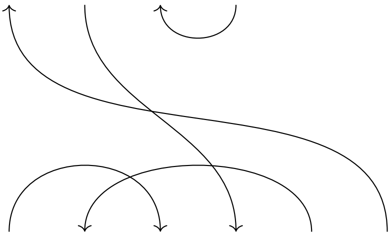
Universal categories



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Outline

Goal: Give overview of universal categories and their importance.

Overview:

1. Universal constructions in algebra
2. Warm-up
3. Monoidal categories and string diagrams
4. More warm-up
5. (Oriented) Temperley–Lieb category
6. (Oriented) Brauer category
7. Current work (very brief)

Free monoid

Let S be a set.

The free monoid M_S on S has

- elements given by words in S,
- multiplication given by concatenation:

$$(ab^2c) \cdot (c^4ba) = ab^2c^5ba$$

Universal property: Given any function $f \colon S \to M$, with M a monoid, there exists a unique monoid homomorphism $\varphi \colon M_S \to M$ such that

$$\begin{array}{ccc} S & \xrightarrow{s \mapsto s} & M_S \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle \varphi} \\ & & M \end{array}$$

commutes.

Free group

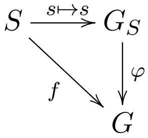
Let S be a set.

The free group G_S on S has

- elements given by words in S ($+$ inverses),
- multiplication given by concatenation with reduction:

$$(ab^2c) \cdot (c^{-1}ba) = ab^2cc^{-1}ba = ab^3a$$

Universal property: Given any function $f \colon S \to G$, with G a group, there exists a unique group homomorphism $\varphi \colon G_S \to G$ such that



commutes.

Free algebras

Let $\Bbbk$ be a commutative ring.

Free commutative algebra

The polynomial ring $\Bbbk[X_1, \dots, X_n]$ is the free commutative $\Bbbk$-algebra on n generators.

Universal property: Given any commutative $\Bbbk$-algebra R and elements $r_1, \dots, r_n \in R$, there exists a unique ring homomorphism

$$\Bbbk[X_1, \dots, X_n] \to R, \quad X_i \mapsto r_i.$$

Free associative algebra

One can also define free associative (not necessarily commutative) algebras.

Warm-up: Some simple universal categories

Free category $\mathcal{F}(X)$ on one object X

- Objects: X
- Morphisms: $\mathrm{Hom}(X, X) = \{1_X\}$

Universal property: If $\mathcal{C}$ is any category and Y is an object of $\mathcal{C}$, then there exists a unique functor

$$\mathcal{F}(X) \to \mathcal{C}, \quad X \mapsto Y.$$

Warm-up: Some simple universal categories

Free category $\mathcal{F}(X, f)$ on one object X and one generating morphism $f\colon X \to X$

- Objects: X
- Morphisms: $\mathrm{Hom}(X, X)$ is the free monoid on the generator f.

Universal property: If $\mathcal{C}$ is any category, Y is an object of $\mathcal{C}$, and $g\colon Y \to Y$ is a morphism in $\mathcal{C}$, then there exists a unique functor

$$\mathcal{F}(X, F) \to \mathcal{C}, \quad X \mapsto Y,\ f \mapsto g.$$

Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbf{1}$,

such that

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all objects A, B, C,
- $\mathbf{1} \otimes A = A = A \otimes \mathbf{1}$ for all objects A,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ for all morphisms f, g, h,
- $1_{\mathbf{1}} \otimes f = f = f \otimes 1_{\mathbf{1}}$ for all morphisms f.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.

$\Bbbk$-linear monoidal categories

Fix a commutative ground ring $\Bbbk$.

A $\Bbbk$-linear category is a category such that

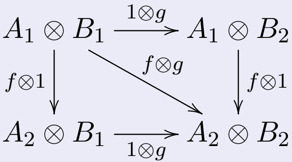
- each morphism space is a $\Bbbk$-module,
- composition of morphisms is $\Bbbk$-bilinear.

In a strict $\Bbbk$-linear monoidal category, we also require that

- tensor product of morphisms is $\Bbbk$-bilinear.

The interchange law

The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:



Warm-up: Some simple universal categories

Free $\mathbb{k}$-linear category $\mathcal{L}(X, f)$ on one object X and one generating morphism $f\colon X \to X$

- Objects: X
- Morphisms: $\mathrm{Hom}(X, X) = \mathbb{k}[f]$.

Universal property: If $\mathcal{C}$ is any $\mathbb{k}$-linear category, Y is an object of $\mathcal{C}$, and $g\colon Y \to Y$ is a morphism in $\mathcal{C}$, then there exists a unique $\mathbb{k}$-linear functor

$$\mathcal{F}(X, F) \to \mathcal{C}, \quad X \mapsto Y,\ f \mapsto g.$$

Warm-up: Some simple universal categories

Free monoidal category $\mathcal{M}(X)$ on one generating object X

- Objects: $X^{\otimes n}$ for $n \geq 0$ (where $X^{\otimes 0} = \mathbf{1}$).
- Morphisms:

$$\mathrm{Hom}(X^{\otimes n}, X^{\otimes m}) = \begin{cases} 1_{X^{\otimes n}} & \text{if } n = m, \\ \varnothing & \text{if } n \neq m. \end{cases}$$

Universal property: If $\mathcal{C}$ is any monoidal category and Y is an object of $\mathcal{C}$, then there exists a unique monoidal functor

$$\mathcal{M}(X) \to \mathcal{C}, \quad X \mapsto Y.$$

Warm-up: Some simple universal categories

Free $\Bbbk$-linear monoidal category $\mathcal{M}(X, f)$ on one generating object X and one generating morphism $f \colon X \to X$

- Objects: $X^{\otimes n}$ for $n \geq 0$.
- Morphisms:

$$\operatorname{Hom}(X^{\otimes n}, X^{\otimes m}) = \begin{cases} \Bbbk[f_1, \dots, f_n] & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases}$$

where $f_i = 1_X^{\otimes(i-1)} \otimes f \otimes 1_X^{\otimes(n-i)}$. (The f_i commute because of the interchange law!)

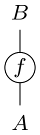
Universal property: If $\mathcal{C}$ is any $\Bbbk$-linear monoidal category, Y is an object of $\mathcal{C}$, and $g \colon Y \to Y$ is a morphism in $\mathcal{C}$, then there exists a unique $\Bbbk$-linear monoidal functor

$$\mathcal{M}(X, f) \to \mathcal{C}, \quad X \mapsto Y, \quad f \mapsto g.$$

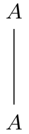
String diagrams

Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:



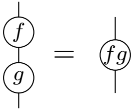
The identity map $1_A\colon A \to A$ is a string with no label:

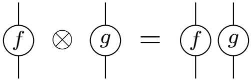


We sometimes omit the object labels when they are clear or unimportant.

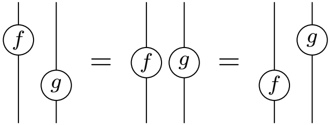
String diagrams

Composition is vertical stacking and tensor product is horizontal juxtaposition:

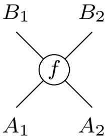




The interchange law then becomes:



A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



Presentations of strict monoidal categories

One can give presentations of some strict $\Bbbk$-linear monoidal categories, just as for monoids, groups, algebras, etc.

Objects: If the objects are generated by some collection A_i, $i \in I$, then we have all possible tensor products of these objects:

$$\mathbf{1}, \quad A_i, \quad A_i \otimes A_j \otimes A_k \otimes A_\ell, \quad \text{etc.}$$

Morphisms: If the morphisms are generated by some collection f_j, $j \in J$, then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

$$1_{A_i}, \quad f_j \otimes (f_i f_k) \otimes (f_\ell), \quad \text{etc.}$$

We then often impose some relations on these morphism spaces.

String diagrams: We can build complex diagrams out of our simple generating diagrams.

Symmetric monoidal categories

A symmetric monoidal category is a monoidal category $\mathcal{C}$ such that, for each pair of objects X, Y in $\mathcal{C}$, there is an isomorphism

$$s_{XY} \colon X \otimes Y \xrightarrow{\cong} Y \otimes X$$

that is

- natural in both X and Y,
- $s_{YX} s_{XY} = 1_{X \otimes Y}$ for all objects X, Y,
- the s_{XY} satisfy certain (associativity and unit) coherence conditions.

Examples

- Sets (cartesian product)
- Groups (cartesian product)
- Vector spaces over a given field (tensor product)

A universal symmetric monoidal category

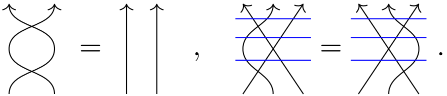
Define a strict monoidal category $\mathcal{S}$ with one generating object $\uparrow$ and denote

$$1_{\uparrow} = \uparrow$$

We have one generating morphism

 $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow .$

We impose the relations:



Then

$$\mathrm{End}_{\mathcal{S}}(\uparrow^{\otimes n}) = S_n$$

is the symmetric group on n letters.

A universal symmetric monoidal category

Universal property: If $\mathcal{C}$ is any symmetric monoidal category and X is an object of $\mathcal{C}$, then there is a unique monoidal functor

$$\mathcal{S} \to \mathcal{C}, \quad \uparrow \mapsto X,$$

$$\mapsto s_{XX}.$$

Immediate consequence: We have an S_n-action on $\mathrm{End}_{\mathcal{C}}(X^{\otimes n})$ coming from the induced map

$$S_n = \mathrm{End}_{\mathcal{S}}(\uparrow^{\otimes n}) \to \mathrm{End}_{\mathcal{C}}(X^{\otimes n}).$$

Linear version: We can repeat the above with $\Bbbk$-linear categories. Then we get an action of the group algebra

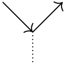
$$\Bbbk S_n = \mathrm{End}_{\mathcal{S}}(\uparrow^{\otimes n}) \to \mathrm{End}_{\mathcal{C}}(X^{\otimes n}).$$

Duals

Suppose a strict monoidal category $\mathcal{C}$ has two objects $\uparrow$ and $\downarrow$, with

$$1_\uparrow = \uparrow \quad , \quad 1_\downarrow = \downarrow .$$

A morphism $\mathbf{1} \to \downarrow \otimes \uparrow$ would have string diagram



, where $= 1_{\mathbf{1}}$.

We typically omit the dotted line and draw:

$: \mathbf{1} \to \downarrow \otimes \uparrow .$

Similarly, we can have

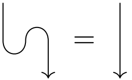
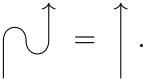
$: \uparrow \otimes \downarrow \to \mathbf{1}.$

Duals

We say that ↓ is right dual to ↑ (and ↑ is left dual to ↓) if there exist morphisms

 $: \mathbf{1} \to \downarrow \otimes \uparrow$. and  $: \uparrow \otimes \downarrow \to \mathbf{1}$.

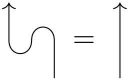
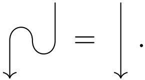
such that

 and 

Objects ↑ and ↓ are both left and right dual to each other if we also have

 $: \mathbf{1} \to \uparrow \otimes \downarrow$ and  $: \downarrow \otimes \uparrow \to \mathbf{1}$

such that

 and 

Duals: example

Let $\Bbbk$ be a field and consider the category $\mathrm{Vect}_\Bbbk$ of f.d. $\Bbbk$-vector spaces.

Unit object: $\Bbbk$

Fix a f.d. $\Bbbk$-vector space V.

Claim: The dual vector space V^* is both left and right dual to V, in the sense mentioned above.

Proof: Fix a basis B of V. Let $\{\delta_v : v \in B\}$ be the dual basis of V^*. Viewing V and $\uparrow$ and V^* as $\downarrow$, we define

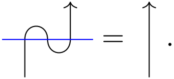
 $: \Bbbk \to V^* \otimes V, \quad 1 \mapsto \sum_{v \in B} \delta_v \otimes v,$

 $: V \otimes V^* \to \Bbbk, \quad v \otimes f \mapsto f(v),$

 $: \Bbbk \to V \otimes V^*, \quad 1 \mapsto \sum_{v \in B} v \otimes \delta_v,$

 $: V^* \otimes V \to \Bbbk, \quad f \otimes v \mapsto f(v).$

Let's check the relation

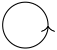


The left-hand side is the composition

$$V \cong V \otimes \Bbbk \xrightarrow{1_V \otimes \cup} V \otimes V^* \otimes V \xrightarrow{\cap \otimes 1_V} \Bbbk \otimes V \cong V,$$
$$w \mapsto w \otimes 1 \mapsto \sum_{v \in B} w \otimes \delta_v \otimes v \mapsto \sum_{v \in B} \delta_v(w) \otimes v \mapsto \sum_{v \in V} \delta_v(w) v = w.$$

The verification of the other relations is analogous.

Note:



$$: \Bbbk \to \Bbbk, \quad 1 \mapsto \sum_{v \in B} \delta_v(v) = \dim V.$$

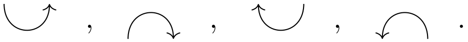
So the "bubble" corresponds to the dimension of the object V.

The oriented Temperley–Lieb category

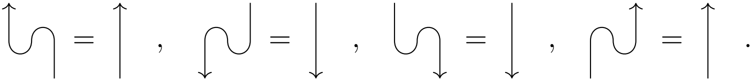
The oriented Temperley-Lieb category $\mathcal{OTL}$ is the free $\Bbbk$-linear monoidal category on a pair of dual objects.

Two generating objects: $\uparrow$ and $\downarrow$

Four generating morphisms:

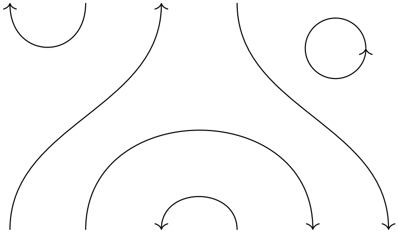


Relations:



The oriented Temperley–Lieb category

An arbitrary morphism is a $\Bbbk$-linear combination of oriented Temperly–Lieb diagrams:



Composition is vertical "gluing".

Universal property: Any $\Bbbk$-linear monoidal category $\mathcal{C}$ with a pair of dual objects V, W admits a $\Bbbk$-linear monoidal functor

$$\mathcal{OTL} \to \mathcal{C}, \quad \uparrow \mapsto V, \quad \downarrow \mapsto W.$$

If we impose an extra relation that bubbles are equal to some δ (giving a category $\mathcal{OTL}(\delta)$), we get a universal property for objects of dimension δ.

The oriented Temperley–Lieb category

Example (Vector spaces)

For any f.d. $\Bbbk$-vector space V, we have a $\Bbbk$-linear monoidal functor

$$\mathcal{OTL} \to \mathrm{Vect}_{\Bbbk}, \quad \uparrow \mapsto V, \quad \downarrow \mapsto V^*.$$

Example ($\mathfrak{gl}_\delta(\Bbbk)$-mod)

Let $V = \Bbbk^\delta$ be the natural $\mathfrak{gl}_\delta(\Bbbk)$-module. We have a $\Bbbk$-linear monoidal functor

$$\mathcal{OTL}(\delta) \to \mathfrak{gl}_\delta(\Bbbk)\text{-mod}, \quad \uparrow \mapsto V, \quad \downarrow \mapsto V^*.$$

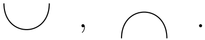
Question: What if we look at self-dual objects?

The Temperley–Lieb category

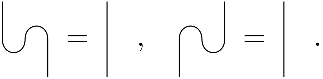
The Temperley-Lieb category $\mathcal{TL}$ is the free $\Bbbk$-linear monoidal category on a self-dual object.

One generating object: $|$

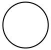
Two generating morphisms:



Relations:



If we impose the extra relation

 $= \delta$

we get $\mathcal{TL}(\delta)$, the free $\Bbbk$-linear monoidal category on a self-dual object of dimension δ.

The Temperley–Lieb category

The endomorphism algebra

$$TL_n(\delta) := \mathrm{End}_{\mathcal{TL}(\delta)}(|^{\otimes n})$$

is the Temperley–Lieb algebra.

Universal property: Any $\Bbbk$-linear monoidal category $\mathcal{C}$ with a self-dual object V of dimension δ admits a $\Bbbk$-linear monoidal functor

$$\mathcal{TL}(\delta) \to \mathcal{C}, \quad | \mapsto V.$$

Corollary: If $\mathcal{C}$ is a $\Bbbk$-linear monoidal category with a self-dual object V of dimension δ, we have an algebra homomorphism

$$TL_n(\delta) \to \mathrm{End}_{\mathcal{C}}(V^{\otimes n}).$$

The Temperley–Lieb category

Example (Inner product spaces)

Consider the category Inn of real inner product spaces. For any object V of dimension δ, we have a functor

$$\mathcal{TL}(\delta) \to \mathrm{Inn}, \quad | \mapsto V$$

and an algebra homomorphism $TL_n(\delta) \to \mathrm{End}_{\mathrm{Inn}}(V^{\otimes n})$.

Example (The orthogonal group)

Let $V = \mathbb{R}^\delta$ be the natural representation of the orthogonal group $O_\delta(\mathbb{R})$.

We have a functor

$$\mathcal{TL}(\delta) \to O_\delta(\mathbb{R})\text{-mod}, \quad | \mapsto V,$$

and an algebra homomorphism $TL_n(\delta) \to \mathrm{End}_{O_\delta(\mathbb{R})}(V^{\otimes n})$.

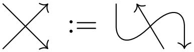
Idea: Let's consider symmetric monoidal categories with duals!

The oriented Brauer category $\mathcal{OB}$ is the free $\Bbbk$-linear symmetric monoidal category on a pair of dual objects.

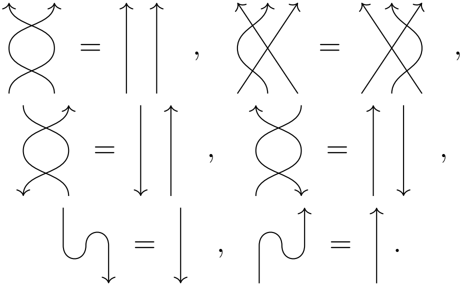
Two generating objects: $\uparrow$ and $\downarrow$

Four generating morphisms:

Define



Relations:



Then one can define

Universal property: Any $\Bbbk$-linear symmetric monoidal category $\mathcal{C}$ with dual objects V, W admits a $\Bbbk$-linear monoidal functor

$$\mathcal{OB} \to \mathcal{C}, \quad \uparrow \mapsto V, \quad \downarrow \mapsto W.$$

The oriented Brauer category

Note: The oriented Temperley–Lieb category is a subcategory of the oriented Brauer category. We can enhance our previous examples.

Example (Vector spaces)

For any f.d. $\Bbbk$-vector space V, we have a $\Bbbk$-linear monoidal functor

$$\mathcal{OB} \to \mathrm{Vect}_{\Bbbk}, \quad \uparrow \mapsto V, \quad \downarrow \mapsto V^*.$$

Example ($\mathfrak{gl}_\delta(\Bbbk)$-mod)

Let $V = \Bbbk^\delta$ be the natural $\mathfrak{gl}_\delta(\Bbbk)$-module. We have a $\Bbbk$-linear monoidal functor

$$\mathcal{OB} \to \mathfrak{gl}_\delta(\Bbbk)\text{-mod}, \quad \uparrow \mapsto V, \quad \downarrow \mapsto V^*.$$

Schur–Weyl duality: This functor is full.

Question: What if we look at self-dual objects?

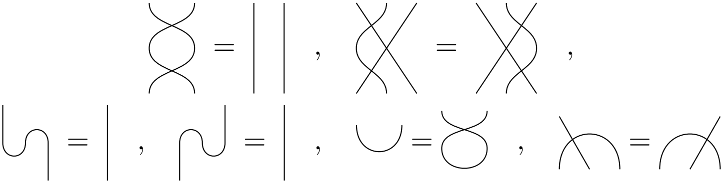
The Brauer category

The Brauer category $\mathcal{B}$ is the free $\Bbbk$-linear symmetric monoidal category on a self-dual object.

One generating object: $|$

Three generating morphisms:

Relations:

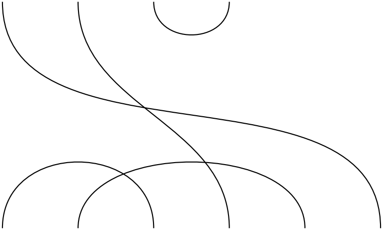


$+$ flips in the horizontal.

Note: Can add relation that bubble is δ to get $\mathcal{B}(\delta)$.

The Brauer category

An arbitrary morphism in $\mathcal{B}(\delta)$ is a $\Bbbk$-linear combination of Brauer diagrams:



Composition is vertical "gluing".

The endomorphism algebra

$$B_n(\delta) := \mathrm{End}_{\mathcal{B}(\delta)}(|^{\otimes n})$$

is the Brauer algebra.

The Brauer category

Universal property: Any $\Bbbk$-linear symmetric monoidal category $\mathcal{C}$ with self-dual object V of dimension δ admits a $\Bbbk$-linear monoidal functor

$$\mathcal{B}(\delta) \to \mathcal{C}, \quad | \mapsto V.$$

Corollary: If $\mathcal{C}$ is a $\Bbbk$-linear symmetric monoidal category with a self-dual object V of dimension δ, we have an algebra homomorphism

$$B_n(\delta) \to \operatorname{End}_{\mathcal{C}}(V^{\otimes n}).$$

Example (The orthogonal group)

Let $V = \mathbb{R}^\delta$ be the natural representation of the orthogonal group $O_\delta(\mathbb{R})$.

We have a functor

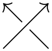
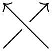
$$\mathcal{B}(\delta) \to O_\delta(\mathbb{R})\text{-mod}, \quad | \mapsto V,$$

and an algebra homomorphism $B_n(\delta) \to \operatorname{End}_{O_\delta(\mathbb{R})}(V^{\otimes n})$.

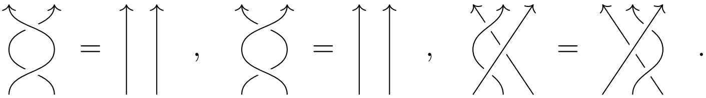
Braided monoidal categories

Some categories are not symmetric monoidal, but they are braided monoidal.

Replace crossing generators with

 and 

subject to the relations



Category of braids: Free braided monoidal category on one generating object.

Category of framed oriented tangles: Free ribbon category (duals + twists) on one generating object (Shum–Reshetikhin–Turaev Theorem).

Even more universal categories

Oriented skein category

- Start with category of framed oriented tangles.
- Impose a skein relation (symmetric groups $\rightsquigarrow$ Hecke algebras), twist relation, and dimension relation.
- Underpins the HOMFLY-PT polynomial (knot invariant).

Super versions

- Can work with supercategories.
- Now objects can be even dual or odd dual.
- Get odd Brauer category, etc.

Current work (joint with J. Brundan and B. Webster)

Affine versions

- Allow strands to carry dots and impose some additional relations.
- Corresponds to symmetric groups $\rightsquigarrow$ degenerate affine Hecke algebras.

Frobenius algebras

- Fix a Frobenius algebra F (associative algebra with a trace map).
- Deform categories: strands can now carry tokens labelled by elements of F.
- Obtain generalizations of (affine) oriented Brauer category, (affine) oriented skein category, etc.

Heisenberg categories

Affine oriented Brauer category and affine oriented skein category are the charge zero case of a more general category: the (quantum) Heisenberg category.