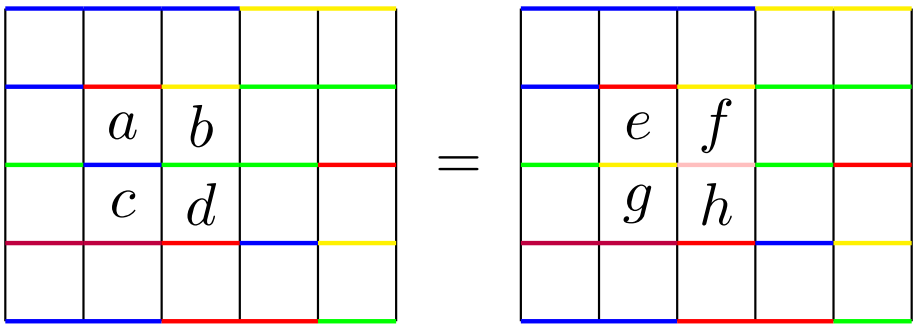
Two-dimensional algebra



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Outline

Goal: Introduce a two-dimensional approach to algebra by playing with blocks.

Overview:

1. Traditional (one-dimensional) algebra
2. Two-dimensional algebra
3. Being precise: monoidal categories
4. Going a bit further

Monoids

Monoid

A monoid is a set M with a binary operation

$$M \times M \to M, \quad (a, b) \mapsto ab,$$

that

- is associative, so $(ab)c = a(bc)$ for all $a, b, c \in M$;
- has an identity element 1, so $1a = a = a1$ for all $a \in M$.

Associativity means we can omit parentheses. E.g. we can write:

$$ab^2ca1ba^3$$

Think of lining up boxes:

a	b	b	c	a	1	b	a	a	a

Games with blocks

We have relations; some sequences of boxes are equal.

Some relations are forced by the monoid axioms:

$$\boxed{1}\boxed{a} = \boxed{a} = \boxed{a}\boxed{1}$$

This relation is local; we can use it inside larger sequences of blocks:

$$\boxed{a}\boxed{b}\boxed{b}\boxed{c}\boxed{a}\boxed{1}\boxed{b}\boxed{a}\boxed{a}\boxed{a} = \boxed{a}\boxed{b}\boxed{b}\boxed{c}\boxed{a}\boxed{b}\boxed{a}\boxed{a}\boxed{a}$$

Particular monoids may have other relations. E.g. if a monoid is commutative, we have the local relation:

$$\boxed{a}\boxed{b} = \boxed{b}\boxed{a} \qquad \text{for all } a, b$$

Note

The axiom of associativity is built into our framework—we just omit parentheses.

More games with blocks

Presentations

We can define a monoid by giving a presentation:

- a set S of generators (the blocks we can use),
- a set R of (local) relations.

Example

Consider the monoid with

- one generator a,
- one relation $a^2 = 1$.

With blocks, we have $\boxed{a}\boxed{a} = \boxed{1}$.

This monoid is the cyclic group with two elements.

Recall: A group is a monoid where every element a has an inverse a^{-1}, so $aa^{-1} = 1 = a^{-1}a$. This is just a type of relation.

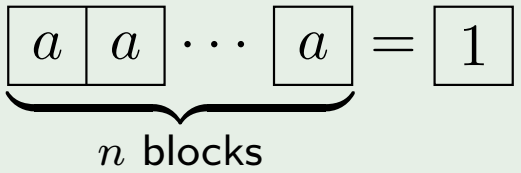
More games with blocks

Example

Consider the monoid with

- one generator a,
- one relation $a^n = 1$.

With blocks, we have:



This monoid is the cyclic group with n elements.

More games with blocks

Example

Fix $n \geq 3$ and consider the monoid with two generators r, s and relations:

$$\underbrace{\boxed{r}\boxed{r}\cdots\boxed{r}}_{n \text{ blocks}} = \boxed{1}\,, \quad \boxed{s}\boxed{s} = \boxed{1}\,, \quad \boxed{r}\boxed{s}\boxed{r}\boxed{s} = \boxed{1}$$

This is the dihedral group (group of symmetries of a regular n-gon).

More games with blocks

Example

Fix $n \geq 1$ and consider the monoid with generators

$$s_1, \ldots, s_{n-1},$$

and relations

$$\boxed{s_i}\boxed{s_i} = \boxed{1}\,, \qquad 1 \leq i \leq n-1,$$

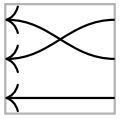
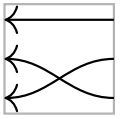
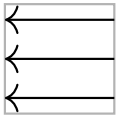
$$\boxed{s_i}\boxed{s_{i+1}}\boxed{s_i} = \boxed{s_{i+1}}\boxed{s_i}\boxed{s_{i+1}}\,, \qquad 1 \leq i \leq n-2,$$

$$\boxed{s_i}\boxed{s_j} = \boxed{s_j}\boxed{s_i}\,, \qquad 1 \leq i, j \leq n-1,\ |i-j| > 1.$$

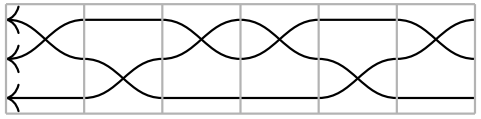
This is the symmetric group S_n.

From blocks to strings

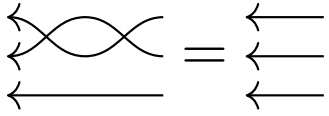
In the symmetric group S_3, let's replace our blocks by string diagrams:

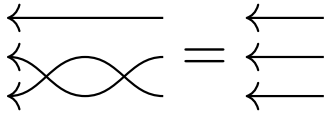
$\boxed{s_1} =$  , $\boxed{s_2} =$  $\boxed{1} =$ 

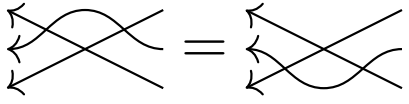
Lining up blocks becomes concatenation of string diagrams:



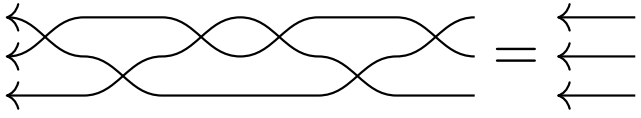
As for relations:

$\boxed{s_1}\boxed{s_1} = \boxed{1}$ becomes 

$\boxed{s_2}\boxed{s_2} = \boxed{1}$ becomes 

$\boxed{s_1}\boxed{s_2}\boxed{s_1} = \boxed{s_2}\boxed{s_1}\boxed{s_2}$ becomes 

Exercise:



Set maps

By definition, the symmetric group S_n consists of bijections

$$\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

The monoid operation is composition. Every bijection is invertible, so S_n is a group.

We could also consider the monoid of all set maps

$$\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

This is no longer a group, but it still a monoid. We can still use blocks:

$$\boxed{f}\boxed{g} = \boxed{f \circ g}$$

Exercise: Come up with a natural way of drawing string diagrams for this monoid.

Composition problems

What if we want to consider maps between different sets?

For $n \geq 0$, let

$$X_n = \{1, 2, \ldots, n\}.$$

By convention, $X_0 = \varnothing$.

Let

$$\mathcal{C}(m,n) = \{f \colon X_m \to X_n\} \quad \text{and} \quad \mathcal{C} = \bigcup_{m,n=0}^{\infty} \mathcal{C}(m,n).$$

Question: Does the operation of composition make this a monoid?

NO! Composition is not always defined. E.g. if

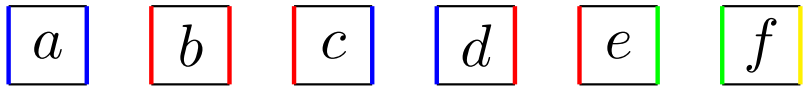
$$f \colon X_2 \to X_3 \quad \text{and} \quad g \colon X_5 \to X_4$$

then $f \circ g$ is not defined.

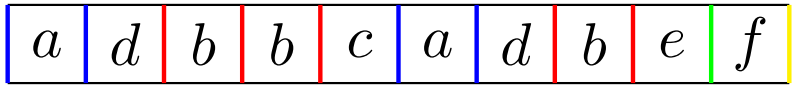
Colourful boxes

So $\mathcal{C}$ is not a monoid. So our boxes description breaks down.

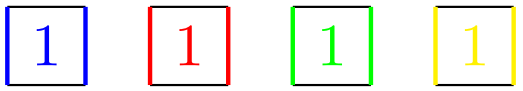
How can we fix it? Colour the sides of boxes!



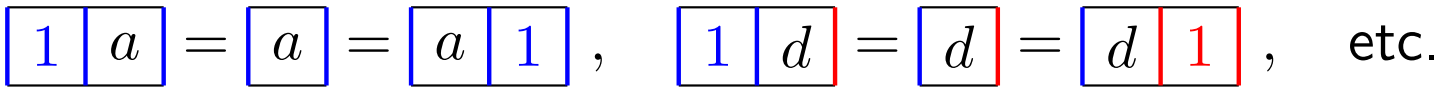
Now we can only compose boxes where the sides match:



The identity element 1 in a monoid is replaced by an identity for each colour:



These satisfy



Back to sets

Recall

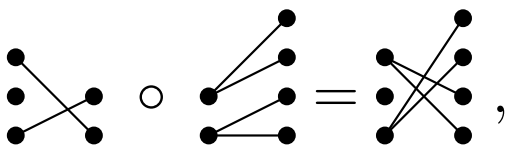
$$\mathcal{C}(m,n) = \{f : X_m \to X_n\} \quad \text{and} \quad \mathcal{C} = \bigcup_{m,n=0}^{\infty} \mathcal{C}(m,n).$$

Now we depict $f : X_m \to X_n$ as a box with labelled/coloured sides

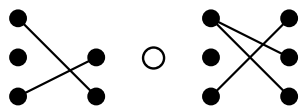
$$n \boxed{f} m$$

We can only compose boxes (i.e. maps) when the labels match.

If we use string diagrams for set maps ($\circ$ = composition):



but



not defined

Making things precise

Category

A (small) category $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}\,\mathcal{C}$ of objects ("colours"),
- for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$, a set of morphisms $\mathcal{C}(X, Y)$ ("boxes with edges coloured X and Y"),

together with composition maps

$$\circ \colon \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

such that

- composition is associative (don't worry about parentheses),
- each $X \in \mathrm{Ob}\,\mathcal{C}$ has a identity 1_X such that

$$1_Y \circ f = f = f \circ 1_X \quad \text{for all } f \in \mathcal{C}(X, Y).$$

Sets

Recall our example

$$\mathcal{C}(m,n) = \{f : X_m \to X_n\} \quad \text{and} \quad \mathcal{C} = \bigcup_{m,n=0}^{\infty} \mathcal{C}(m,n).$$

This is a category with

- Objects: X_n, $n \geq 0$
- For $X_n, X_m \in \mathrm{Ob}\,\mathcal{C}$, the set of morphisms from X_m to X_n is $\mathcal{C}(m,n)$.

Composition is usual composition of set maps.

- Composition is associative.
- The identity $1_{X_n} : X_n \to X_n$ is the identity map.

Categories

Example (Sets)

- Objects: sets
- Morphisms: set maps

Example (Vector spaces)

- Objects: real vector spaces
- Morphisms: linear maps

Example (Groups)

- Objects: groups
- Morphisms: group homomorphisms

Categories

Example (Rings)

- Objects: rings
- Morphisms: ring homomorphisms

Example (Topological spaces)

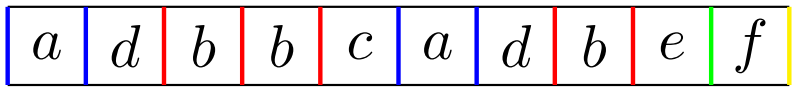
- Objects: topological spaces
- Morphisms: continuous maps

Other examples

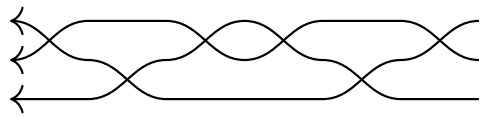
- modules over a fixed ring
- smooth manifolds
- algebraic varieties
- ...

Stuck in one dimension

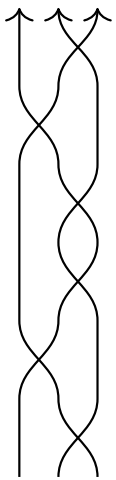
So far, everything is one-dimensional. We compose blocks in a line:



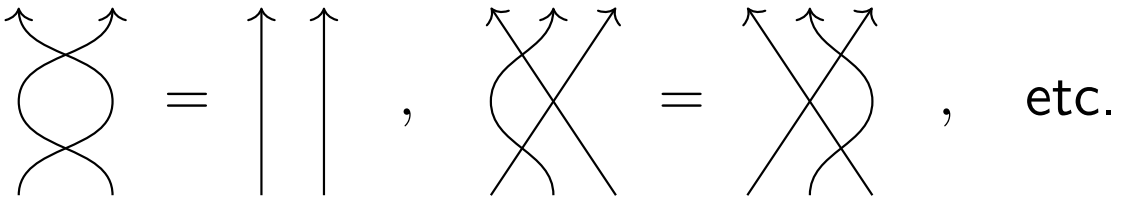
Or string diagrams:



From now on, we'll rotate these pictures clockwise $90°$:



Relations :



But we're still stuck in one dimension—only have vertical composition $\circ$.

Moving to two dimensions

Let's introduce a new horizontal associative composition $\otimes$!

$$\begin{array}{|c|}\hline e \\ \hline b \\ \hline d \\ \hline a \\ \hline \end{array} \otimes \begin{array}{|c|}\hline f \\ \hline e \\ \hline d \\ \hline c \\ \hline \end{array} = \begin{array}{|c|c|}\hline e & f \\ \hline b & e \\ \hline d & d \\ \hline a & c \\ \hline \end{array} \tag{1}$$

How should we interpret the right-hand side? We have objects

blue, red, green, yellow, etc.

However, (1) seems to be a morphism

blue-red $\to$ green-yellow.

So we need a way of "combining colours".

Precisely, we want an associative map

$$\otimes\colon \operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{C}.$$

Moving to two dimensions

We want the associative map

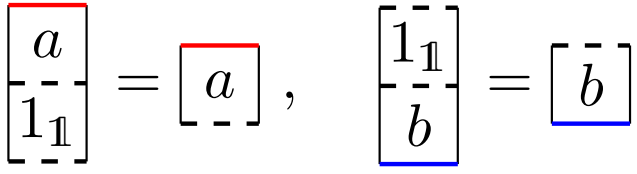
$$\otimes\colon \operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{C}$$

to have an identity. Precisely, we want an identity object $\mathbb{1}$ such that

$$\mathbb{1} \otimes X = X = X \otimes \mathbb{1} \quad \text{for all } X \in \operatorname{Ob}\mathcal{C}.$$

Let's denote this identity object by a dashed edge, e.g.

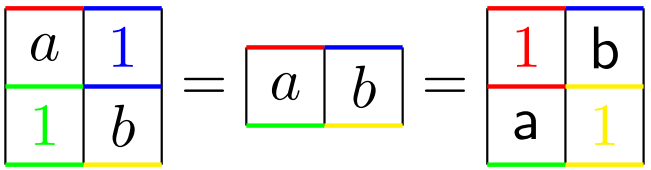
The identity object also has an identity morphism $1_{\mathbb{1}}\colon \mathbb{1} \to \mathbb{1}$:



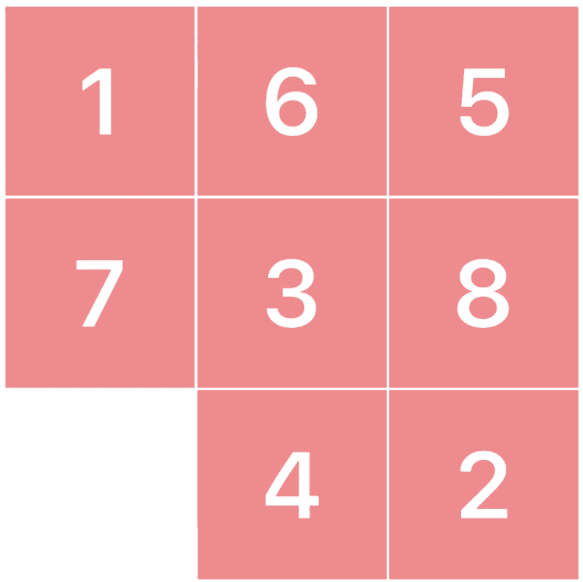
Moving to two dimensions

We want $1_{\mathbb{1}}$ to also be an identity for horizontal composition:

For technical reasons, we also want the interchange relation:

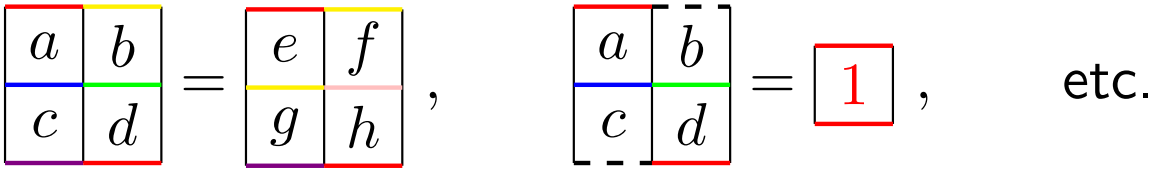


You can think of identities a bit like an empty slot in a slide puzzle:

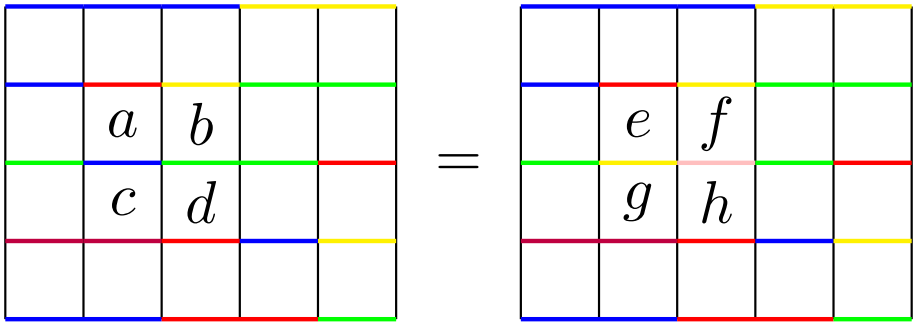


2D local relations

We can now impose two-dimensional relations:



We can use these relations anywhere in a wall of blocks:



Making things precise

Strict monoidal category

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for all objects A, B, C,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all objects A, B, C,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$ for all objects A,

and, for all morphisms a, b, c,

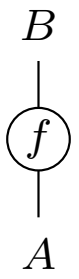
- $(a \otimes b) \otimes c = a \otimes (b \otimes c)$,
- $1_{\mathbb{1}} \otimes a = a = 1_{\mathbb{1}} \otimes a$.

$\circ$: vertical composition

$\otimes$: horizontal composition

String diagrams

We will denote a morphism $f : A \to B$ by:

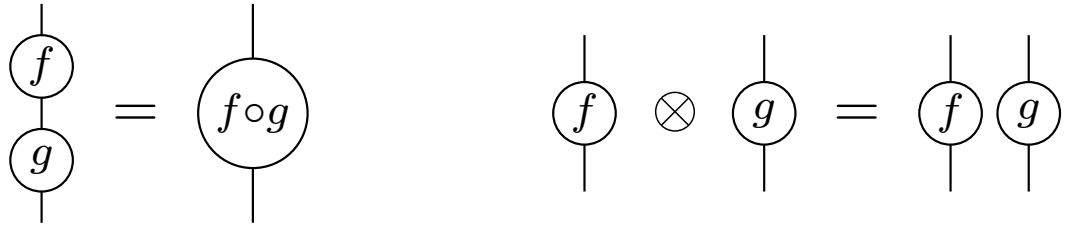


The identity map $1_A : A \to A$ is a string with no label:

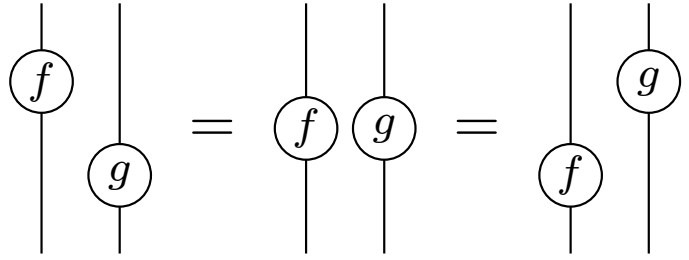
We sometimes omit the object labels when they are clear or unimportant.

String diagrams

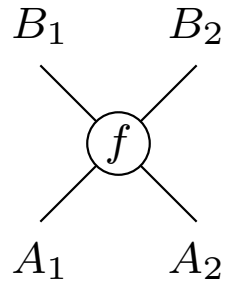
Composition is vertical stacking and tensor product is horizontal juxtaposition:



The interchange law then becomes:



A morphism $f : A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



Presentations of strict monoidal categories

One can give presentations of strict monoidal categories, just as for monoids, groups, etc.

Objects: If the objects are generated by some collection A_i, $i \in I$, then we have all possible tensor products of these objects:

$$\mathbf{1}, \quad A_i, \quad A_i \otimes A_j \otimes A_k \otimes A_\ell, \quad \text{etc.}$$

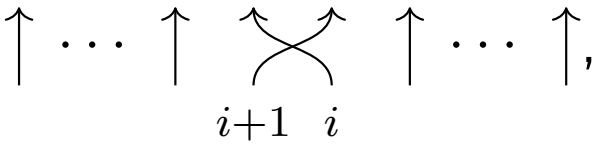
Morphisms: If the morphisms are generated by some collection f_j, $j \in J$, then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

$$1_{A_i}, \quad f_j \otimes (f_i f_k) \otimes (f_\ell), \quad \text{etc.}$$

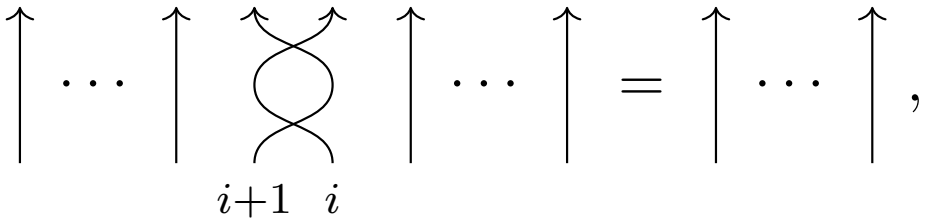
We then often impose some relations on these morphism spaces.

String diagrams: We can build complex diagrams out of our simple generating diagrams.

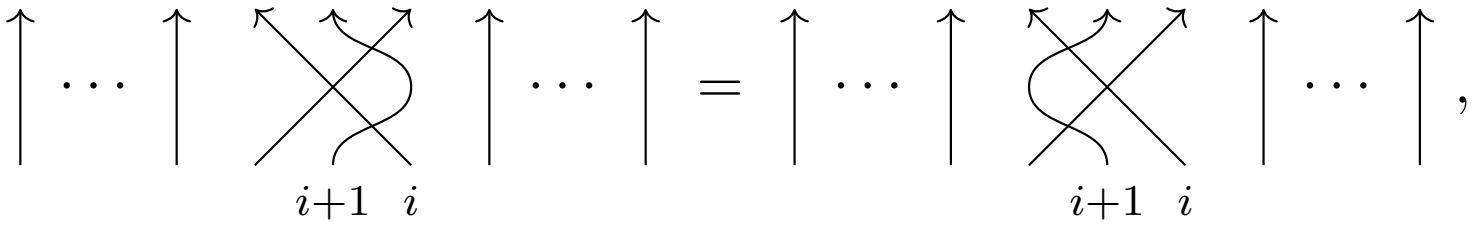
Recall the "one-dimensional" symmetric group S_n

We needed $n-1$ generators $s_i =$ 

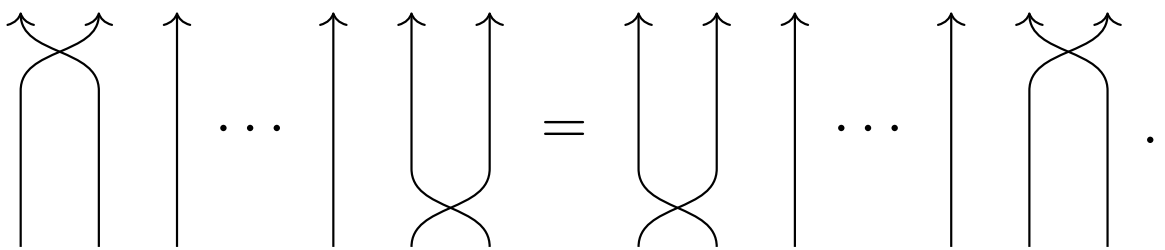
the $n-1$ quadratic relations



the $n-2$ braid relations



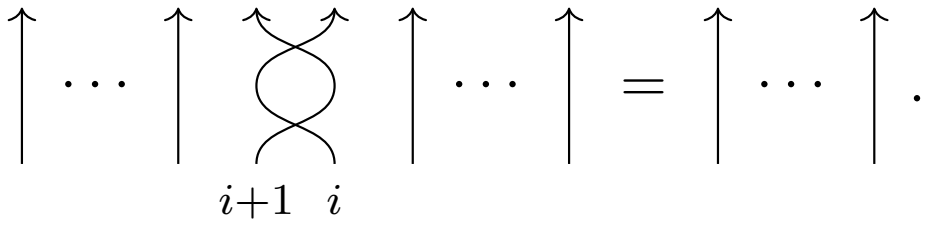
and the (order n^2) distant braid relations



Recall the "one-dimensional" symmetric group S_n

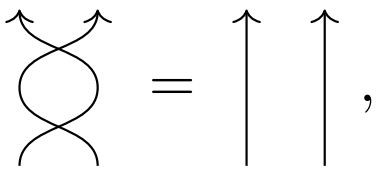
Too many generators!!

Many of the relations are just "shifts" of each other, e.g.



If we can compose horizontally, we just need one generator

and one relation



and similarly for the other relations.

Monoidally generated symmetric groups

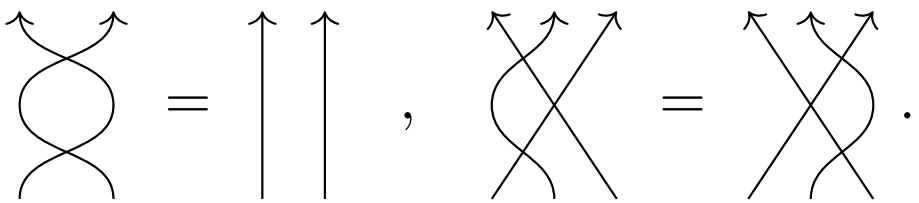
Define a strict monoidal category $\mathcal{S}$ with one generating object X and denote

$$1_X = \uparrow$$

We have one generating morphism


$: X \otimes X \to X \otimes X.$

We impose the relations:



Then

$$\mathcal{S}(X^{\otimes n}, X^{\otimes n}) = S_n$$

is the symmetric group on n letters.

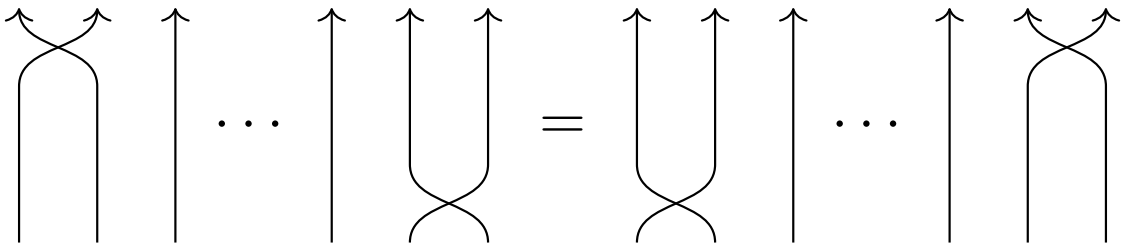
Monoidally generated symmetric groups

This monoidal presentation of S_n is very efficient! We only needed

- one generating morphism, and
- two relations,

to get all the symmetric groups.

Note that the distant braid relation follows for free from the interchange law:



Monoidal categories in the wild

Example (Sets I)

- Objects: finite sets
- Morphisms: set maps
- Vertical composition: composition of set maps
- Horizontal composition: cartesian product
- Unit object $\mathbb{1}$: one element set $\{\star\}$

Example (Sets II)

- Objects: finite sets
- Morphisms: set maps
- Vertical composition: composition of set maps
- Horizontal composition: disjoint union
- Unit object $\mathbb{1}$: empty set $\varnothing$

Monoidal categories in the wild

Example (Vector spaces)

- Objects: $\mathbb{R}$-vector spaces
- Morphisms: linear transformations
- Vertical composition: composition of linear transformations
- Horizontal composition: tensor product $\otimes$
- Unit object $\mathbb{1}$: $\mathbb{R}$

Example (Abelian groups)

- Objects: abelian groups
- Morphisms: group homomorphisms
- Vertical composition: composition of group homomorphisms
- Horizontal composition: tensor product $\otimes_{\mathbb{Z}}$
- Unit object $\mathbb{1}$: $\mathbb{Z}$

Example (Rings)

- Objects: rings
- Morphisms: ring homomorphisms
- Vertical composition: composition of ring homomorphisms
- Horizontal composition: tensor product $\otimes_{\mathbb{Z}}$
- Unit object $\mathbb{1}$: $\mathbb{Z}$

Technicality

The above examples are not really strict monoidal categories. E.g. for sets X, Y, Z, we don't have $(X \times Y) \times Z = X \times (Y \times Z)$.

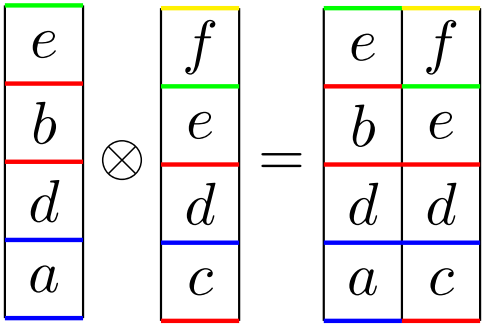
However, we have a natural isomorphism

$$(X \times Y) \times Z \cong X \times (Y \times Z).$$

So the above are ~~strict~~ monoidal categories. Every monoidal category is equivalent to a strict one.

More colours please!

Recall our horizontal composition



Why are only the horizontal edges coloured?

Indeed, we can colour the vertical edges as well:

Now we can only compose horizontally when colours match (just like for vertical composition)!

2-categories

2-category

- objects (vertical edges of boxes),
- 1-morphisms (horizontal edges of boxes),
- 2-morphisms (boxes).

Example

Bimodules over rings

- Objects: rings
- 1-Morphisms: bimodules
- 2-morphisms: bimodule homomorphisms.

A monoidal category is a 2-category with one object (one vertical edge colour):

- 1-morphisms of 2-category $\rightsquigarrow$ objects of monoidal category
- 2-morphisms of 2-category $\rightsquigarrow$ morphisms of monoidal category