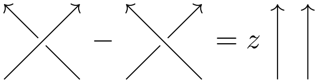
Heisenberg and Kac–Moody categorification



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Slides: alistairsavage.ca/talks

Preprint: 1907.11988 (with J. Brundan and B. Webster)

Outline

Goal: Explain how categorical Heisenberg actions give rise to categorical Kac–Moody actions.

Overview:

1. Background: strict monoidal categories and string diagrams
2. Definition of the quantum Heisenberg category
3. Categorical actions
4. Key Theorem
5. Examples

Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.

$\Bbbk$-linear monoidal categories

Fix a commutative ground ring $\Bbbk$.

A strict $\Bbbk$-linear monoidal category is a strict monoidal category such that

- each morphism space is a $\Bbbk$-module,
- composition of morphisms is $\Bbbk$-bilinear,
- tensor product of morphisms is $\Bbbk$-bilinear.

The interchange law

The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

$$\begin{array}{ccc} A_1 \otimes B_1 & \xrightarrow{1 \otimes g} & A_1 \otimes B_2 \\ {\scriptstyle f\otimes 1}\downarrow & \searrow^{f \otimes g} & \downarrow{\scriptstyle f\otimes 1} \\ A_2 \otimes B_1 & \xrightarrow[1 \otimes g]{} & A_2 \otimes B_2 \end{array}$$

Strict monoidal categories

Example (Monoids)

A (strict) monoidal category with one object is simply a commutative monoid. More precisely, the endomorphisms of $\mathbb{1}$ form a commutative monoid.

Conversely, every commutative monoid gives rise to a one-object monoidal category.

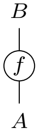
Example (Associative algebras)

A (strict) $\mathbb{k}$-linear monoidal category with one object is simply a commutative associative unital $\mathbb{k}$-algebra.

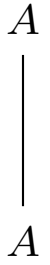
String diagrams

Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:



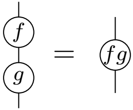
The identity map $1_A\colon A \to A$ is a string with no label:

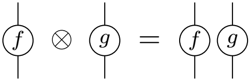


We sometimes omit the object labels when they are clear or unimportant.

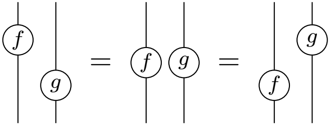
String diagrams

Composition is vertical stacking and tensor product is horizontal juxtaposition:

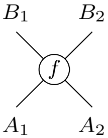




The interchange law then becomes:



A morphism $f \colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



Example: monoidally generated symmetric groups

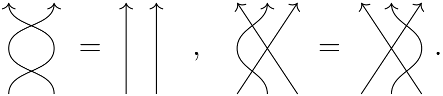
Define a strict monoidal category $\mathcal{S}$ with one generating object $\uparrow$ and denote

$$1_\uparrow = \uparrow$$

We have one generating morphism

 $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow .$

We impose the relations:



Then

$$\mathrm{End}_{\mathcal{S}}(\uparrow^{\otimes n}) = S_n$$

is the symmetric group on n letters.

Example: monoidally generated symmetric groups

This monoidal presentation of S_n is very efficient! We only needed

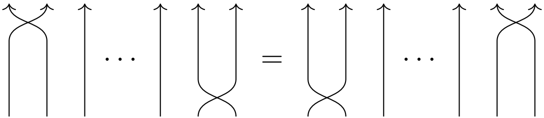
- one generating morphism, and
- two relations,

to get all the symmetric groups.

Note that the "distant braid relation"

$$s_i s_j = s_j s_i, \qquad |i-j| > 1$$

for simple transpositions follows for free from the interchange law:



Note: If we define $\mathcal{S}$ to be $\Bbbk$-linear, then $\mathrm{End}_{\mathcal{S}}(\uparrow^{\otimes n}) = \Bbbk S_n$.

Overview

We are interested in:

- the quantum Heisenberg category (Licata–S., Brundan–S.–Webster)
- the Kac–Moody 2-category (Khovanov–Lauda, Rouquier)

These categorify (i.e. their Grothendieck rings are isomorphic to)

- the Heisenberg algebra
- Kac–Moody algebras

We will:

- define the quantum Heisenberg category
- explain how categorical actions of the quantum Heisenberg category yield categorical actions of the Kac–Moody 2-category

Note: Analogous results exist for the (non-quantum) Heisenberg category.

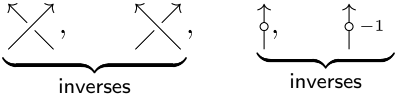
The affine Hecke category

Fix an algebraically closed ground field $\Bbbk$ and $1 \neq q^2 \in \Bbbk^\times$.
Set $z = q - q^{-1}$.

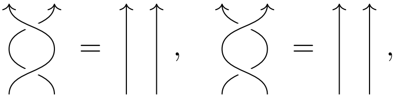
Consider the strict $\Bbbk$-linear monoidal category $\mathcal{AH}$ with:

Generating object: $\uparrow$

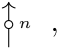
Generating morphisms:



So



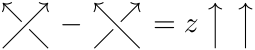
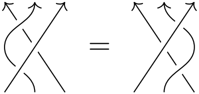
and we have



$n \in \mathbb{Z}$.

The affine Hecke category

Relations:

- HOMFLY skein relation: 
- Braid relation: 
- Dot-slide relation: 

We have

$$\mathrm{Hom}_{\mathcal{AH}}(\uparrow^{\otimes n}, \uparrow^{\otimes m}) = 0 \quad \text{if } n \neq m,$$

and

$$\mathrm{End}_{\mathcal{AH}}(\uparrow^{\otimes n}) = AH_n$$

is the affine Hecke algebra associated to GL_n.

The quantum Heisenberg category $\mathcal{H}eis_k$

Data

- central charge: $k \in \mathbb{Z}$
- parameter $t \in \Bbbk^\times$

For simplicity, assume $k < 0$.

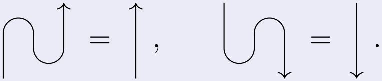
- Case $k > 0$ is similar: there is a symmetry $\mathcal{H}eis_{-k} \cong \mathcal{H}eis_k^{\mathrm{op}}$ given by reflection in the horizontal.
- Case $k = 0$ is easier: it is the affine HOMFLY-PT skein category.

Step 1: Add a right dual

Adjoin a right dual object $\downarrow$ to $\mathcal{AH}$, with morphisms

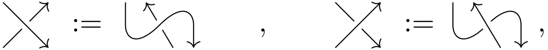
 $: \mathbb{1} \to \downarrow \otimes \uparrow$ and  $: \uparrow \otimes \downarrow \to \mathbb{1}$

subject to the relations

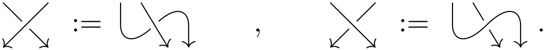


The quantum Heisenberg category

Define the positive and negative right crossings by

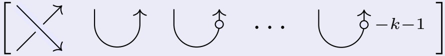


and positive and negative downwards crossings by



Step 2: Inversion relation

Impose the relation that



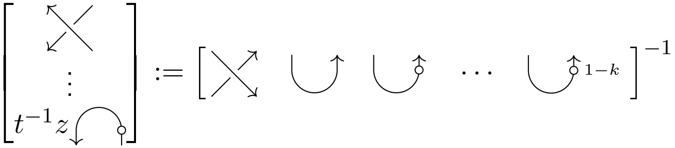
$$: \uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus(-k)} \to \downarrow \otimes \uparrow$$

is invertible.

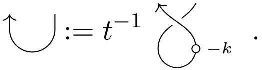
Motivation: Canonical commutation relation in Heisenberg algebra:

$$pq = qp + k$$

Let

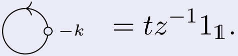


and define



Step 3

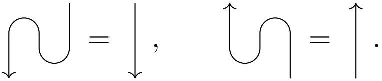
Impose the relation



Denote by $\mathcal{H}eis_k$ the resulting quantum Heisenberg category.

The quantum Heisenberg category

One can show that



So ↓ is both right and left dual to ↑.

In fact $\mathcal{H}eis_k$ is strictly pivotal: morphisms are invariant under isotopy.

Connection to previous definitions

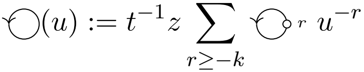
When $k = -1$, $\mathcal{H}eis_k$ is closely related to the q-deformed Heisenberg category (Licata–S.).

Essentially the q-deformed Heisenberg category is obtained from $\mathcal{H}eis_{-1}$ by throwing out the generator  .

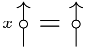
Generating functions

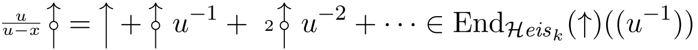
It is very helpful to work with generating functions!!

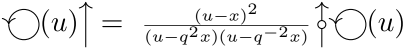
Let u be an indeterminate and define



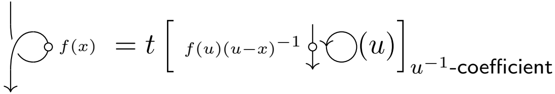
$$= u^k 1_{\mathbb{1}} + \text{lower terms} \in \mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1})((u^{-1})).$$

Defining , we can label dots by generating functions. E.g.



Bubble slide relation: 

Curl relation: For $f(x) \in \Bbbk[x]$,



Categorical actions

Let $\mathcal{M}$ be a locally finite abelian $\Bbbk$-linear category:

- all objects have finite length,
- all morphism spaces are finite dimensional.

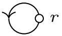
A categorical action on $\mathcal{M}$ is a monoidal functor

$$\mathcal{H}eis_k \to \mathcal{E}nd(\mathcal{M}),$$

where $\mathcal{E}nd(\mathcal{M})$ is the monoidal category of $\Bbbk$-linear endofunctors and natural transformations.

$\uparrow \mapsto$ "induction functor" E,

$\downarrow \mapsto$ "restriction functor" F,



$\mapsto$ central element in $Z(\mathcal{M}) := \mathrm{End}_{\mathbb{1}}(\mathcal{M})$.

Categorical actions

For a simple object $L \in \mathcal{M}$, let

$$m_L(x) = \text{min. poly. of } \overset{\bullet}{\uparrow} \colon EL \to EL,$$
$$n_L(x) = \text{min. poly. of } \overset{\bullet}{\downarrow} \colon FL \to FL.$$

Theorem

$\circlearrowright(u)$ acts on L as $\frac{n_L(u)}{m_L(u)} = u^k + \text{lower terms} \in \Bbbk((u^{-1}))$.

Spectrum I := set of all roots of $m_L(x)$ or $n_L(x)$ for L a simple object.

So

$$m_L(x) = \prod_{i \in I} (x - i)^{\varepsilon_i(L)}, \quad n_L(x) = \prod_{i \in I} (x - i)^{\varphi_i(L)}$$

for some $\varepsilon_i(L), \varphi_i(L)$.

Categorical actions

Let

$$E_i = \text{generalized eigenspace of } \overset{\circ}{\uparrow} \colon E \Rightarrow E,$$
$$F_i = \text{generalized eigenspace of } \overset{\circ}{\downarrow} \colon F \Rightarrow F.$$

Then

$$(x-i)^{\varepsilon_i(L)} \text{ is the min. poly. of } \overset{\circ}{\uparrow} \colon E_i L \to E_i L,$$
$$(x-i)^{\varphi_i(L)} \text{ is the min. poly. of } \overset{\circ}{\downarrow} \colon F_i L \to F_i L.$$

Note: We interpret natural transformations of (compositions of) E_i, F_i using

$$E_i \hookrightarrow E \twoheadrightarrow E_i \quad \text{and} \quad F_i \hookrightarrow F \twoheadrightarrow F_i.$$

Block decompostion

Let $\mathfrak{g}$ be the Kac–Moody algebra associated to the graph with

- set of vertices I,
- edges $i \to q^2 i$, $i \in I$.

For

$$\lambda \in X = \text{weight lattice of } \mathfrak{g},$$

let $\mathcal{M}_\lambda$ be the Serre subcategory of $\mathcal{M}$ generated by simple objects L such that

$$\lambda = \sum_{i \in I} (\varphi_i(L) - \varepsilon_i(L)) \Lambda_i.$$

So $\langle h_i, \lambda \rangle$ is the multiplicity of i as a zero/pole of the rational function

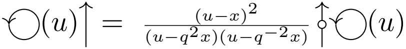
$$\frac{n_L(u)}{m_L(u)} = \bigcirc(u)\big|_L \in \Bbbk((u^{-1})).$$

We have a block decomposition $\mathcal{M} = \bigoplus_{\lambda \in X} \mathcal{M}_\lambda$.

Block decomposition

Recall:  (u) acts on simple L as $\frac{n_L(u)}{m_L(u)} = \prod_{i \in I} (u-i)^{\varepsilon_i(L) - \varphi_i(L)}$.

Recall the bubble slide relation:



It follows that

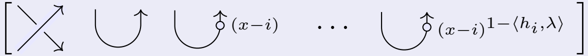
$$E_i \colon \mathcal{M}_\lambda \to \mathcal{M}_{\lambda+\alpha_i}.$$

Similarly,

$$F_i \colon \mathcal{M}_\lambda \to \mathcal{M}_{\lambda-\alpha_i}.$$

Key Theorem (Brundan–S.–Webster 2019)

For $\lambda \in X$ and $i \in I$ with $\langle h_i, \lambda \rangle \leq 0$, the natural transformation



$$: E_i F_i \oplus \mathrm{Id}^{\oplus(-\langle h_i,\lambda\rangle)} \Rightarrow F_i E_i$$

is invertible, and similarly when $\langle h_i, \lambda \rangle \geq 0$.

Corollary

There is an induced categorical action of the Kac–Moody 2-category $\mathcal{U}(\mathfrak{g})$ of Khovanov–Lauda–Rouquier on $\mathcal{M}$.

Application: Constructing categorical Kac–Moody actions

Above theorem gives a replacement for Rouquier's "control by K_0" result, the method which always had been used previously.

Categorical Heisenberg actions are easy to construct!!

Example: Ariki–Koike algebras

Fix

$$m(x) = \prod (x-i)^{m_i} \in \Bbbk[x] \text{ of degree } -k.$$

Let $t^2 = m(0)$.

Cyclotomic quotient (aka Ariki–Koike algebra)

$$H_d^m = AH_d/(m(x_1)),$$

where x_1 is a dot on the rightmost string.

Then $\mathcal{H}eis_k$ acts on $\bigoplus_{d\geq 0} H_d^m$-mod.

If L is the trivial H_0^m-module, then $m_L(x) = m(x)$, $n_L(x) = 1$.

So (u) acts on L as $\frac{1}{m(u)} \in \Bbbk[\![u^{-1}]\!]$.

L has weight $\lambda = -\sum m_i \Lambda_i$.

Ariki's Theorem: $\bigoplus_{d\geq 0} H_d^m$-mod categorifies the l.w. $\mathfrak{g}$-module $V(\lambda)$.

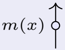
Example: Generalized cyclotomic quotients

Fix

$$m(x), n(x) \in \Bbbk[x] \text{ monic}$$
$$k = \deg n(x) - \deg m(x) \in \mathbb{Z}, \quad t^2 = \frac{m(0)}{n(0)}.$$

Generalized cyclotomic quotient (GCQ)

Let $\mathcal{H}(m|n)$ be the quotient of $\mathcal{H}eis_k$ by the left tensor ideal generated by



and coefficients of  $(u) - \frac{n(u)}{m(u)} 1_{\mathbb{1}}$.

- GCQs are universal cyclic module categories.
- Using Key Theorem, one can identify $\mathcal{H}(m|n)$ with an analogously defined GCQ of the Kac–Moody 2-category $\mathcal{U}(\mathfrak{g})$ introduced by Webster.
- GCQs categorify lowest$\otimes$highest weight integrable reps of $\mathfrak{g}$.

More examples

Reps of symmetric groups, cyclotomic Hecke algebras of type A

$\uparrow$= induction, $\downarrow$= restriction

Reps of general linear groups/Lie algebras of different sorts

$$\uparrow = V \otimes -, \qquad \downarrow = V^* \otimes -$$

(V = natural GL_n- or $\mathfrak{gl}_n$-module. Here $k = 0$.)

Reps of $U_q(\mathfrak{gl}_n)$, including situations where q is a root of unity

$$\uparrow = V \otimes -, \qquad \downarrow = V^* \otimes -$$

(V = natural $U_q(\mathfrak{gl}_n)$-module. Here $k = 0$.)

More examples

Category $\mathcal{O}$ for rational Cherednik algebras of type $G(\ell, 1, n)$

$\uparrow$ and $\downarrow$ act as Bezrukavnikov–Etingof induction and restriction functors

Special case: modules over cyclotomic q-Schur algebras

Generalized cyclotomic quotients of $\mathcal{H}eis_k$

Examples include Deligne categories $\underline{\mathrm{Rep}}\, GL_t$ and $\underline{\mathrm{Rep}}\, U_q(\mathfrak{gl}_t)$.