Embedding Deligne’s category $\text{Rep}(S_t)$ in the Heisenberg category

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Goal: Define an embedding from the partition category and Deligne’s category $\text{Rep}(S_t)$ into the Heisenberg category.

Overview:

1. Strict monoidal categories & string diagrams
2. The partition category & Deligne’s category $\text{Rep}(S_t)$
3. The Heisenberg category
4. The embedding
5. Future directions
Strict monoidal categories

A strict monoidal category is a category $C$ equipped with

- a bifunctor (the tensor product) $\otimes : C \times C \to C$, and
- a unit object $1$,

such that

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all objects $A, B, C$,
- $1 \otimes A = A = A \otimes 1$ for all objects $A$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.
**\(k\)-linear monoidal categories**

Fix a commutative ground ring \(k\).

A **strict \(k\)-linear monoidal category** is a strict monoidal category such that

- each morphism space is a \(k\)-module,
- composition of morphisms is \(k\)-bilinear,
- tensor product of morphisms is \(k\)-bilinear.

**The interchange law**

The axioms of a strict monoidal category imply the **interchange law**: For \(A_1 \xrightarrow{f} A_2\) and \(B_1 \xrightarrow{g} B_2\), the following diagram commutes:

\[
\begin{array}{ccc}
A_1 \otimes B_1 & \xrightarrow{1 \otimes g} & A_1 \otimes B_2 \\
\downarrow f \otimes 1 & & \downarrow f \otimes 1 \\
A_2 \otimes B_1 & \xrightarrow{1 \otimes g} & A_2 \otimes B_2
\end{array}
\]
Example (Monoids)

A (strict) monoidal category with one object is simply a commutative monoid. More precisely, the endomorphisms of \( 1 \) form a commutative monoid.

Conversely, every commutative monoid gives rise to a one-object monoidal category.

Example (Associative algebras)

A (strict) \( \mathbb{k} \)-linear monoidal category with one object is simply a commutative associative unital \( \mathbb{k} \)-algebra.
String diagrams

Let’s draw pictures! Fix a strict monoidal category $C$.

We will denote a morphism $f : A \to B$ by:

![String diagram of $f : A \to B$]

The identity map $1_A : A \to A$ is a string with no label:

![String diagram of the identity map $1_A$]

We sometimes omit the object labels when they are clear or unimportant.
String diagrams

Composition is **vertical stacking** and tensor product is **horizontal juxtaposition**:

\[
f \circ g = f_g = f \otimes g = f g
\]

The **interchange law** then becomes:

\[
f \circ g = f \circ g = g
\]

A morphism \( f: A_1 \otimes A_2 \to B_1 \otimes B_2 \) can be depicted:
Let $V = \mathbb{C}^n$ be the permutation representation of the symmetric group $S_n$.

We have a Schur–Weyl type duality

$$\mathbb{C}S_n \to \text{End}_\mathbb{C}(V^\otimes k) \xleftarrow{g} P_k(n),$$

where $P_k(n)$ is the partition algebra.

In particular:

- the image of $g$ is $\text{End}_{\mathbb{C}S_n}(V^\otimes k)$,
- $g$ is injective if and only if $2k \leq n$.

We can describe $P_k(n)$ in terms of partition diagrams.
Partition diagrams

For $k, \ell \in \mathbb{N}$, a partition of type $\binom{\ell}{k}$ is a partition of the set 

$$\{1, \ldots, k, 1', \ldots, \ell'\}.$$ 

Consider the associated partition diagram.

The partition diagram of 

$$\{\{1, 5\}, \{2\}, \{3, 1'\}, \{4, 4', 7'\}, \{2', 3'\}, \{5'\}, \{6'\}\}$$ 

of type $\binom{7}{5}$

is

$$D = \begin{array}{cccccc}
7' & 6' & 5' & 4' & 3' & 2' & 1' \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
5 & 4 & 3 & 2 & 1 & & \\
\end{array} : 5 \rightarrow 7.$$ 

We often omit the labels of the vertices.
Partition diagrams

We have unique partition diagrams of types \( \binom{1}{0} \) and \( \binom{0}{1} \):

\[
\uparrow: 0 \to 1 \quad \text{and} \quad \downarrow: 1 \to 0.
\]

Fix a commutative ring \( k \) and \( t \in k \).

**Composition:** We compose partition diagrams by vertical stacking. If

\[
D' = \quad \text{and} \quad D = \quad
\]

then

\[
D \circ D' = \quad, \quad D \circ D' = t^2.
\]

The exponent of \( t \) is the number of connected components in the “center”.
The partition category is the strict $k$-linear monoidal category $\mathcal{P}ar(t)$ with:

- **Objects**: $k \in \mathbb{N}$

- **Morphisms**: $\text{Hom}_{\mathcal{P}ar(t)}(k, \ell) = \text{Span}_k \left\{ \text{partition diag. of type } \binom{\ell}{k} \right\}$

- **Composition**: $\circ$ (vertical stacking)

- **Tensor product**: horizontal juxtaposition

Then

$$P_k(t) = \text{End}_{\mathcal{P}ar(t)}(k)$$

is the partition algebra.

**Note**: We do not require $t \in \mathbb{N}$. 
Presentations of strict monoidal categories

One can give presentations of some strict \( k \)-linear monoidal categories, just as for monoids, groups, algebras, etc.

**Objects:** If the objects are generated by some collection \( A_i, \ i \in I \), then we have all possible tensor products of these objects:

\[
1, \quad A_i, \quad A_i \otimes A_j \otimes A_k \otimes A_\ell, \quad \text{etc.}
\]

**Morphisms:** If the morphisms are generated by some collection \( f_j, \ j \in J \), then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

\[
1_{A_i}, \quad f_j \otimes (f_i \circ f_k) \otimes (f_\ell), \quad \text{etc.}
\]

We then often impose some relations on these morphism spaces.

**String diagrams:** We can build complex diagrams out of our simple generating diagrams.
**Presentation of the partition category**

\( \mathcal{P}ar(t) \) is generated by the object 1 and the morphisms

\[
\mu = \begin{array}{c}
\bullet \\
\end{array}, \quad \delta = \begin{array}{c}
\bullet \\
\end{array}, \quad s = \begin{array}{c}
\bullet \\
\end{array}, \quad \eta = \begin{array}{c}
\bullet \\
\end{array}, \quad \varepsilon = \begin{array}{c}
\bullet \\
\end{array},
\]

subject to the relations:

\[
\begin{align*}
\begin{array}{c}
\bullet \\
\end{array} & = \begin{array}{c}
\bullet \\
\end{array} = \begin{array}{c}
\bullet \\
\end{array}, \\
\begin{array}{c}
\bullet \\
\end{array} & = \begin{array}{c}
\bullet \\
\end{array} = \begin{array}{c}
\bullet \\
\end{array}, \\
\begin{array}{c}
\bullet \\
\end{array} & = \begin{array}{c}
\bullet \\
\end{array} = \begin{array}{c}
\bullet \\
\end{array},
\end{align*}
\]

Consequence: \( \mathcal{P}ar(t) \) is the free \( k \)-linear symmetric monoidal category generated by a \( t \)-dimensional special commutative Frobenius object.
Action of the partition category

Suppose $\mathbb{k}$ is a field. Let

- $V = \mathbb{k}^n$ be the permutation representation of $S_n$,
- $1_n$ be the one-dimensional trivial $S_n$-module.

There is a strong monoidal functor

$$\Phi_n : \mathcal{P}ar(n) \to S_n\text{-mod}$$

given by $\Phi_n(1) = V$ and

\[
\begin{align*}
\Phi_n(\mu) & : V \otimes V \to V, & v_i \otimes v_j & \mapsto \delta_{i,j}v_i, \\
\Phi_n(\eta) & : 1_n \to V, & 1 & \mapsto \sum_{i=1}^n v_i, \\
\Phi_n(\delta) & : V \to V \otimes V, & v_i & \mapsto v_i \otimes v_i, \\
\Phi_n(\varepsilon) & : V \to 1_n, & v_i & \mapsto 1, \\
\Phi_n(s) & : V \otimes V \to V \otimes V, & v_i \otimes v_j & \mapsto v_j \otimes v_i 
\end{align*}
\]

where $v_1, \ldots, v_n$ is the standard basis of $V = \mathbb{k}^n$. 

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Deligne $\hookrightarrow$ Heisenberg

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Action of the partition category

**Proposition (Comes 2016)**

1. $\Phi_n : \mathcal{P}ar(n) \to S_n \text{-mod}$ is full.
2. The induced map

$$\text{Hom}_{\mathcal{P}ar(n)}(k, \ell) \to \text{Hom}_{S_n}(V^\otimes k, V^\otimes \ell)$$

is an isomorphism if and only if $k + \ell \leq n$.

This is a generalization of the Schur–Weyl duality property of the partition algebra mentioned earlier.
Additive envelope

Suppose $C$ is some $\mathbb{k}$-linear monoidal category.

Its additive envelope is the category whose:

- objects are formal finite direct sums $\bigoplus_i X_i$ of objects $X_i$ in $C$,
- morphisms

$$f : \bigoplus_{i=1}^n X_i \to \bigoplus_{j=1}^m Y_j$$

are $m \times n$ matrices, where the $(j, i)$-entry is a morphism

$$f_{i,j} : X_i \to Y_j.$$

Composition is given by matrix multiplication.
**Idempotent completion**

**Definition: idempotent completion (Karoubi envelope)**

The idempotent completion of a category \( C \) is the category whose

- objects are pairs \((A, e)\) where \( A \in \text{Ob} \, C \) and \( e \in \text{Mor}_C(A, A) \) is an idempotent \((e^2 = e)\), and
- morphisms from \((A, e)\) to \((B, f)\) are elements of \( f \text{Mor}_C(A, B)e \).

**Intuition:** One thinks of passing to the idempotent completion as adding in objects such that the idempotents correspond to projections onto direct summands.

\[
\begin{align*}
A & \cong X \oplus Y \\
& \overset{e}{\longrightarrow} \\
& \longrightarrow X \\
& \longrightarrow X \oplus Y \cong A
\end{align*}
\]

We let \( \text{Kar}(C) \) denote the Karoubi envelope of the additive envelope of \( C \).
Deligne’s category $\text{Rep}(S_t)$

We define

$$\text{Rep}(S_t) := \text{Kar}(\mathcal{P}ar(t)).$$

**Notes**

- $\text{Rep}(S_t)$ is semisimple when $t \notin \mathbb{N}$.
- When $t = n \in \mathbb{N}$, $S_n$-mod is the quotient of $\text{Rep}(S_t)$ by the negligible morphisms.
- $\text{Rep}(S_t)$ “interpolates” between the categories $\text{Rep}(S_n)$.

**Intuition:** $\text{Rep}(S_t)$ describes the representation of the $S_n$ in a uniform way, but with $n$ fixed.
The Heisenberg category: Motivation

Geissinger (1970s) constructed an isomorphism of bialgebras

\[ \bigoplus_{n=0}^{\infty} K_0(S_n\text{-mod}) \cong \text{Sym.} \]

Multiplication given by

\[ [\text{Ind}] : K_0(S_n\text{-mod}) \otimes K_0(S_m\text{-mod}) \to K_0(S_{n+m}\text{-mod}). \]

Comultiplication given by restriction.

Operators of mult. by \([K] \in K_0(S_n\text{-mod})\), together with their adjoints, defines an action of the infinite rank Heisenberg algebra, and

\[ \bigoplus_{n=0}^{\infty} K_0(S_n\text{-mod}) \]

is the Fock space module.

Further study of these ideas leads to the Heisenberg category.
The Heisenberg category \( \text{Heis} \) (Khovanov 2014) is the strict \( k \)-linear monoidal category generated by two objects \( \uparrow, \downarrow \), morphisms

\[
\begin{align*}
\uparrow\uparrow & : \uparrow\uparrow \to \uparrow\uparrow, \\
\downarrow\uparrow & : 1 \to \downarrow\uparrow, \\
\uparrow\downarrow & : \uparrow\downarrow \to 1, \\
\downarrow & : 1 \to \downarrow\uparrow, \\
\uparrow & : \downarrow\uparrow \to 1,
\end{align*}
\]

and relations

\[
\begin{align*}
\uparrow\uparrow & = \uparrow\uparrow, \\
\downarrow\uparrow & = \downarrow\uparrow, \\
\uparrow\downarrow & = \uparrow\downarrow, \\
\downarrow\downarrow & = \downarrow\downarrow, \\
\downarrow\uparrow & = \downarrow\uparrow - \downarrow\uparrow, \\
\uparrow\downarrow & = 0, \\
\circ & = 1_1. \\
\end{align*}
\]

Here the other crossings are defined by

\[
\begin{align*}
\downarrow\uparrow & := \downarrow\uparrow, \\
\uparrow\downarrow & := \uparrow\downarrow.
\end{align*}
\]

The relations \((\star)\) imply that \( \downarrow\uparrow \cong \downarrow\downarrow \oplus 1 \).
The Heisenberg category

One can prove that we have the following bubble slide relations:

\[ \text{\includegraphics[width=1cm]{up.pdf}} + \text{\includegraphics[width=1cm]{down.pdf}} = \text{\includegraphics[width=1cm]{up_up.pdf}} \quad \text{and} \quad \text{\includegraphics[width=1cm]{down_down.pdf}} = \text{\includegraphics[width=1cm]{down_up.pdf}} - \text{\includegraphics[width=1cm]{down_down.pdf}}. \]

Therefore

\[ \text{\includegraphics[width=1cm]{up_down.pdf}} = \text{\includegraphics[width=1cm]{up_up.pdf}} - \text{\includegraphics[width=1cm]{up_down.pdf}}. \]

Let \( \mathcal{H}eis_{\uparrow\downarrow} \) be the full \( \mathbb{k} \)-linear monoidal subcategory of \( \mathcal{H}eis \) generated by \( \uparrow\downarrow \).

Since the clockwise bubble is strictly central in \( \mathcal{H}eis_{\uparrow\downarrow} \), we can define \( \mathcal{H}eis_{\uparrow\downarrow}(t) \) to be the quotient of \( \mathcal{H}eis_{\uparrow\downarrow} \) by the additional relation

\[ \text{\includegraphics[width=1cm]{up.pdf}} = t \mathbb{1}. \]
Action of the Heisenberg category

There is a \( \mathbb{k} \)-linear monoidal functor

\[
\Theta : \text{Heis} \to \prod_{m \in \mathbb{N}} \left( \bigoplus_{n \in \mathbb{N}} (S_n, S_m)\text{-bimod} \right).
\]

For \( 0 \leq m \leq n \), we have a natural embedding \( S_m \subseteq S_n \). Let

\[
(n)_m = \mathbb{k}S_n, \text{ as a } (S_n, S_m)\text{-bimodule},
\]

\[
m(n) = \mathbb{k}S_n, \text{ as a } (S_m, S_n)\text{-bimodule}.
\]

On objects,

\[
\Theta(\uparrow) = \bigoplus_{n \geq 1} (n)_{n-1}, \quad \Theta(\downarrow) = \bigoplus_{n \geq 1} n_{n-1}(n).
\]

On morphisms

\[
\Theta \left( \begin{array}{c} \downarrow \\ \uparrow \end{array} \right) = \left( (n)_{n-2} \to (n)_{n-2}, \ g \mapsto gs_{n-1} \right)_{n \geq 2},
\]

and \( \Theta \) sends cups/caps to the bimodule homomorphisms making \( \Theta(\uparrow) \) and \( \Theta(\downarrow) \) into a biadjoint pair (Frobenius reciprocity).
Restricting to $\mathcal{H}eis_{\uparrow \downarrow}$ gives a functor

$$\Theta: \mathcal{H}eis_{\uparrow \downarrow} \to \bigoplus_{m \in \mathbb{N}} (S_m, S_m)\text{-bimod}.$$ 

Recall

$$1_n = \text{trivial } S_n\text{-module}.$$ 

Consider the composition

$$\mathcal{H}eis_{\uparrow \downarrow} \xrightarrow{\Theta} \bigoplus_{m \in \mathbb{N}} (S_m, S_m)\text{-bimod} \xrightarrow{-\otimes S_n 1_n} S_n\text{-mod}.$$ 

This factors through $\mathcal{H}eis_{\uparrow \downarrow}(n)$ to give an action functor

$$\Omega_n: \mathcal{H}eis_{\uparrow \downarrow}(n) \to S_n\text{-mod}.$$
Theorem (Nyobe Likeng–Ryba–S. 2019)

There is a faithful strict linear monoidal functor \( \Psi_t : \mathcal{P}ar(t) \hookrightarrow \mathcal{H}eis_{\uparrow\downarrow}(t) \) defined on objects by

\[ k \mapsto (\uparrow\downarrow)^k \]

and on generating morphisms by

\[
\begin{align*}
\begin{array}{c}
\bullet \\
\circ \\
\circ
\end{array}
\rightsquigarrow
\begin{array}{c}
\downarrow \\
\circ
\end{array}
, \\
\begin{array}{c}
\circ \\
\bullet
\end{array}
\rightsquigarrow
\begin{array}{c}
\circ \\
\uparrow \\
\circ
\end{array}
, \\
\begin{array}{c}
\bullet
\end{array}
\rightsquigarrow
\begin{array}{c}
\circ \\
\circ
\end{array}
, \\
\begin{array}{c}
\circ
\end{array}
\rightsquigarrow
\begin{array}{c}
\circ \\
\circ
\end{array}
, \\
\begin{array}{c}
\bullet
\end{array}
\rightsquigarrow
\begin{array}{c}
\circ \\
\circ
\end{array}
. \\
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\rightsquigarrow
\begin{array}{c}
\downarrow \\
\circ \\
\circ
\end{array}
+ \\
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\rightsquigarrow
\begin{array}{c}
\circ \\
\downarrow \\
\circ
\end{array}
.
\end{align*}
\]
Recall that
\[
\text{Rep}(S_t) = \text{Kar}(\mathcal{P} ar(t)).
\]

**Corollary**

There is a faithful strict linear monoidal functor
\[
\text{Rep}(S_t) \hookrightarrow \text{Kar}(\mathcal{H}eis_{\uparrow\downarrow}(t)).
\]

When $t$ is generic (i.e. we work over $\mathbb{k}[t]$), we can extend our embedding to
\[
\Psi_t : \text{Rep}(S_t) \hookrightarrow \text{Kar}(\mathcal{H}eis).
\]
Compatibility of the actions

Recall that

\[ V = \mathbb{k}^n \] is the permutation representation of \( S_n \),
\[ 1_n = \mathbb{k} \] is the trivial representation of \( S_n \).

Let \( v_1, \ldots, v_n \) be the standard basis of \( V \).

The elements

\[ g_i := s_i s_{i+1} \cdots s_{n-1}, \quad i = 1, \ldots, n, \]
form a complete set of \( S_n/S_{n-1} \)-coset representatives.

We have an isomorphism of \( S_n \)-modules

\[ \mathbb{k} S_n \otimes_{S_{n-1}} 1_n \xrightarrow{\sim} V, \quad g_i \otimes 1 \mapsto v_i = g_i v_n. \]

In other words, \( \text{Ind}_{n-1}^n \text{Res}_{n-1}^n 1_n = \text{Ind}_{n-1}^n 1_{n-1} \cong V. \)
Compatibility of the actions

More generally, we have an isomorphism of $S_n$-modules

$$V \otimes^k \cong \mathbb{k} S_n \otimes S_{n-1} \cdots \otimes S_{n-1} \mathbb{k} S_n \otimes S_{n-1} 1_n,$$

$\star$ factors

\hspace{1cm}

**Theorem (Nyobe Likeng–S. 2019)**

Fix $n \in \mathbb{N}$, and consider the following functors:

$$\mathcal{P}ar(n) \xrightarrow{\Psi_n} \mathcal{H}eis_{\uparrow \downarrow}(n)$$

$\Phi_n$

$$\Omega_n$$

$S_n$-mod

The morphisms $\star$ give a natural isomorphism of functors $\Omega_n \circ \Psi_n \cong \Phi_n$. 
Grothendieck rings

If $C$ is an additive linear monoidal category, let

$$K_0(C) = \text{split Grothendieck ring of } C$$

$$= \text{Span}_\mathbb{Z}(\text{isom classes of objects in } C)/(\left[ X \oplus Y \right] - \left[ X \right] - \left[ Y \right]).$$

Multiplication in $K_0(C)$ is given by

$$[X][Y] = [X \otimes Y].$$

Our embedding

$$\Psi_t : \text{Rep}(S_t) \to \text{Kar}(\mathcal{H}eis)$$

induces a ring homomorphism

$$[\Psi_t] : K_0(\text{Rep}(S_t)) \to K_0(\text{Kar}(\mathcal{H}eis)).$$

Goal: Describe this ring homomorphism algebraically.
The Heisenberg algebra

Let

\[ \text{Sym} = \text{ring of symmetric functions with coefficients in } \mathbb{Z}. \]

The infinite-dimensional Heisenberg Lie algebra \( \mathfrak{h} \) is the Lie algebra over \( \mathbb{Q} \) with

- generators \( c, p_n^\pm, n \geq 1 \),
- relations

\[
[p_m^-, p_n^-] = [p_m^+, p_n^+] = [c, p^\pm_n] = 0, \quad [p_m^+, p_n^-] = \delta_{m,n} nc.
\]

Then central reduction \( U(\mathfrak{h})/(c + 1) \) is the Heisenberg double \( \text{Sym}_\mathbb{Q} \#_\mathbb{Q} \text{Sym}_\mathbb{Q} \) with respect to the pairing

\[
\langle -, - \rangle : \text{Sym}_\mathbb{Q} \times \text{Sym}_\mathbb{Q}, \quad \langle p_m, p_n \rangle = \delta_{m,n} n.
\]

We have

\[
\text{Sym}_\mathbb{Q} \#_\mathbb{Q} \text{Sym}_\mathbb{Q} \cong \text{Sym}_\mathbb{Q} \otimes_\mathbb{Q} \text{Sym}_\mathbb{Q} \text{ as } \mathbb{Q}\text{-modules.}
\]
The Heisenberg algebra

We can restrict to obtain a \( \mathbb{Z} \)-form

\[
\text{Heis} := \text{Sym} \#_{\mathbb{Z}} \text{Sym} \cong \text{Sym} \otimes_{\mathbb{Z}} \text{Sym} \quad (\text{as } \mathbb{Z}\text{-modules}).
\]

For \( f \in \text{Sym} \), let

\[
f^- := f \otimes 1 \quad f^+ := 1 \otimes f.
\]

Then

\[
s^+_\lambda s^-_\mu, \quad \lambda, \mu \in \mathcal{P},
\]

is a \( \mathbb{Z} \)-basis for \( \text{Heis} \), where

- \( \mathcal{P} \) is the set of partitions,
- \( s_\lambda \) is the Schur function corresponding to \( \lambda \in \mathcal{P} \).

Theorem (Brundan–S.–Webster 2018, Conjecture by Khovanov 2014)

There is an explicit isomorphism of rings

\[
\text{Heis} \cong K_0(\text{Kar}(\mathcal{H}\text{eis})).
\]
The Kronecker coproduct is

$$\Delta_{\text{Kr}} : \text{Sym}_Q \to \text{Sym}_Q \otimes_Q \text{Sym}_Q,$$

$$\Delta_{\text{Kr}} (p_\lambda) = p_\lambda \otimes p_\lambda, \quad \lambda \in \mathcal{P},$$

where

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell},$$

and $p_n$ is the $n$-th power sum.

Restriction gives a coproduct

$$\Delta_{\text{Kr}} : \text{Sym} \to \text{Sym} \otimes_{\mathbb{Z}} \text{Sym}.$$
Assume \( \text{char } k = 0 \). There is an isomorphism of Hopf algebras

\[
\bigoplus_{n=0}^{\infty} K_0(S_n\text{-mod}) \cong \text{Sym}, \quad [S_\lambda] \mapsto s_\lambda, \quad (\clubsuit)
\]

where \( S_\lambda \) is the Specht module.

Consider the diagonal embedding

\[
kS_n \rightarrow kS_n \otimes_k kS_n.
\]

Under \((\clubsuit)\), the functor

\[
S_n\text{-mod} \rightarrow (S_n \times S_n)\text{-mod}, \quad M \mapsto \text{Ind}^{S_n \times S_n}_{S_n}(M),
\]

corresponds precisely to \( \Delta_{Kr} \) after passing to Grothendieck groups.
Grothendieck rings

Recall our embedding functor

$$\Psi_t : \text{Rep}(S_t) \hookrightarrow \text{Kar}(H e i s).$$

This induces a ring homomorphism

$$[\Psi_t] : K_0(\text{Rep}(S_t)) \rightarrow K_0(\text{Kar}(H e i s)) \cong \text{Heis}.$$ 

As \(\mathbb{Z}\)-modules,

$$\text{Heis} \cong \text{Sym} \otimes_{\mathbb{Z}} \text{Sym}.$$ 

**Theorem (Nyobe Likeng–S. 2019)**

The map \([\Psi_t]\) is injective and

$$[\Psi_t] (K_0(\text{Rep}(S_t))) = \Delta_{K_r}(\text{Sym}) \subseteq \text{Heis}.$$
Future directions: Frobenius generalization

To any Frobenius (super)algebra $A$, one can define a Frobenius Heisenberg category (Rosso–S. 2017)

$$\mathcal{Heis}_A.$$ 

This corresponds to

symmetric groups $S_n \rightsquigarrow$ wreath algebras $A^\otimes n \rtimes S_n$.

Then

$$K_0(\mathcal{Heis}_A) \cong \text{lattice Heisenberg algebra corr. to } K_0(A).$$

Generalizing above work should relate these to colored partition algebras and wreath Deligne categories.

Work in progress (S. Nyobe–Likeng).
**McKay correspondence:** finite subgroups $\Gamma \subseteq SL_2(\mathbb{C})$ parametrized by simply-laced affine Dynkin diagrams (affine type $ADE$).

To $\Gamma$, one can associate a *zigzag algebra* $A$.

**Cautis–Licata 2012:** $H_{\text{eis}} A$ acts on derived category of coherent sheaves on Hilbert schemes of points on the resolution $\hat{\mathbb{C}^2}/\Gamma$ of the quotient $\mathbb{C}^2/\Gamma$.

**Work in progress** (Reeks–S.): Compute traces of $H_{\text{eis}} A$. Should be related to AGT correspondence.
Future directions: Quantum version

There is a quantum Heisenberg category (Licata–S. 2013, Brundan–S.–Webster 2018)

\[ \mathcal{H}eis(z, t). \]

This corresponds to symmetric groups \( S_n \) \( \rightsquigarrow \) Iwahori–Hecke algebras of type \( A \).

Still have (conjecturally)

\[ K_0(\mathcal{H}eis(z, t)) \cong \text{Heis}. \]

Should be related to \( q \)-partition algebras and a quantum analogue of Deligne’s category.

Work in progress (Y. Moussaid).
Future directions: General central charge

The Heisenberg categories above are central charge $-1$.

In general, for a central charge $k \in \mathbb{Z}$, we have Heisenberg categories

$$H_{eis_k}, \quad H_{eis_A, k}, \quad H_{eis_A, k}(z, t).$$

Corresponds to replacing symmetric groups by

- degenerate cyclotomic Hecke algebras,
- cyclotomic wreath algebras,
- cyclotomic quantum wreath algebras.

It would be interesting to generalize above picture to arbitrary level.