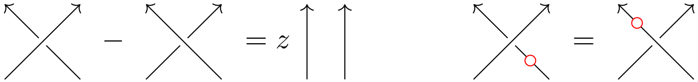
Quantum Heisenberg categorification



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Slides available online: alistairsavage.ca/talks

Preprint: 1812.04779 (with J. Brundan and B. Webster)

Outline

Goal:

1. Define a family of quantum Heisenberg categories categorifying the Heisenberg algebra
2. Study categorical actions and applications in representation theory

Overview:

1. Strict monoidal categories and string diagrams
2. Quantum Heisenberg category
3. Categorical actions
4. Future directions

Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.

$\Bbbk$-linear monoidal categories

Fix a commutative ground ring $\Bbbk$.

A strict $\Bbbk$-linear monoidal category is a strict monoidal category such that

- each morphism space is a $\Bbbk$-module,
- composition of morphisms is $\Bbbk$-bilinear,
- tensor product of morphisms is $\Bbbk$-bilinear.

The interchange law

The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

$$\begin{tikzcd}
A_1 \otimes B_1 \arrow[r, "1\otimes g"] \arrow[d, "f\otimes 1"'] \arrow[dr, "f\otimes g"] & A_1 \otimes B_2 \arrow[d, "f\otimes 1"] \\
A_2 \otimes B_1 \arrow[r, "1\otimes g"'] & A_2 \otimes B_2
\end{tikzcd}$$

Strict monoidal categories

Example (Monoids)

A (strict) monoidal category with one object is simply a commutative monoid. More precisely, the endomorphisms of $\mathbb{1}$ form a commutative monoid.

Conversely, every commutative monoid gives rise to a one-object monoidal category.

Example (Associative algebras)

A (strict) $\Bbbk$-linear monoidal category with one object is simply a commutative associative unital $\Bbbk$-algebra.

Categorification via split Grothendieck group

Suppose $\mathcal{C}$ is an additive category (i.e. have $\oplus$).

$\mathrm{Iso}_{\mathbb{Z}}(\mathcal{C})$ = free abelian group generated by isom. classes of objects in $\mathcal{C}$.

The split Grothendieck group of $\mathcal{C}$ is

$$K_0(\mathcal{C}) = \mathrm{Iso}_{\mathbb{Z}}(\mathcal{C}) / \langle [X \oplus Y] = [X] + [Y] \mid X, Y \in \mathcal{C} \rangle.$$

If $\mathcal{C}$ is monoidal, then $K_0(\mathcal{C})$ is a ring:

$$[X] \cdot [Y] = [X \otimes Y].$$

Categorification

For our purposes, to categorify a ring R is to find an additive monoidal category $\mathcal{C}$ such that

$$K_0(\mathcal{C}) \cong R \quad \text{as rings.}$$

The Heisenberg algebra

Fix a central charge $k \in \mathbb{Z}$.

Definition

The rank one Heisenberg algebra has generators p^+ and p^- and relation

$$p^+ p^- = p^- p^+ + k.$$

This is called the canonical commutation relation in physics.

Definition

The infinite rank Heisenberg algebra Heis_k has generators $p_n^\pm$, $n \geq 1$, and relations

$$p_m^+ p_n^+ = p_n^+ p_m^+, \quad p_m^- p_n^- = p_n^- p_m^-, \quad p_m^+ p_n^- = p_n^- p_m^+ + \delta_{n,m} kn.$$

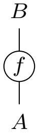
Plays a fundamental role in QFT and the theory of affine Lie algebras.

Goal: Categorify the infinite rank Heisenberg algebra.

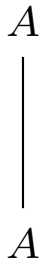
String diagrams

Let's draw pictures! Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f \colon A \to B$ by:



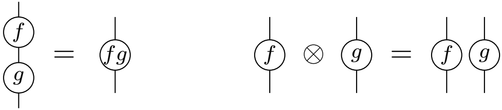
The identity map $1_A \colon A \to A$ is a string with no label:



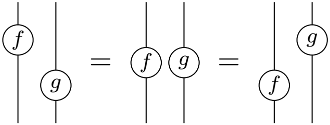
We sometimes omit the object labels when they are clear or unimportant.

String diagrams

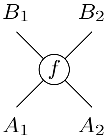
Composition is vertical stacking and tensor product is horizontal juxtaposition:



The interchange law then becomes:



A morphism $f : A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



Presentations of strict monoidal categories

One can give presentations of some strict $\Bbbk$-linear monoidal categories, just as for monoids, groups, algebras, etc.

Objects: If the objects are generated by some collection A_i, $i \in I$, then we have all possible tensor products of these objects:

$$\mathbb{1}, \quad A_i, \quad A_i \otimes A_j \otimes A_k \otimes A_\ell, \quad \text{etc.}$$

Morphisms: If the morphisms are generated by some collection f_j, $j \in J$, then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

$$1_{A_i}, \quad f_j \otimes (f_i f_k) \otimes (f_\ell), \quad \text{etc.}$$

We then often impose some relations on these morphism spaces.

String diagrams: We can build complex diagrams out of our simple generating diagrams.

Monoidally generated symmetric groups

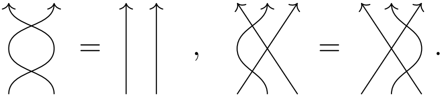
Define a strict $\Bbbk$-linear monoidal category $\mathcal{S}$ with one generating object $\uparrow$ and denote

$$1_{\uparrow} = \uparrow$$

We have one generating morphism

$$: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow .$$

We impose the relations:



It is straightforward to verify that

$$\mathrm{End}_{\mathcal{S}}(\uparrow^{\otimes n}) = \Bbbk S_n$$

is the group algebra of the symmetric group on n letters.

Monoidally generated symmetric groups

This monoidal presentation of $\Bbbk S_n$ is very efficient! We only needed

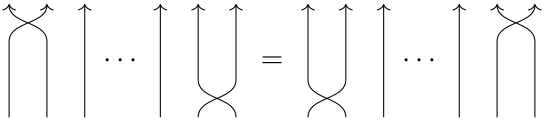
- one generating morphism, and
- two relations,

to get all the symmetric groups.

Note that the "distant braid relation"

$$s_i s_j = s_j s_i, \qquad |i-j| > 1$$

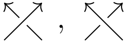
for simple transpositions follows for free from the interchange law:



Monoidally generated braid groups

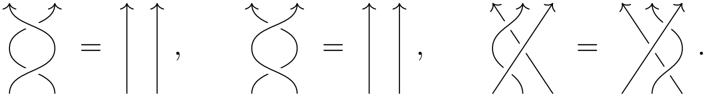
Consider the strict $\Bbbk$-linear monoidal category $\mathcal{B}raid$ generated by

- one object $\uparrow$, and
- two morphisms



$$: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow .$$

subject to the relations



Then

$$\mathrm{End}_{\mathcal{B}raid}(\uparrow^{\otimes n})$$

is the group algebra of the braid group of type A_{n-1}.

Again, this presentation is very efficient.

Monoidally generated Hecke algebras

Fix a parameter $z \in \Bbbk^\times$.

Let $\mathcal{H}(z)$ be the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{B}raid$ by imposing one additional relation:

(skein relation)

Then

$$\mathrm{End}_{\mathcal{H}(z)}(\uparrow^{\otimes n})$$

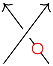
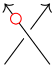
is the Iwahori–Hecke algebra of type A_{n-1}.

Monoidally generated affine Hecke algebras

Let $\mathcal{AH}(z)$ be the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{H}(z)$ by adding one more invertible generator:

$\colon \uparrow \to \uparrow$

and one additional relation:

 $=$  .

Then

$$\mathrm{End}_{\mathcal{AH}(z)}(\uparrow^{\otimes n})$$

is the affine Hecke algebra of type A_{n-1}.

Algebraic versus diagrammatic

Let's match the diagrammatic description with the usual algebraic definition of the affine Hecke algebra.

Algebraic generators:

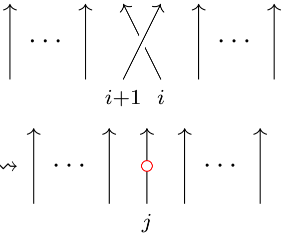
- T_i, $1 \le i \le n-1$
- $X_j^{\pm 1}$, $1 \le j \le n$

Conventions:

- Number strands right to left
- Read diagrams bottom to top

$T_i \leftrightsquigarrow$

$X_j \leftrightsquigarrow$



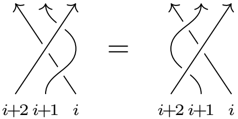
Algebraic versus diagrammatic

Relations:

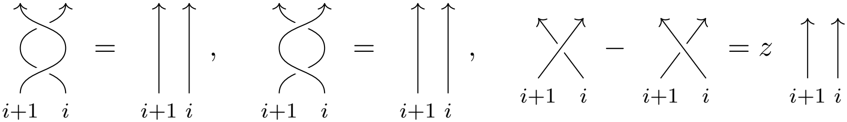
1. $T_i T_j = T_j T_i$ when $|i-j| > 1$

   Follows from interchange relation

2. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$



3. $T_i^2 = zT_i + 1$
   Equivalent to: T_i is invertible and $T_i - T_i^{-1} = z$

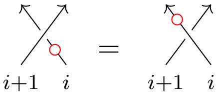


Algebraic versus diagrammatic

4. $T_i X_j = X_j T_i$ when $j \neq i, i+1$

   Follows from the interchange relation

5. $T_i X_i = X_{i+1} T_i^{-1}$

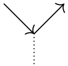


Duals

Suppose a strict monoidal category $\mathcal{C}$ has two objects $\uparrow$ and $\downarrow$, with

$$1_\uparrow = \uparrow \quad , \quad 1_\downarrow = \downarrow .$$

A morphism $\mathbb{1} \to \downarrow \otimes \uparrow$ would have string diagram

 , where $= 1_{\mathbb{1}}$.

We typically omit the dotted line and draw:

 $: \mathbb{1} \to \downarrow \otimes \uparrow .$

Similarly, we can have

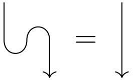
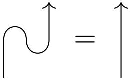
 $: \uparrow \otimes \downarrow \to \mathbb{1}.$

Duals

We say that ↓ is right dual to ↑ (and ↑ is left dual to ↓) if there exist morphisms

 $: \mathbb{1} \to \downarrow \otimes \uparrow.$ and  $: \uparrow \otimes \downarrow \to \mathbb{1}.$

such that

 and  .

(Analogous to the unit-counit formulation of adjunction of functors.)

The quantum Heisenberg category

Recall the canonical commutation relation of the Heisenberg algebra:

$$p^- p^+ = p^+ p^- - k.$$

(Assume $k < 0$ for simplicity.)

We want to categorify this relation. So we want objects $\uparrow$ and $\downarrow$ such that

$$\downarrow \otimes \uparrow \;\cong \uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus(-k)}. \qquad \text{(Canonical commutation isom)}$$

Then, in the Grothendieck ring, we have

$$[\downarrow][\uparrow] = [\uparrow][\downarrow] - k.$$

This is the canonical commutation relation with

$$[\uparrow] \leftrightsquigarrow p^+ \quad \text{and} \quad [\downarrow] \leftrightsquigarrow p^-.$$

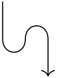
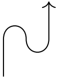
The quantum Heisenberg category

To obtain the quantum Heisenberg category $\mathcal{H}eis_k(z,t)$ from $\mathcal{AH}(z)$ we perform two steps:

1. We adjoin a right dual $\downarrow$ to $\uparrow$. Precisely, we add a generating object $\downarrow$ and additional generating morphisms

    $: \mathbb{1} \to \downarrow \otimes \uparrow$ and  $: \uparrow \otimes \downarrow \to \mathbb{1}$

   such that

    $= \downarrow$ and  $= \uparrow$.

2. We add more generating morphisms and relations ensuring that the resulting monoidal category is pivotal and that

$$\downarrow \otimes \uparrow \;\cong\; \uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus(-k)}.$$

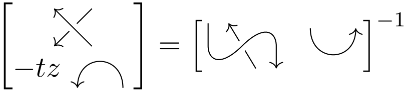
(Canonical commutation isom)

There are three equivalent ways to do this. For simplicity, suppose $k = -1$.

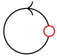
First approach: Add generating morphisms

and

and relations



and

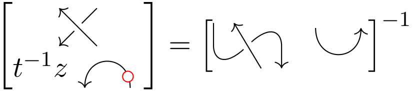


$= tz^{-1} 1_{\mathbb{1}}$.

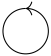
Second approach: Add generating morphisms

and

and relations



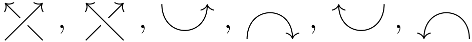
and



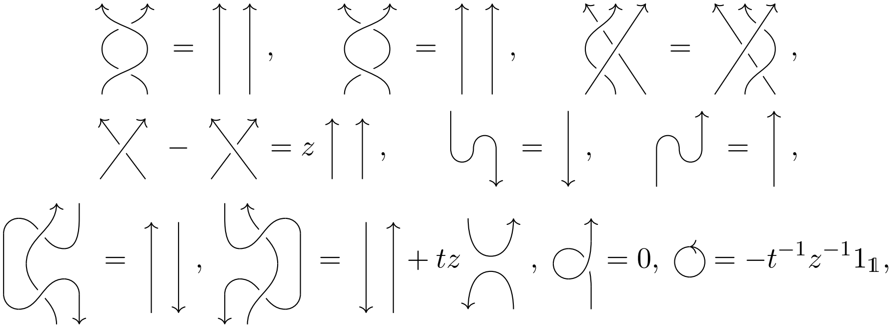
$= -t^{-1} z^{-1} 1_{\mathbb{1}}$.

The quantum Heiseberg category

Third approach: Generating morphisms



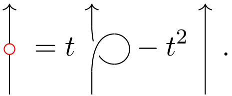
subject to the relations



and one more relation (†).

The quantum Heisenberg category

Third approach: Note that we do not need the dot generator. It can be recovered via



The extra relation (†) is that this dot is invertible.

Theorem (Brundan–S.–Webster)

1. All three approaches define isomorphic categories ($\mathcal{H}eis_k(z,t)$).
2. $\mathcal{H}eis_k(z,t)$ is strictly pivotal (i.e. we have isotopy invariance for morphisms).

Special cases

Deformed Heisenberg category ($k = -1$)

$\mathcal{H}eis_{-1}(z,t)$ is closely related to a deformed Heisenberg category $\mathcal{H}(q^2)$ introduced by Licata–S. (2013).

Precisely, $\mathcal{H}(q^2)$ is the monoidal subcategory of

$$\mathcal{H}eis_{-1}(z,-z^{-1}), \quad z = q - q^{-1},$$

consisting of all objects and morphisms that do not involve negative powers of the dots.

Affine oriented skein category ($k = 0$)

$\mathcal{H}eis_0(z,t)$ is the affine oriented skein category, an affinization of the HOMFLY-PT skein category.

Heisenberg categorification

We want an algebra isomorphism

$$K_0(\mathcal{H}eis_k(z,t)) \cong \mathrm{Heis}_k.$$

Under this isomorphism, we should have

$$[\uparrow] \mapsto p_1^+, \quad [\downarrow] \mapsto p_1^-.$$

Questions/problems

1. $K_0(\mathcal{H}eis_k(z,t))$ doesn't make sense! $\mathcal{H}eis_k(z,t)$ is not an additive category!
2. What about the other generators $p_n^\pm$, $n > 1$?

Answers

1. Formally add in direct sums.
2. Enlarge the category by taking the idempotent completion.

Additive envelope

Suppose $\mathcal{C}$ is some $\Bbbk$-linear monoidal category.

Its additive envelope is the category whose:

- objects are formal finite direct sums $\bigoplus_i X_i$ of objects X_i in $\mathcal{C}$,
- morphisms

$$f\colon \bigoplus_{i=1}^{n} X_i \to \bigoplus_{j=1}^{m} Y_j$$

  are $m \times n$ matrices, where the (j,i)-entry is a morphism

$$f_{i,j}\colon X_i \to Y_j .$$

  Composition is given by matrix multiplication.

Idempotent completion

Definition: idempotent completion

The idempotent completion of a category $\mathcal{C}$ is the category whose

- objects are pairs (A, e) where $A \in \mathrm{Ob}\,\mathcal{C}$ and $e \in \mathrm{Mor}_{\mathcal{C}}(A, A)$ is an idempotent ($e^2 = e$), and
- morphisms from (A, e) to (B, f) are elements of $f\,\mathrm{Mor}_{\mathcal{C}}(A, B)e$.

Intuition: One thinks of passing to the idempotent completion as adding in objects such that the idempotents correspond to projections onto direct summands.

$$\begin{tikzcd} A \cong X \oplus Y \arrow[r, two heads] \arrow[rr, bend left, "e"] & X \arrow[r, hook] & X \oplus Y \cong A \end{tikzcd}$$

Idempotent completion

Recall that we have a natural homomorhpism

$$\text{Hecke algebra} \to \mathrm{End}(\uparrow^{\otimes n}).$$

Result: Idempotents in the Hecke algebra yield idempotents in $\mathrm{End}(\uparrow^{\otimes n})$.

The idempotents in the Hecke algebra are well known. These give us additional objects in the idempotent completion corresponding to the $p_n^{\pm}$.

For the experts (knowing about symmetric functions)

- To each partition λ of n, we have a Young idempotent e_λ.
- The object $(\uparrow^{\oplus n}, e_\lambda)$ corresponds to the Schur function s_λ.
- The $p_n^{\pm}$ are the power sums.

Categorification of the Heisenberg algebra

There exists an injective algebra homomorphism

$$\mathrm{Heis}_k \to K_0(\mathsf{Kar}(\mathcal{H}eis_k(z,t))).$$

Conjecture: This is an isomorphism.

Difficulty: Passage to the Karoubi envelope. How do we know we've found all idempotents?

Degenerate case

- Previously, Khovanov defined a Heisenberg category which is essentially the $k=-1$, $z=0$ version of the category defined here.
- Khovanov also conjectured the isomorphism.
- Then degenerate setting extended to general k (Mackaay-S., Brundan).
- Categorification conjecture recently proved in the degenerate setting (Brundan–Webster–S.).

Categorical actions ($k \neq 0$)

When $k \neq 0$, the category $\mathcal{H}eis_k(z,t)$ acts naturally on modules for cyclotomic Hecke algebras H_n^f of level $|k|$.

We have a chain of algebras

$$\Bbbk = H_0^f \subseteq H_1^f \subseteq H_2^f \subseteq \cdots .$$

If $k < 0$, then

- $\uparrow$ acts by induction from H_n^f-mod to H_{n+1}^f-mod,
- $\downarrow$ acts by restriction from H_n^f-mod to H_{n-1}^f-mod.

The morphisms (diagrams) act by certain natural transformations.

Fact that $\mathcal{H}eis_k(z,t)$ is pivotal corresponds to fact that induction and restriction are biadjoint.

In other words H_n^f is a Frobenius extension of H_{n-1}^f.

Categorical actions ($k = 0$)

Suppose $k = 0$ and $t = q^n$.

$\mathcal{H}eis_0(z,t)$ acts on representations of $U_q(\mathfrak{gl}_n)$:

- $\uparrow$ tensors with natural module V,
- $\downarrow$ tensors with dual V^*.

This action extends the monoidal functor

$$\text{HOMFLY-PT skein category} \to \text{cat of fd } U_q(\mathfrak{gl}_n)\text{-modules}$$

originally constructed by Turaev.

The center of $\mathcal{H}eis_0(z,t)$ maps surjectively to the center of $U_q(\mathfrak{gl}_n)$. So we get a diagrammatic calculus for this center.

Basis theorem

For many applications, one needs to know a basis for morphism spaces in $\mathcal{H}eis_k(z,t)$.

Usual approach: Use

- categorical actions described above,
- known bases for the algebras involved (H_n^f and $U_q(\mathfrak{gl}_n)$),
- asymptotic faithfulness (as $n \to \infty$).

However, this approach fails for $\mathcal{H}eis_k(z,t)$, $k \neq 0$, due to $\mathcal{H}eis_k(z,t)$ having a larger center than expected.

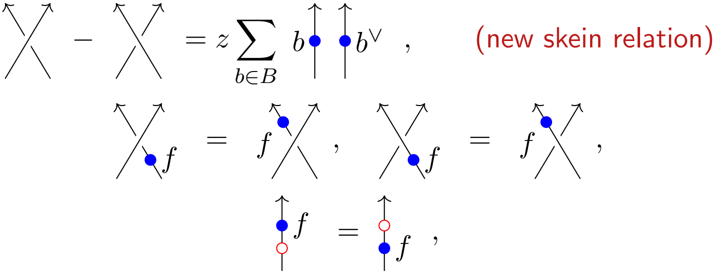
Solution: Define a categorical coproduct ("unfurling" technique of B. Webster) to define a large representation.

Generally, can incorporate a Frobenius superalgebra F to get a more general quantum Frobenius Heisenberg category.

Strand can now carry tokens:

$$, \quad f \in F.$$

We have additional/modified relations:



$+$ inversion, etc.

Categorical actions

Categorical actions: Largely unexplored.

Case: $k = 0$

Obtain a Frobenius deformation of the affine oriented skein category.

Natural action is an open question for general F.

Should act on modules for some F-deformation of $U_q(\mathfrak{gl}_n)$.

Case: $k \neq 0$

Acts on cyclotomic quotients of quantum affine wreath algebras (generalize affine Hecke algebras, affine Sergeev algebras, etc.).

The structure theory of these algebras has been explored with D. Rosso.

Future directions

Traces

The trace of the deformed Heisenberg category $\mathcal{H}(q^2)$ was computed by Cautis–Lauda–Licata–Samuelson–Sussan.

It is related to the elliptic Hall algebra.

One should be able to extend this description to the larger quantum Heisenberg category $\mathcal{H}eis_k(z,t)$.

Connections to Kac–Moody 2-categories (with Brundan & Webster)

Given certain categorical Heisenberg actions, one can define a categorical Kac-Moody action.

Conversely, given certain categorical Kac–Moody actions, one can define a categorical Heisenberg action.

This extends work with Queffelec and Yacobi, which considered the level one case.