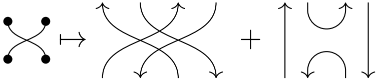
Embedding Deligne's category $\mathrm{Rep}(S_t)$ in the Heisenberg category



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Preprint: 1905.05620 (with S. Nyobe Likeng and C. Ryba)

Outline

Goal: Define an embedding from the partition category and Deligne's category $\mathrm{Rep}(S_t)$ into the Heisenberg category.

Overview:

1. Strict monoidal categories & string diagrams
2. The partition category & Deligne's category $\mathrm{Rep}(S_t)$
3. The Heisenberg category
4. The embedding
5. Future directions

Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.

$\Bbbk$-linear monoidal categories

Fix a commutative ground ring $\Bbbk$.

A strict $\Bbbk$-linear monoidal category is a strict monoidal category such that

- each morphism space is a $\Bbbk$-module,
- composition of morphisms is $\Bbbk$-bilinear,
- tensor product of morphisms is $\Bbbk$-bilinear.

The interchange law

The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

$$\begin{tikzcd} A_1 \otimes B_1 \arrow[r, "1\otimes g"] \arrow[d, "f\otimes 1"'] \arrow[dr, "f\otimes g"] & A_1 \otimes B_2 \arrow[d, "f\otimes 1"] \\ A_2 \otimes B_1 \arrow[r, "1\otimes g"'] & A_2 \otimes B_2 \end{tikzcd}$$

Strict monoidal categories

Example (Monoids)

A (strict) monoidal category with one object is simply a commutative monoid. More precisely, the endomorphisms of $\mathbb{1}$ form a commutative monoid.

Conversely, every commutative monoid gives rise to a one-object monoidal category.

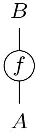
Example (Associative algebras)

A (strict) $\Bbbk$-linear monoidal category with one object is simply a commutative associative unital $\Bbbk$-algebra.

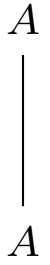
String diagrams

Let's draw pictures! Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:



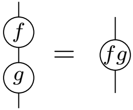
The identity map $1_A\colon A \to A$ is a string with no label:

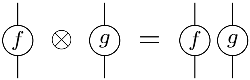


We sometimes omit the object labels when they are clear or unimportant.

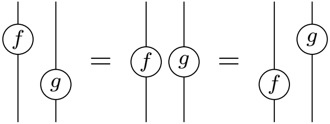
String diagrams

Composition is vertical stacking and tensor product is horizontal juxtaposition:

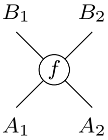




The interchange law then becomes:



A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



Partition algebras: Motivation

Let $V = \mathbb{C}^n$ be the permutation representation of the symmetric group S_n.

We have a Schur–Weyl type duality

$$\mathbb{C}S_n \to \operatorname{End}_{\mathbb{C}}(V^{\otimes k}) \xleftarrow{g} P_k(n),$$

where $P_k(n)$ is the partition algebra.

In particular:

- the image of g is $\operatorname{End}_{\mathbb{C}S_n}(V^{\otimes k})$,
- g is injective if and only if $2k \leq n$.

We can describe $P_k(n)$ in terms of partition diagrams.

Partition diagrams

For $k, \ell \in \mathbb{N}$, a partition of type $\binom{\ell}{k}$ is a partition of the set

$$\{1, \dots, k, 1', \dots, \ell'\}.$$

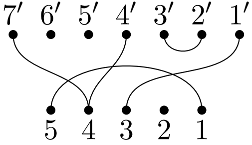
Consider the associated partition diagram.

The partition diagram of

$$\big\{\{1,5\},\{2\},\{3,1'\},\{4,4',7'\},\{2',3'\},\{5'\},\{6'\}\big\} \text{ of type } \binom{7}{5}$$

is

$D =$



$: 5 \to 7.$

We often omit the labels of the vertices.

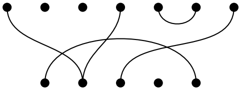
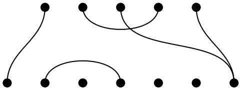
Partition diagrams

We have unique partition diagrams of types $\binom{1}{0}$ and $\binom{0}{1}$:

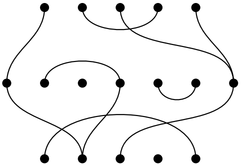
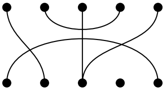
 $: 0 \to 1$ and  $: 1 \to 0.$

Fix a commutative ring $\Bbbk$ and $t \in \Bbbk$.

Composition: We compose partition diagrams by vertical stacking. If

$D' =$  and $D =$ 

then

$\begin{matrix} D \\ D' \end{matrix} =$ , $\quad D \circ D' = t^2$ .

The exponent of t is the number of connected components in the "center".

The partition category

The partition category is the strict $\Bbbk$-linear monoidal category $\mathcal{P}ar(t)$ with:

- Objects: $k \in \mathbb{N}$
- Morphisms: $\mathrm{Hom}_{\mathcal{P}ar(t)}(k,\ell) = \mathrm{Span}_{\Bbbk}\left\{\text{partition diag. of type } \binom{\ell}{k}\right\}$
- Composition: $\circ$ (vertical stacking)
- Tensor product: horizontal juxtaposition

Then

$$P_k(t) = \mathrm{End}_{\mathcal{P}ar(t)}(k)$$

is the partition algebra.

Note: We do not require $t \in \mathbb{N}$.

Presentations of strict monoidal categories

One can give presentations of some strict $\Bbbk$-linear monoidal categories, just as for monoids, groups, algebras, etc.

Objects: If the objects are generated by some collection A_i, $i \in I$, then we have all possible tensor products of these objects:

$$\mathbb{1}, \quad A_i, \quad A_i \otimes A_j \otimes A_k \otimes A_\ell, \quad \text{etc.}$$

Morphisms: If the morphisms are generated by some collection f_j, $j \in J$, then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

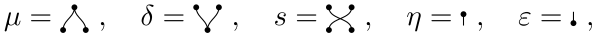
$$1_{A_i}, \quad f_j \otimes (f_i \circ f_k) \otimes (f_\ell), \quad \text{etc.}$$

We then often impose some relations on these morphism spaces.

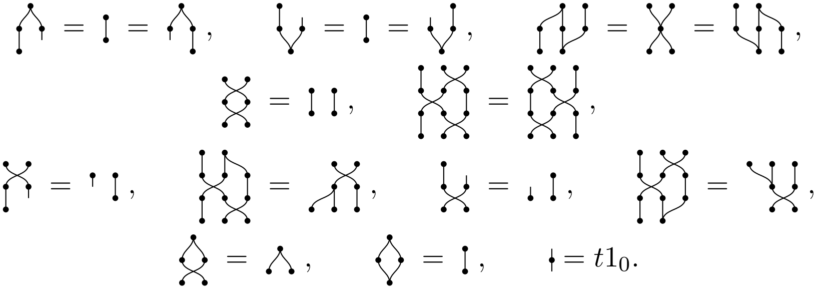
String diagrams: We can build complex diagrams out of our simple generating diagrams.

Presentation of the partition category

$\mathcal{P}ar(t)$ is generated by the object 1 and the morphisms



subject to the relations:



Consequence: $\mathcal{P}ar(t)$ is the free $\Bbbk$-linear symmetric monoidal category generated by a t-dimensional special commutative Frobenius object.

Action of the partition category

Suppose $\Bbbk$ is a field. Let

- $V = \Bbbk^n$ be the permutation representation of S_n,
- $\mathbf{1}_n$ be the one-dimensional trivial S_n-module.

There is a strong monoidal functor

$$\Phi_n \colon \mathcal{P}ar(n) \to S_n\text{-mod}$$

given by $\Phi_n(1) = V$ and

$$\begin{aligned}
\Phi_n(\mu) &\colon V \otimes V \to V, & v_i \otimes v_j &\mapsto \delta_{i,j} v_i, \\
\Phi_n(\eta) &\colon \mathbf{1}_n \to V, & 1 &\mapsto \textstyle\sum_{i=1}^n v_i, \\
\Phi_n(\delta) &\colon V \to V \otimes V, & v_i &\mapsto v_i \otimes v_i, \\
\Phi_n(\varepsilon) &\colon V \to \mathbf{1}_n, & v_i &\mapsto 1, \\
\Phi_n(s) &\colon V \otimes V \to V \otimes V, & v_i \otimes v_j &\mapsto v_j \otimes v_i
\end{aligned}$$

where $v_1, \dots, v_n$ is the standard basis of $V = \Bbbk^n$.

Action of the partition category

Proposition (Comes 2016)

1. $\Phi_n \colon \mathcal{P}ar(n) \to S_n$-mod is full.
2. The induced map

$$\mathrm{Hom}_{\mathcal{P}ar(n)}(k, \ell) \to \mathrm{Hom}_{S_n}(V^{\otimes k}, V^{\otimes \ell})$$

is an isomorphism if and only if $k + \ell \leq n$.

This is a generalization of the Schur–Weyl duality property of the partition algebra mentioned earlier.

Additive envelope

Suppose $\mathcal{C}$ is some $\Bbbk$-linear monoidal category.

Its additive envelope is the category whose:

- objects are formal finite direct sums $\bigoplus_i X_i$ of objects X_i in $\mathcal{C}$,
- morphisms

$$f \colon \bigoplus_{i=1}^{n} X_i \to \bigoplus_{j=1}^{m} Y_j$$

  are $m \times n$ matrices, where the (j,i)-entry is a morphism

$$f_{i,j} \colon X_i \to Y_j.$$

  Composition is given by matrix multiplication.

Idempotent completion

Definition: idempotent completion (Karoubi envelope)

The idempotent completion of a category $\mathcal{C}$ is the category whose

- objects are pairs (A, e) where $A \in \operatorname{Ob}\mathcal{C}$ and $e \in \operatorname{Mor}_{\mathcal{C}}(A, A)$ is an idempotent ($e^2 = e$), and
- morphisms from (A, e) to (B, f) are elements of $f \operatorname{Mor}_{\mathcal{C}}(A, B) e$.

Intuition: One thinks of passing to the idempotent completion as adding in objects such that the idempotents correspond to projections onto direct summands.

$$\begin{tikzcd} A \cong X \oplus Y \arrow[r, two heads] \arrow[rr, bend left, "e"] & X \arrow[r, hook] & X \oplus Y \cong A \end{tikzcd}$$

We let $\operatorname{Kar}(\mathcal{C})$ denote the Karoubi envelope of the additive envelope of $\mathcal{C}$.

Deligne's category $\mathrm{Rep}(S_t)$

We define

$$\mathrm{Rep}(S_t) := \mathrm{Kar}(\mathcal{P}ar(t)).$$

Notes

- $\mathrm{Rep}(S_t)$ is semisimple when $t \notin \mathbb{N}$.
- When $t = n \in \mathbb{N}$, S_n-mod is the quotient of $\mathrm{Rep}(S_t)$ by the negligible morphisms.
- $\mathrm{Rep}(S_t)$ "interpolates" between the categories $\mathrm{Rep}(S_n)$.

Intuition: $\mathrm{Rep}(S_t)$ describes the representation of the S_n in a uniform way, but with n fixed.

The Heisenberg category: Motivation

Geissinger (1970s) constructed an isomorphism of bialgebras

$$\bigoplus_{n=0}^{\infty} K_0(S_n\text{-mod}) \cong \mathsf{Sym}.$$

Multiplication given by

$$[\mathrm{Ind}] \colon K_0(S_n\text{-mod}) \otimes K_0(S_m\text{-mod}) \to K_0(S_{n+m}\text{-mod}).$$

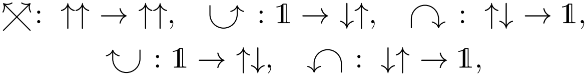
Comultiplication given by restriction.

Operators of mult. by $[K] \in K_0(S_n\text{-mod})$, together with their adjoints, defines an action of the infinite rank Heisenberg algebra, and $\bigoplus_{n=0}^{\infty} K_0(S_n\text{-mod})$ is the Fock space module.

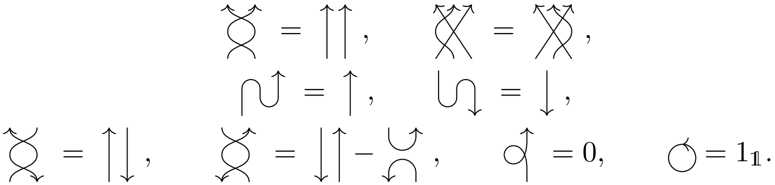
Further study of these ideas leads to the Heisenberg category.

The Heisenberg category

The Heisenberg category $\mathcal{H}eis$ (Khovanov 2014) is the strict $\Bbbk$-linear monoidal category generated by two objects $\uparrow$, $\downarrow$, morphisms



and relations



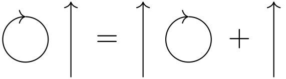
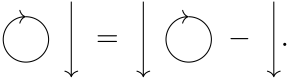
$(\star)$

Here the other crossings are defined by

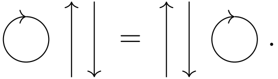
The relations $(\star)$ imply that $\downarrow\uparrow \cong \uparrow\downarrow \oplus \mathbb{1}$.

The Heisenberg category

One can prove that we have the following bubble slide relations:

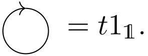

and


Therefore



Let $\mathcal{H}eis_{\uparrow\downarrow}$ be the full $\Bbbk$-linear monoidal subcategory of $\mathcal{H}eis$ generated by $\uparrow\downarrow$.

Since the clockwise bubble is strictly central in $\mathcal{H}eis_{\uparrow\downarrow}$, we can define $\mathcal{H}eis_{\uparrow\downarrow}(t)$ to be the quotient of $\mathcal{H}eis_{\uparrow\downarrow}$ by the additional relation



Action of the Heisenberg category

There is a $\Bbbk$-linear monoidal functor

$$\Theta\colon \mathcal{H}eis \to \prod_{m\in\mathbb{N}} \left(\bigoplus_{n\in\mathbb{N}} (S_n, S_m)\text{-bimod} \right).$$

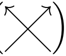
For $0 \le m \le n$, we have a natural embedding $S_m \subseteq S_n$. Let

$$\begin{aligned} (n)_m &= \Bbbk S_n, \text{ as a } (S_n, S_m)\text{-bimodule}, \\ {}_m(n) &= \Bbbk S_n, \text{ as a } (S_m, S_n)\text{-bimodule}. \end{aligned}$$

On objects,

$$\Theta(\uparrow) = \bigoplus_{n\ge 1} (n)_{n-1}, \qquad \Theta(\downarrow) = \bigoplus_{n\ge 1} {}_{n-1}(n).$$

On morphisms

$\Theta\Big($  $\Big) = \Big((n)_{n-2} \to (n)_{n-2},\ g \mapsto g s_{n-1} \Big)_{n\ge 2},$

and Θ sends cups/caps to the bimodule homomorphisms making $\Theta(\uparrow)$ and $\Theta(\downarrow)$ into a biadjoint pair (Frobenius reciprocity).

Action of the Heisenberg category

Restricting to $\mathcal{H}eis_{\uparrow\downarrow}$ gives a functor

$$\Theta \colon \mathcal{H}eis_{\uparrow\downarrow} \to \bigoplus_{m \in \mathbb{N}} (S_m, S_m)\text{-bimod}.$$

Recall

$$\mathbf{1}_n = \text{trivial } S_n\text{-module}.$$

Consider the composition

$$\mathcal{H}eis_{\uparrow\downarrow} \xrightarrow{\Theta} \bigoplus_{m \in \mathbb{N}} (S_m, S_m)\text{-bimod} \xrightarrow{- \otimes_{S_n} \mathbf{1}_n} S_n\text{-mod}.$$

This factors through $\mathcal{H}eis_{\uparrow\downarrow}(n)$ to give an action functor

$$\Omega_n \colon \mathcal{H}eis_{\uparrow\downarrow}(n) \to S_n\text{-mod}.$$

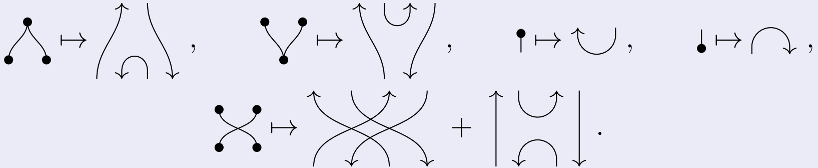
The embedding functor

Theorem (Nyobe Likeng–Ryba–S. 2019)

There is a faithful strict linear monoidal functor $\Psi_t \colon \mathcal{P}ar(t) \hookrightarrow \mathcal{H}eis_{\uparrow\downarrow}(t)$ defined on objects by

$$k \mapsto (\uparrow\downarrow)^k$$

and on generating morphisms by



The embedding functor

Recall that

$$\mathrm{Rep}(S_t) = \mathrm{Kar}(\mathcal{P}ar(t)).$$

Corollary

There is a faithful strict linear monoidal functor

$$\mathrm{Rep}(S_t) \hookrightarrow \mathrm{Kar}(\mathcal{H}eis_{\uparrow\downarrow}(t)).$$

When t is generic (i.e. we work over $\Bbbk[t]$), we can extend our embedding to

$$\Psi_t \colon \mathrm{Rep}(S_t) \hookrightarrow \mathrm{Kar}(\mathcal{H}eis).$$

Compatibility of the actions

Recall that

$$V = \Bbbk^n \text{ is the permutation representation of } S_n,$$
$$\mathbf{1}_n = \Bbbk \text{ is the trivial representation of } S_n.$$

Let $v_1, \dots, v_n$ be the standard basis of V.

The elements

$$g_i := s_i s_{i+1} \cdots s_{n-1}, \quad i = 1, \dots, n,$$

form a complete set of S_n/S_{n-1}-coset representatives.

We have an isomorphism of S_n-modules

$$\Bbbk S_n \otimes_{S_{n-1}} \mathbf{1}_n \xrightarrow{\cong} V, \quad g_i \otimes 1 \mapsto v_i = g_i v_n.$$

In other words, $\mathrm{Ind}_{n-1}^n \mathrm{Res}_{n-1}^n \mathbf{1}_n = \mathrm{Ind}_{n-1}^n \mathbf{1}_{n-1} \cong V$.

Compatibility of the actions

More generally, we have an isomorphism of S_n-modules

$$V^{\otimes k} \xrightarrow{\cong} \underbrace{\Bbbk S_n \otimes_{S_{n-1}} \cdots \otimes_{S_{n-1}} \Bbbk S_n}_{k \text{ factors}} \otimes_{S_{n-1}} \mathbf{1}_n, \qquad (✠)$$

Theorem (Nyobe Likeng–S. 2019)

Fix $n \in \mathbb{N}$, and consider the following functors:

$$\begin{tikzcd} \mathcal{P}ar(n) \arrow[r, "\Psi_n"] \arrow[dr, "\Phi_n"'] & \mathcal{H}eis_{\uparrow\downarrow}(n) \arrow[d, "\Omega_n"] \\ & S_n\text{-mod} \end{tikzcd}$$

The morphisms (✠) give a natural isomorphism of functors $\Omega_n \circ \Psi_n \cong \Phi_n$.

Grothendieck rings

If $\mathcal{C}$ is an additive linear monoidal category, let

$$\begin{aligned} K_0(\mathcal{C}) &= \text{split Grothendieck ring of } \mathcal{C} \\ &= \mathrm{Span}_{\mathbb{Z}}(\text{isom classes of objects in } \mathcal{C})/([X \oplus Y] - [X] - [Y]). \end{aligned}$$

Multiplication in $K_0(\mathcal{C})$ is given by

$$[X][Y] = [X \otimes Y].$$

Our embdedding

$$\Psi_t \colon \mathrm{Rep}(S_t) \to \mathrm{Kar}(\mathcal{H}eis)$$

induces a ring homomorphism

$$[\Psi_t] \colon K_0(\mathrm{Rep}(S_t)) \to K_0(\mathrm{Kar}(\mathcal{H}eis)).$$

Goal: Describe this ring homomorphism algebraically.

The Heisenberg algebra

Let

$$\mathsf{Sym} = \text{ring of symmetric functions with coefficients in } \mathbb{Z}.$$

The infinite-dimensional Heisenberg Lie algebra $\mathfrak{h}$ is the Lie algebra over $\mathbb{Q}$ with

- generators c, $p_n^\pm$, $n \ge 1$,
- relations

$$[p_m^-, p_n^-] = [p_m^+, p_n^+] = [c, p_n^\pm] = 0, \quad [p_m^+, p_n^-] = \delta_{m,n} nc.$$

Then central reduction $U(\mathfrak{h})/(c+1)$ is the Heisenberg double $\mathsf{Sym}_\mathbb{Q} \#_\mathbb{Q} \mathsf{Sym}_\mathbb{Q}$ with respect to the pairing

$$\langle -, - \rangle \colon \mathsf{Sym}_\mathbb{Q} \times \mathsf{Sym}_\mathbb{Q}, \quad \langle p_m, p_n \rangle = \delta_{m,n} n.$$

We have

$$\mathsf{Sym}_\mathbb{Q} \#_\mathbb{Q} \mathsf{Sym}_\mathbb{Q} \cong \mathsf{Sym}_\mathbb{Q} \otimes_\mathbb{Q} \mathsf{Sym}_\mathbb{Q} \quad \text{as } \mathbb{Q}\text{-modules}.$$

The Heisenberg algebra

We can restrict to obtain a $\mathbb{Z}$-form

$$\mathrm{Heis} := \mathsf{Sym}\#_{\mathbb{Z}}\mathsf{Sym} \cong \mathsf{Sym} \otimes_{\mathbb{Z}} \mathsf{Sym} \quad \text{(as } \mathbb{Z}\text{-modules).}$$

For $f \in \mathsf{Sym}$, let

$$f^- := f \otimes 1 \qquad f^+ := 1 \otimes f.$$

Then

$$s_\lambda^+ s_\mu^-, \quad \lambda, \mu \in \mathcal{P},$$

is a $\mathbb{Z}$-basis for Heis, where

- $\mathcal{P}$ is the set of partitions,
- s_λ is the Schur function corresponding to $\lambda \in \mathcal{P}$.

Theorem (Brundan–S.–Webster 2018, Conjecture by Khovanov 2014)

There is an explicit isomorphism of rings

$$\mathrm{Heis} \cong K_0(\mathrm{Kar}(\mathcal{H}eis)).$$

Kronecker coproduct

The Kronecker coproduct is

$$\Delta_{\mathrm{Kr}} \colon \mathsf{Sym}_{\mathbb{Q}} \to \mathsf{Sym}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathsf{Sym}_{\mathbb{Q}}, \quad \Delta_{\mathrm{Kr}}(p_\lambda) = p_\lambda \otimes p_\lambda, \quad \lambda \in \mathcal{P},$$

where

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell},$$

and p_n is the n-th power sum.

Restriction gives a coproduct

$$\Delta_{\mathrm{Kr}} \colon \mathsf{Sym} \to \mathsf{Sym} \otimes_{\mathbb{Z}} \mathsf{Sym}.$$

Kronecker coproduct

Assume $\operatorname{char} \Bbbk = 0$. There is an isomorphism of Hopf algebras

$$\bigoplus_{n=0}^{\infty} K_0(S_n\text{-mod}) \cong \mathsf{Sym}, \quad [S_\lambda] \mapsto s_\lambda, \tag{$\clubsuit$}$$

where S_λ is the Specht module.

Consider the diagonal embedding

$$\Bbbk S_n \to \Bbbk S_n \otimes_\Bbbk \Bbbk S_n.$$

Under ($\clubsuit$), the functor

$$S_n\text{-mod} \to (S_n \times S_n)\text{-mod}, \quad M \mapsto \operatorname{Ind}_{S_n}^{S_n \times S_n}(M),$$

corresponds precisely to Δ_{Kr} after passing to Grothendieck groups.

Grothendieck rings

Recall our embedding functor

$$\Psi_t \colon \operatorname{Rep}(S_t) \hookrightarrow \operatorname{Kar}(\mathcal{H}eis).$$

This induces a ring homomorphism

$$[\Psi_t] \colon K_0(\operatorname{Rep}(S_t)) \to K_0(\operatorname{Kar}(\mathcal{H}eis)) \cong \mathrm{Heis}.$$

As $\mathbb{Z}$-modules,

$$\mathrm{Heis} \cong \mathsf{Sym} \otimes_{\mathbb{Z}} \mathsf{Sym}.$$

Theorem (Nyobe Likeng–S. 2019)

The map $[\Psi_t]$ is injective and

$$[\Psi_t]\,(K_0(\operatorname{Rep}(S_t))) = \Delta_{\mathrm{Kr}}(\mathsf{Sym}) \subseteq \mathrm{Heis}.$$

Future directions: Frobenius generalization

To any Frobenius (super)algebra A, one can define a Frobenius Heisenberg category (Rosso–S. 2017)

$$\mathcal{H}eis_A.$$

This corresponds to

$$\text{symmetric groups } S_n \rightsquigarrow \text{wreath algebras } A^{\otimes n} \rtimes S_n.$$

Then

$$K_0(\mathcal{H}eis_A) \cong \text{lattice Heisenberg algebra corr. to } K_0(A).$$

Generalizing above work should relate these to colored partition algebras and wreath Deligne categories.

Work in progress (S. Nyobe–Likeng).

Connections to Hilbert schemes and quiver varieties

McKay correspondence: finite subgroups $\Gamma \subseteq SL_2(\mathbb{C})$ parametrized by simply-laced affine Dynkin diagrams (affine type ADE).

To Γ, one can associate a zigzag algebra A.

Cautis–Licata 2012: $\mathcal{H}eis_A$ acts on derived category of coherent sheaves on Hilbert schemes of points on the resolution $\widehat{\mathbb{C}^2/\Gamma}$ of the quotient $\mathbb{C}^2/\Gamma$.

Work in progress (Reeks–S.): Compute traces of $\mathcal{H}eis_A$. Should be related to AGT correspondence.

Future directions: Quantum version

There is a quantum Heisenberg category (Licata–S. 2013, Brundan–S.–Webster 2018)

$$\mathcal{H}eis(z,t).$$

This corresponds to

$$\text{symmetric groups } S_n \rightsquigarrow \text{Iwahori–Hecke algebras of type } A.$$

Still have (conjecturally)

$$K_0(\mathcal{H}eis(z,t)) \cong \mathrm{Heis}.$$

Should be related to q-partition algebras and a quantum analogue of Deligne's category.

Work in progress (Y. Moussaid).

Future directions: General central charge

The Heisenberg categories above are central charge -1.

In general, for a central charge $k \in \mathbb{Z}$, we have Heisenberg categories

$$\mathcal{H}eis_k, \qquad \mathcal{H}eis_{A,k}, \qquad \mathcal{H}eis_{A,k}(z,t).$$

Corresponds to replacing symmetric groups by

- degenerate cyclotomic Hecke algebras,
- cyclotomic wreath algebras,
- cyclotomic quantum wreath algebras.

It would be interesting to generalize above picture to arbitrary level.