elements of $\mathbb{C}S_{n-1}$.
- the J_i generate a commutative subalgebra of $\mathbb{C}S_n$.
- The basis elements of Young's seminormal representation are eigenvectors for the J_i.

Theorem (Jucys)

The center of $\mathbb{C}S_n$ is generated by symmetric polynomials in the J_i.

Jucys–Murphy elements play a central role in the Okounkov–Vershik approach to the representation theory of symmetric groups.

Jucys–Murphy elements

Recall the degenerate affine Hecke algebra:

$$H_n = \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}S_n$$

with relations

$$s_i x_i = x_{i+1} s_i - 1, \quad s_i x_j = x_j s_i, \quad j \neq i, i+1.$$

Clearly we have an injection

$$\mathbb{C}S_n \hookrightarrow H_n.$$

We also have a surjection

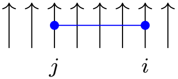
$$H_n \twoheadrightarrow \mathbb{C}S_n, \quad x_i \mapsto J_i.$$

This is an example of a cyclotomic quotient. Map is uniquely determined by $x_1 \mapsto 0$. In general, we can quotient by any polynomial in x_1.

Jucys–Murphy elements

For $1 \leq i < j \leq n$, define

$$t_{i,j} =$$



In the wreath product algebra, we define the Jucys–Murphy elements

$$J_1 = 0, \quad J_i = t_{1,i}(1\,i) + t_{2,i}(2\,i) + \cdots + t_{i-1,i}(i-1\,i), \quad 1 \leq i \leq n.$$

We have a surjection

$$\mathrm{Wr}_n^{\mathsf{aff}}(A) \twoheadrightarrow \mathrm{Wr}_n(A), \quad x_i \mapsto J_i,$$

where x_i is a dot on the i-th strand.

Quantum version: Can also define Jucys–Murphy elements in $\mathrm{Wr}_n(A, z)$ generalizing usual Jucys–Murphy elements for the Hecke algebra.

Other structure theory results

One can prove many general structure theory results in a uniform way:

- Demazure operators (aka divided difference operators)
- Basis theorem:

$$\mathrm{Wr}_n^{\mathsf{aff}}(A) \cong A^{\otimes n} \otimes \Bbbk[x_1, \dots, x_n] \otimes \Bbbk S_n$$
$$\mathrm{Wr}_n^{\mathsf{aff}}(A, z) \cong A^{\otimes n} \otimes \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \otimes H_n(z)$$

  as $\Bbbk$-modules.
- Center:

$$Z(\mathrm{Wr}_n^{\mathsf{aff}}(A)) = \left(Z(A)^{\otimes n} \otimes \Bbbk[x_1, \dots, x_n]\right)^{S_n}$$
$$Z(\mathrm{Wr}_n^{\mathsf{aff}}(A, z)) = \left(Z(A)^{\otimes n} \otimes \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]\right)^{S_n}$$

- Mackey theorem
- Cyclotomic quotients (basis theorem, Mackey theorem, etc.)

Heisenberg categorification

Fix $k \in \mathbb{Z}$. The Heisenberg category (Khovanov, Mackaay–S., Brundan) is defined by adjoining to the degenerate affine Hecke category $\mathcal{H}$ an object

$$\downarrow$$

and morphisms and relations so that

- $\uparrow$ is right dual to $\downarrow$,
- we have an isomorphism

$$\begin{aligned} \uparrow \otimes \downarrow &\cong \downarrow \otimes \uparrow \oplus \mathbb{1}^{\oplus k} \quad \text{(when } k \geq 0\text{)}, \\ \uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus(-k)} &\cong \downarrow \otimes \uparrow \quad \text{(when } k \leq 0\text{)} \end{aligned}$$

  (the inversion relation).

Acts on modules for degenerate cyclotomic Hecke algebras, categorifies the Heisenberg algebra.

Heisenberg categorification

We can now repeat this with our (quantum) affine wreath categories!

We get:

- quantum Heisenberg category (Licata–S., Brundan–S.–Webster)
- Frobenius Heisenberg category (Rosso-S., S.)
- quantum Frobenius Heisenberg category (Brundan–S.–Webster, work in progress)

These act on modules for the corresponding cyclotomic quotients.

Can also define an odd quantum Frobenius Heisenberg category...