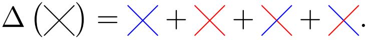
Categorical comultiplication



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Slides available online: alistairsavage.ca/talks

Based on joint work with Jon Brundan and Ben Webster.

Outline

Goal: Discuss concept of categorical comultipication, with examples.

Overview:

1. Motivation: Why categorical comultiplcation?
2. Example 1: The symmetric group category
3. Example 2: The affine Hecke algebra category
4. Example 3: The quantum Heisenberg category

Motivation

Suppose R is an associative $\mathbb{C}$-algebra and M, N are R-modules.

We have the outer tensor product $R \otimes R$-module $M \boxtimes N$:

$$(r \otimes s)(m \otimes n) = rm \otimes sn.$$

If we want an internal tensor product R-module $M \otimes N$, we need a comultiplication

$$\delta \colon R \to R \otimes R,$$

so that we can define the R-action

$$r \cdot (m \otimes n) = \delta(r)(m \otimes n).$$

Motivating observation:

- If $\mathfrak{g}$ is a f.d. simple Lie algebra, then $x \in U(\mathfrak{g})$ annihilates every f.d. simple module iff $x = 0$.
- For arbitrary Kac–Moody algebras this is false. But it's true if we consider (highest weight) $\otimes$ (lowest weight) modules.

Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

A strict $\mathbb{C}$-linear monoidal category is a strict monoidal category such that

- each morphism space is a $\mathbb{C}$-module,
- composition of morphisms is $\mathbb{C}$-bilinear,
- tensor product of morphisms is $\mathbb{C}$-bilinear.

Motivation

Suppose $\mathcal{C}$ is a strict $\mathbb{C}$-linear monoidal category.

A module category over $\mathcal{C}$ is a $\mathbb{C}$-linear category $\mathcal{V}$, together with a $\mathbb{C}$-linear monoidal functor

$$\mathcal{C} \to \mathcal{E}nd(\mathcal{V}),$$

where $\mathcal{E}nd(\mathcal{V})$ is the strict $\mathbb{C}$-linear monoidal category of $\mathbb{C}$-linear endofunctors and natural transformations.

If $\mathcal{V}, \mathcal{W}$ are $\mathcal{C}$-module categories, then $\mathcal{V} \boxtimes \mathcal{W}$ is a module category over the symmetric product category $\mathcal{C} \odot \mathcal{C}$.

To form the tensor product of $\mathcal{C}$-module categories, we need a functor

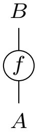
$$\Delta \colon \mathcal{C} \to \mathcal{C} \odot \mathcal{C}.$$

Goal

See some interesting examples of such categorical comultiplications.

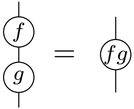
String diagrams for strict monoidal categories

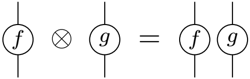
We use string diagrams to represent morphisms:

$f\colon A \to B \rightsquigarrow$ 

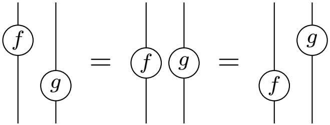
, $1_A\colon A \to A \rightsquigarrow$ 

Composition is vertical stacking and tensor product is horizontal juxtaposition:





We have the interchange law:



The symmetric group category

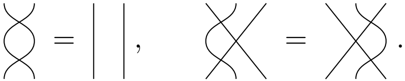
Let $\mathcal{Sym}$ be the free $\mathbb{C}$-linear symmetric monoidal category on one object. So $\mathcal{Sym}$ has one generating object I and we denote

$$1_{\mathsf{I}} = \Big|$$

We have one generating morphism

$$: \mathsf{I} \otimes \mathsf{I} \to \mathsf{I} \otimes \mathsf{I}.$$

We impose the relations:



We have that

$$\mathrm{End}_{\mathcal{Sym}}(\mathsf{I}^{\otimes n}) = \Bbbk S_n$$

is the group algebra of the symmetric group on n letters.

Categorification of symmetric functions

We have

$$K_0(\mathrm{Kar}(\mathcal{Sym})) \cong \bigoplus_{n=0}^{\infty} K_0(\Bbbk S_n) \cong \mathrm{Sym}, \qquad [S_\lambda] \mapsto s_\lambda,$$

where, for a partition λ,

$$S_\lambda = \text{Specht module}, \qquad s_\lambda = \text{Schur function}.$$

Multiplication on $\mathcal{Sym}$-modules is "induction product"

$$X \circ Y = \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} X \boxtimes Y.$$

This is a categorification of the ring of symmetric functions.

Question: Can we categorify the well-known coproduct on symmetric functions?

Symmetric product of monoidal categories

The symmetric product of strict $\mathbb{C}$-linear monoidal categories $\mathcal{C} \odot \mathcal{D}$ is obtained by:

1. taking the free product of $\mathcal{C}$ and $\mathcal{D}$, i.e. the strict $\mathbb{C}$-linear monoidal category defined by the disjoint union of the given generators and relations of $\mathcal{C}$ and $\mathcal{D}$;
2. adjoining isomorphisms $\sigma_{X,Y} \colon X \otimes Y \xrightarrow{\cong} Y \otimes X$ for each $X \in \mathcal{C}$, $Y \in \mathcal{D}$, subject to the relations

$$\begin{aligned}
\sigma_{X_1 \otimes X_2, Y} &= (\sigma_{X_1,Y} \otimes 1_{X_2}) \circ (1_{X_1} \otimes \sigma_{X_2,Y}),\\
\sigma_{X, Y_1 \otimes Y_2} &= (1_{Y_1} \otimes \sigma_{X,Y_2}) \circ (\sigma_{X,Y_1} \otimes 1_{Y_2}),\\
\sigma_{X_2,Y} \circ (f \otimes 1_Y) &= (1_Y \otimes f) \circ \sigma_{X_1,Y},\\
\sigma_{X,Y_2} \circ (1_X \otimes g) &= (g \otimes 1_X) \circ \sigma_{X,Y_1},
\end{aligned}$$

for all $X, X_1, X_2 \in \mathcal{C}$, $Y, Y_1, Y_2 \in \mathcal{D}$ and $f \colon X_1 \to X_2$, $g \colon Y_1 \to Y_2$.

We have

$$K_0(\mathcal{C} \odot \mathcal{D}) \cong K_0(\mathcal{C}) \otimes K_0(\mathcal{D}).$$

$\mathcal{S}ym \odot \mathcal{S}ym$

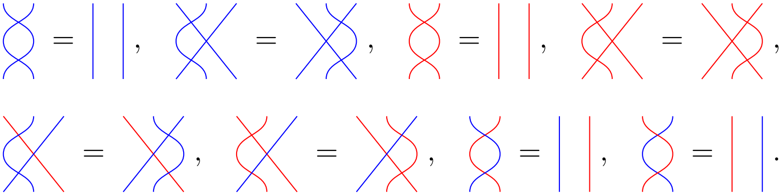
The symmetric product $\mathcal{S}ym \odot \mathcal{S}ym$ has two generating objects,

 and ,

and generating morphisms

, , $\sigma_{\mathsf{I},\mathsf{I}} =$ , $\sigma_{\mathsf{I},\mathsf{I}}^{-1} =$ ,

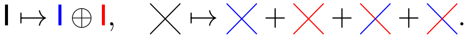
subject to the relations



Categorical comultiplication on $\mathcal{S}ym$

Consider the strict $\mathbb{C}$-linear monoidal functor

$$\Delta \colon \mathcal{S}ym \to \mathrm{Add}(\mathcal{S}ym \odot \mathcal{S}ym),$$



This induces a strict $\mathbb{C}$-linear monoidal functor

$$\Delta \colon \ \mathrm{Kar}(\mathcal{S}ym) \to \mathrm{Kar}(\mathcal{S}ym \odot \mathcal{S}ym).$$

Passing to split Grothendieck groups yields

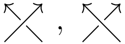
$$\begin{array}{ccc} K_0(\mathcal{S}ym) & \xrightarrow{[\Delta]} & K_0(\mathcal{S}ym \odot \mathcal{S}ym) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Sym} & \xrightarrow{\quad \delta \quad} & \mathrm{Sym} \otimes \mathrm{Sym} \end{array}$$

where δ is the usual comultiplication on Sym:

$$\delta(h_n) = \sum_{r=0}^{n} h_{n-r} \otimes h_r, \quad \delta(e_n) = \sum_{r=0}^{n} e_{n-r} \otimes e_r, \quad \delta(p_n) = p_n \otimes 1 + 1 \otimes p_n.$$

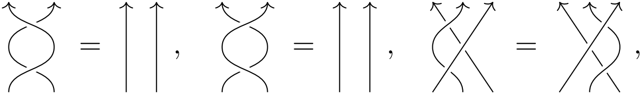
Hecke algebras

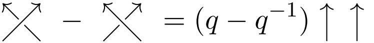
Fix $q \in \mathbb{C}^\times$. Let $\mathcal{H}(q)$ be the strict $\mathbb{C}$-linear monoidal category with one generating object $\uparrow$, generating morphisms



$$: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow$$

and relations





(Conway skein relation).

Then

$$H_n(q) := \mathrm{End}_{\mathcal{H}(q)}(\uparrow^{\otimes n})$$

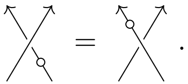
is the Iwahori–Hecke algebra of type A_{n-1}.

Affine Hecke algebras

Define $\mathcal{H}^{\mathsf{aff}}(q)$ by adjoining to $\mathcal{H}(q)$ the invertible morphism

$$\uparrow\!\!\!\circ \colon \uparrow \to \uparrow$$

and relation



Then

$$H_n^{\mathsf{aff}}(q) := \mathrm{End}_{\mathcal{H}^{\mathsf{aff}}(q)}(\uparrow^{\otimes n})$$

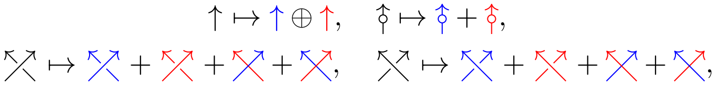
is the affine Hecke algebra of type A_{n-1}.

Comultiplication on $\mathcal{H}^{\mathsf{aff}}(q)$

We'd like to define a categorical comultiplication:

$$\Delta\colon \mathcal{H}^{\mathsf{aff}}(q) \to \mathrm{Add}(\mathcal{H}^{\mathsf{aff}}(q) \odot \mathcal{H}^{\mathsf{aff}}(q)).$$

However, naive guesses such as

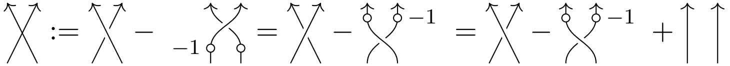


do not work! They don't respect the relations.

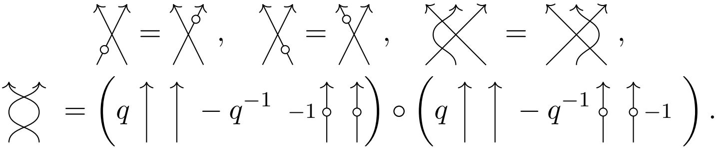
It turns out we'll have to localize to define the categorical comultiplication.

The neutral crossing

Consider the neutral crossing

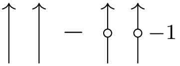


(Related to intertwining operators for affine Hecke algebras.) Direct computation shows that



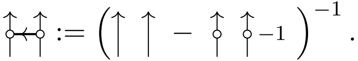
We'd like to work with a new presentation of $\mathcal{H}^{\mathrm{aff}}(q)$ involving .

To do this, we need to be able to express  and  in terms of . This means we need to invert



A new presentation

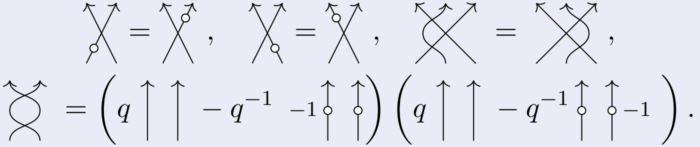
Define $\overline{\mathcal{H}^{\mathsf{aff}}}(q)$ by adjoining to $\mathcal{H}^{\mathsf{aff}}(q)$ a morphism



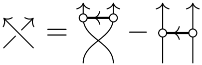
(✠)

Proposition

The strict $\mathbb{C}$-linear monoidal category $\overline{\mathcal{H}^{\mathsf{aff}}}(q)$ is generated by one object $\uparrow$ and morphisms (crossing), (dot), (dotted pair), subject to the relation (✠) and

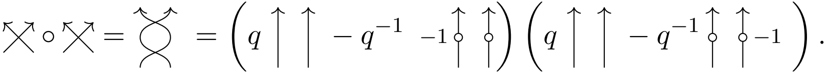


Sketch of proof:



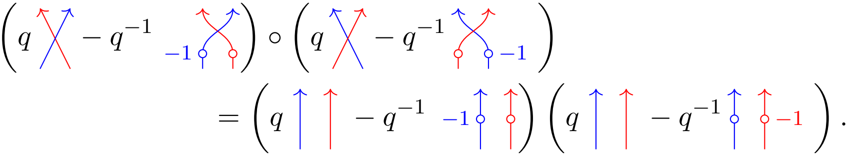
Comultiplication on $\mathcal{H}^{\mathsf{aff}}(q)$

Recall:



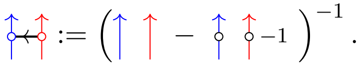
However the two-colored crossings behave differently:

We can “fix” this:



Comultiplication on $\mathcal{H}^{\mathrm{aff}}(q)$

Let $\mathcal{H}^{\mathrm{aff}}(q)\overline{\odot}\mathcal{H}^{\mathrm{aff}}(q)$ denote the symmetric product $\mathcal{H}^{\mathrm{aff}}(q) \odot \mathcal{H}^{\mathrm{aff}}(q)$ where we have adjoined an inverse

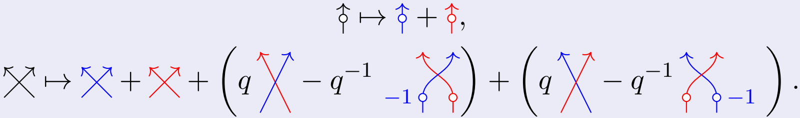


Proposition

There is a strict $\mathbb{C}$-linear monoidal functor

$$\Delta \colon \overline{\mathcal{H}^{\mathrm{aff}}}(q) \to \mathrm{Add}\left(\mathcal{H}^{\mathrm{aff}}(q)\overline{\odot}\mathcal{H}^{\mathrm{aff}}(q)\right)$$

given on objects by $\uparrow \mapsto \uparrow \oplus \uparrow$, and on morphisms by



Quantum Heisenberg category

Fix $k \in \mathbb{Z}$. The quantum Heisenberg category (Licata–S., Brundan–S.–Webster) is defined by adjoining to $\mathcal{H}^{\text{aff}}(q)$ an object

$$\downarrow$$

and morphisms and relations so that

- $\downarrow$ is right dual to $\uparrow$, meaning we have morphisms  and  such that

- we have an isomorphism

$$\uparrow \otimes \downarrow \;\cong \downarrow \otimes \uparrow \oplus \mathbb{1}^{\oplus k} \quad (\text{when } k \geq 0),$$
$$\uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus(-k)} \;\cong \downarrow \otimes \uparrow \quad (\text{when } k \leq 0)$$

  (the inversion relation).

Acts on modules for cyclotomic Hecke algebras.

Comultiplication on the quantum Heisenberg category

We can extend the categorical comultiplication on $\mathcal{H}^{\mathsf{aff}}(q)$ to one on the quantum Heisenberg category, with

$$\downarrow \mapsto \downarrow \oplus \downarrow,$$

This allows us to take tensor products of module categories.

In particular, we can form generalized cyclotomic quotients, which are large representations on which the quantum Heisenberg category acts faithfully.

We use these large representations to prove a basis theorem for the quantum Heisenberg category.

Technique can be used for other categories: degenerate Heisenberg category, categorified quantized enveloping algebras,...