Heisenberg categorification

\[ \sum_{b \in B} b \lor b^\vee \]

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Goals:
1. Explain diagrammatic categorification
2. Describe a family of categories that categorify the Heisenberg algebra

Overview:
1. Strict monoidal categories and string diagrams
2. Monoidally presented algebras
3. Adjunction and pivotal categories
4. The Frobenius Heisenberg category
A **strict monoidal category** is a category $\mathcal{C}$ equipped with
- a bifunctor (the tensor product) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $1$,

such that
- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all objects $A$, $B$, $C$,
- $1 \otimes A = A = A \otimes 1$ for all objects $A$.

**Remark: Non-strict monoidal categories**

In a (not necessarily strict) **monoidal category**, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.
$\mathbb{k}$-linear monoidal categories

Fix a commutative ground ring $\mathbb{k}$.

A strict $\mathbb{k}$-linear monoidal category is a strict monoidal category such that
- each morphism space is a $\mathbb{k}$-module,
- composition of morphisms is $\mathbb{k}$-bilinear,
- tensor product of morphisms is $\mathbb{k}$-bilinear.

The interchange law

The axioms of a strict monoidal category imply the **interchange law**: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

\[
\begin{array}{c}
A_1 \otimes B_1 & \xrightarrow{1 \otimes g} & A_1 \otimes B_2 \\
\downarrow f \otimes 1 & & \downarrow f \otimes 1 \\
A_2 \otimes B_1 & \xrightarrow{1 \otimes g} & A_2 \otimes B_2 \\
\end{array}
\]

\[
f \otimes g
\]

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Example (Monoids)

A (strict) monoidal category with one object is simply a commutative monoid. More precisely, the endomorphisms of 1 form a commutative monoid.

Conversely, every commutative monoid gives rise to a one-object monoidal category.

Example (Associative algebras)

A (strict) $\mathbb{k}$-linear monoidal category with one object is simply a commutative associative unital $\mathbb{k}$-algebra.
Categorification via split Grothendieck group

Suppose $\mathcal{C}$ is an additive category (i.e. have $\oplus$).

$\text{Iso}_\mathbb{Z}(\mathcal{C}) =$ free abelian group generated by isom. classes of objects in $\mathcal{C}$.

The split Grothendieck group of $\mathcal{C}$ is

$$K_0(\mathcal{C}) = \text{Iso}_\mathbb{Z}(\mathcal{C}) / \langle [X \oplus Y] = [X] + [Y] \mid X, Y \in \mathcal{C} \rangle.$$ 

If $\mathcal{C}$ is monoidal, then $K_0(\mathcal{C})$ is a ring:

$$[X] \cdot [Y] = [X \otimes Y].$$

Categorification

For our purposes, to categorify a ring $R$ is to find an additive monoidal category $\mathcal{C}$ such that

$$K_0(\mathcal{C}) \cong R \text{ as rings.}$$
The Heisenberg algebra

Let $\mathfrak{h}$ be the infinite-dimensional Heisenberg Lie algebra.

Thus, $\mathfrak{h}$ is the complex Lie algebra with basis

$$\{ c, q_n^\pm : n \geq 1 \}$$

and product

$$[q_m^+, q_n^+] = [q_m^-, q_n^-] = [c, q_n^\pm] = 0, \quad [q_m^+, q_n^-] = \delta_{m,n} nc.$$ 

The associative Heisenberg algebra at central charge $\xi \in \mathbb{Z}$ is

$$U(\mathfrak{h})/\langle c - \xi \rangle.$$ 

We will describe categories that categorify these algebras.
String diagrams

Let’s draw pictures! Fix a strict monoidal category $C$.

We will denote a morphism $f : A \to B$ by:

![Diagram of a morphism $f : A \to B$](image)

The identity map $1_A : A \to A$ is a string with no label:

![Diagram of the identity map $1_A : A \to A$](image)

We sometimes omit the object labels when they are clear or unimportant.
String diagrams

Composition is **vertical stacking** and tensor product is **horizontal juxtaposition**:

\[
\begin{array}{c}
\begin{array}{c}
f \\
g
\end{array}
\end{array}
= \begin{array}{c}
f g \\
\end{array}
\begin{array}{c}
f \\
\otimes \\
g
\end{array}
= \begin{array}{c}
f g \\
\end{array}
\]

The **interchange law** then becomes:

\[
\begin{array}{c}
\begin{array}{c}
f \\
g
\end{array}
\end{array}
= \begin{array}{c}
f g \\
\end{array}
\begin{array}{c}
f \\
\otimes \\
g
\end{array}
= \begin{array}{c}
g \\
\end{array}
\]

A morphism \( f : A_1 \otimes A_2 \rightarrow B_1 \otimes B_2 \) can be depicted:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B_1 \\
\begin{array}{c}
\begin{array}{c}
f \\
\end{array}
\end{array}
\end{array}
A_1 \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B_2 \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
f \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
A_2
\end{array}
\end{array}
\end{array}
\end{array}
\]
Presentations of strict monoidal categories

One can give presentations of some strict \( k \)-linear monoidal categories, just as for monoids, groups, algebras, etc.

**Objects:** If the objects are generated by some collection \( A_i, i \in I \), then we have all possible tensor products of these objects:

\[
1, \quad A_i, \quad A_i \otimes A_j \otimes A_k \otimes A_\ell, \quad \text{etc.}
\]

**Morphisms:** If the morphisms are generated by some collection \( f_j, j \in J \), then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

\[
1_{A_i}, \quad f_j \otimes (f_i f_k) \otimes (f_\ell), \quad \text{etc.}
\]

We then often impose some relations on these morphism spaces.

**String diagrams:** We can build complex diagrams out of our simple generating diagrams.
Define a strict monoidal category $\mathcal{S}$ with one generating object $Q_+$ and denote

$$1_{Q_+} = \uparrow$$

We have one generating morphism

$$\begin{array}{ccc}
\begin{array}{ccc}
\textbf{X} & : & Q_+ \otimes Q_+ \rightarrow Q_+ \otimes Q_+. \\
\end{array}
\end{array}$$

We impose the relations:

$$\begin{array}{ccc}
\begin{array}{ccc}
\textbf{X} & = & \uparrow \uparrow \\
\textbf{X} & = & \begin{array}{ccc}
\textbf{X}
\end{array}
\end{array}
\end{array}$$

It is straightforward to verify that

$$\text{End}_\mathcal{S}(Q_+^\otimes n) = S_n$$

is the symmetric group on $n$ letters.
Monoidally generated symmetric groups

This monoidal presentation of $S_n$ is very efficient! We only needed

- one generating morphism, and
- two relations,

to get all the symmetric groups.

Note that the “distant braid relation”

$$s_i s_j = s_j s_i, \quad |i - j| > 1$$

for simple transpositions follows for free from the interchange law:

\[
\begin{array}{ccccc}
\uparrow & \uparrow & \cdots & \uparrow & \uparrow \\
\downarrow & \downarrow & & \downarrow & \downarrow \\
\end{array}
\quad = \quad
\begin{array}{ccccc}
\uparrow & \uparrow & \cdots & \uparrow & \uparrow \\
\uparrow & \uparrow & & \uparrow & \uparrow \\
\end{array}
\]

Note: If we define $S$ to be $k$-linear, then $\text{End}_S(Q^\otimes n) = kS_n$. 
Start again with the strict $\kappa$-linear monoidal category $S$, but add a morphism:

$$\uparrow : Q_+ \to Q_+$$

We impose the additional relation:

$$\begin{array}{ccc}
\begin{array}{ccc}
\circ & \to & \circ \\
\downarrow & & \downarrow \\
& & \\
\end{array}
- 
\begin{array}{ccc}
\circ & \to & \circ \\
\downarrow & & \downarrow \\
& & \\
\end{array}
= 
\begin{array}{ccc}
\uparrow & & \uparrow \\
\end{array}
\end{array}.$$  

Now

$$\text{End}(Q_+^\otimes n)$$

is the degenerate affine Hecke algebra (of type $A$).
The wreath product category

Fix an associative $k$-algebra $F$. We add an endomorphism of $Q_+$ for each element of $F$.

More precisely, let $\mathcal{W}(F)$ be the strict $k$-linear monoidal category obtained from $S$ by adding morphisms such that we have an algebra homomorphism:

$$F \to \text{End } Q_+, \quad f \mapsto \bullet f$$

We impose the additional relations:

$$f \bullet \bullet = f, \quad f \in F$$

Example of diagrammatic proof:
The wreath product category

\[
\text{End}_{\mathcal{W}(F)}(Q^n_+) = F^n \rtimes S_n
\]

is a wreath product algebra.

As a vector space,

\[
F^n \rtimes S_n = F^n \otimes_k kS_n.
\]

Multiplication is determined by

\[
(f_1 \otimes \pi_1)(f_2 \otimes \pi_2) = f_1(\pi_1 \cdot f_2) \otimes \pi_1 \pi_2,
\]

where \(\pi_1 \cdot f_2\) denotes the natural action of \(S_n\) on \(F^n\) by permutation of the factors.

Note: \(\mathcal{W}(k) = S\), the symmetric group category.

Want: An affine version of the wreath product category. \(F = k\) should recover the degenerate affine Hecke category.
A Frobenius algebra is a f.d. associative algebra $F$ together with a linear trace map

$$\text{tr}: F \to k$$

such that the induced map

$$F \to \text{Hom}_k(F, k), \quad f \mapsto (g \mapsto \text{tr}(gf)),$$

is an isomorphism.

For simplicity, we assume that the trace is symmetric:

$$\text{tr}(fg) = \text{tr}(gf), \quad \text{for all } f, g \in F.$$
Frobenius algebras: Examples

Example ($k$)

$k$ is a Frobenius algebra with $tr = 1_k$.

Example (Matrix algebra)

Any matrix algebra over a field is a Frobenius algebra with the usual trace.

Example ($k[x]/(x^k)$)

$k[x]/(x^k)$ is a Frobenius algebra with

$$tr(x^\ell) = \delta_{\ell,k-1}.$$
Example (Group algebra)
Suppose $G$ is a finite group.

The group algebra $\mathbb{k}G$ is a Frobenius algebra with

$$\text{tr}(g) = \delta_{g,1_G}, \quad g \in G.$$  

Example (Zigzag algebra)
Associated to every quiver is a zigzag algebra. These are Frobenius algebras.

Example (Hopf algebras)
Every f.d. Hopf algebra is a Frobenius algebra.

From now on: $F$ is a Frobenius algebra with trace $\text{tr}$. 
Fix a basis $B$ of $F$. The left dual basis is

$$B^\vee = \{ b^\vee \mid b \in B \}$$

defined by

$$\text{tr} \left( b^\vee c \right) = \delta_{b,c}, \quad b, c \in B.$$ 

It is easy to check that

$$\sum_{b \in B} b \otimes b^\vee \in F \otimes F$$

is independent of the basis $B$. 

Frobenius algebras: dual bases
Affine wreath product category

Start with the wreath product category $\mathcal{W}(F)$, but add a morphism:

$$\uparrow : Q_+ \to Q_+$$

We impose the additional relations:

$$-\quad = \sum_{b \in B} b \uparrow \quad b^\vee \quad , \quad f \uparrow = \quad f \quad , \quad f \in F$$

Call the resulting category $\mathcal{AW}(F)$ the affine wreath product category.

Now

$$\text{End}_{\mathcal{AW}(F)}(Q_+^n)$$

is an affine wreath product algebra.

Note: $\mathcal{AW}(\mathbb{K})$ is the degenerate affine Hecke category.
Suppose a strict monoidal category $\mathcal{C}$ has two objects $Q_+$ and $Q_-$, with

$$1_{Q_+} = \uparrow, \quad 1_{Q_-} = \downarrow.$$ 

A morphism $1 \to Q_- \otimes Q_+$ would have string diagram

$$
\begin{array}{c}
\vdots \\
\downarrow \\
\end{array}
\quad , \quad \text{where} \quad \vdots = 1_1.
$$

We typically omit the dotted line and draw:

$$
\begin{array}{c}
\cup \\
: \ 1 \to Q_- \otimes Q_+.
\end{array}
$$

Similarly, we can have

$$
\begin{array}{c}
\cup \\
: \ Q_+ \otimes Q_- \to 1.
\end{array}
$$
Adjunction

We say that $Q_-$ is right adjoint to $Q_+$ (and $Q_+$ is left adjoint to $Q_-$) if there exist morphisms

\[ \bigodot \colon 1 \to Q_- \otimes Q_+ \quad \text{and} \quad \bigodot \colon Q_+ \otimes Q_- \to 1 \]

such that

\[ \begin{array}{ccc} \odot & = & \downarrow \\ \downarrow & & \downarrow \end{array} \quad \text{and} \quad \begin{array}{ccc} \odot & = & \uparrow \\ \uparrow & & \uparrow \end{array} \]

(This is analogous to the unit-counit formulation of adjunction of functors.)

We say $Q_+$ and $Q_-$ are biadjoint if they are both left and right adjoint to each other. So we also have

\[ \bigodot \colon 1 \to Q_+ \otimes Q_- \quad \text{and} \quad \bigodot \colon Q_- \otimes Q_+ \to 1 \]

such that

\[ \begin{array}{ccc} \odot & = & \uparrow \\ \uparrow & & \uparrow \end{array} \quad \text{and} \quad \begin{array}{ccc} \odot & = & \downarrow \\ \downarrow & & \downarrow \end{array} \]
If $Q_-$ is right adjoint to $Q_+$, then every $f \in \text{End } Q_+$ has right mate $\overset{f}{\rightarrow} \in \text{End } Q_-$. This gives an antihomomorphism $\text{End } Q_+ \rightarrow \text{End } Q_-$. 

If $Q_-$ is left adjoint to $Q_+$, then every $f \in \text{End } Q_+$ has left mate $\overset{f}{\leftarrow} \in \text{End } Q_-$. This gives another antihomomorphism $\text{End } Q_+ \rightarrow \text{End } Q_-$. 
A strict monoidal category is **strictly pivotal** if every object has a biadjoint and right mates are always equal to left mates:

\[
\begin{array}{c}
\downarrow \quad f \\
\quad = \\
\downarrow 
\end{array} =
\begin{array}{c}
\downarrow \\
\quad f
\end{array}
\]

**Isotopy invariance**: In a strictly pivotal category, isotopic string diagrams represent the same morphism!

This allows us to use geometric intuition and topological arguments in the study of such categories.
Suppose $C$ is some $\mathbb{k}$-linear monoidal category.

Its additive envelope is the category whose:

- **objects** are formal finite direct sums $\bigoplus_i X_i$ of objects $X_i$ in $C$,
- **morphisms**

\[
f : \bigoplus_{i=1}^n X_i \to \bigoplus_{j=1}^m Y_j
\]

are $m \times n$ matrices, where the $(j, i)$-entry is a morphism

\[
f_{i,j} : X_i \to Y_j.
\]

Composition is given by matrix multiplication.
Recall the affine wreath product category $\mathcal{AW}(F)$. It is the strict $k$-linear monoidal category with:

**Objects:** Generated by object $Q_+$. 

**Morphisms:** Generated by

\[ \begin{align*}
&\begin{array}{c}
\text{\ }
\end{array}
\end{align*} : Q_+ \otimes Q_+ \to Q_+ \otimes Q_+, \\
\begin{array}{c}
\text{\ }
\end{array} : Q_+ \to Q_+, \\
\begin{array}{c}
\text{\ }
\end{array} f : Q_+ \to Q_+, \quad f \in F,
\end{align*} \]

with relations

\[\begin{align*}
&\begin{array}{c}
\text{\ }
\end{array} = \begin{array}{c}
\text{\ }
\end{array}, \\
\begin{array}{c}
\text{\ }
\end{array} = \begin{array}{c}
\text{\ }
\end{array}, \\
\begin{array}{c}
\text{\ }
\end{array} f = \begin{array}{c}
\text{\ }
\end{array} f, \quad f \in F, \\
\begin{array}{c}
\text{\ }
\end{array} - \begin{array}{c}
\text{\ }
\end{array} = \sum_{b \in B} b \begin{array}{c}
\text{\ }
\end{array} + b^\vee, \\
\begin{array}{c}
\text{\ }
\end{array} f = \begin{array}{c}
\text{\ }
\end{array} f, \quad f \in F.
\end{align*}\]

For $n \in \mathbb{N}$, define

\[\begin{array}{c}
\text{\ }
\end{array} = \begin{array}{c}
\text{\ }
\end{array} \text{\ } n \text{ dots.}\]
The Frobenius Heisenberg category

Fix a central charge $\xi \in \mathbb{Z}$, $\xi \leq 0$.

(Actually, we can take any $\xi \in \mathbb{Z}$, but we choose $\xi \leq 0$ for simplicity of exposition.)

To $\mathcal{AW}(F')$ we add another object $Q_-$ that is right adjoint to $Q_+$:

\[
\begin{align*}
\downarrow & = \downarrow \\
\uparrow & = \uparrow
\end{align*}
\]

and

\[
\begin{align*}
\begin{array}{c}
\, \\
\end{array} & \quad = \quad \\
\begin{array}{c}
\, \\
\end{array}
\end{align*}
\]

We can then define right crossings:

\[
\begin{align*}
\begin{array}{c}
\, \\
\end{array} & \quad := \quad \\
\begin{array}{c}
\, \\
\end{array}
\end{align*}
\]

: $Q_+ Q_- \rightarrow Q_- Q_+$.

(We start denoting tensor product by juxtaposition: $Q_+ Q_- := Q_+ \otimes Q_-$.)
The Frobenius Heisenberg category

We then impose the crucial inversion relation:

The following matrix of morphisms is an isomorphism in the additive envelope:

\[
\begin{bmatrix}
\begin{array}{c}
k \\
\end{array}
\end{bmatrix}^\dagger, \quad 0 \leq k \leq -\xi - 1, \quad b \in B
\]

: \mathbb{Q}_+ \mathbb{Q}_- \oplus 1^{\oplus (-\xi \dim F)} \rightarrow \mathbb{Q}_- \mathbb{Q}_+.

More precisely, we add in some other morphisms that are the matrix components of an inverse to the above morphism.

We call the resulting category \( \mathcal{H}_{\text{Heis}_F,\xi} \) the Frobenius Heisenberg category.
The Frobenius Heisenberg category

**Theorem (S. 2018)**

There are unique morphisms

\[
\begin{align*}
\uparrow \bigcup : 1 & \rightarrow Q_+ Q_-, \\
\downarrow \bigcup : Q_- Q_+ & \rightarrow 1
\end{align*}
\]  

such that the following relations hold:

\[
\begin{align*}
\begin{array}{ccc}
\begin{array}{c}
\lambda^- \\
\end{array} & = & \begin{array}{c}
\lambda^+ \\
\end{array} \\
\begin{array}{c}
\lambda^+ \\
\end{array} & = & \begin{array}{c}
\lambda^- \\
\end{array} \\
\end{array} & + & \sum_{k,s \geq 0} \sum_{a,b \in B} (k-s-2) \quad \text{if } 0 \leq r < -\xi.
\end{align*}
\]

In addition, \( H_{ Eis F, \xi} \) can be presented equivalently by replacing the inversion relation with the existence of morphisms (1) and above relations.
The Frobenius Heisenberg category

The previous theorem involves left crossings

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{left crossing}
\end{array}
\end{array}
\end{array} := \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{right crossing}
\end{array}
\end{array}
\end{array}$$

and negatively dotted bubbles

$$\begin{array}{c}
r + \xi - 1 \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{bubble}
\end{array}
\end{array}
\end{array}
\end{array} f := (-1)^{r+1} \sum_{b_1,\ldots,b_{r-1} \in B} \det \left( b_{j-1}^\vee b_j \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{bubble}
\end{array}
\end{array}
\end{array} i - j - \xi \right)^r_{i,j=1},
\end{array}$$

if $r \leq -\xi$.

**Theorem (S. 2018)**

1. The objects $Q_-$ and $Q_+$ are biadjoint.
2. The category $\mathcal{H}eis_{F,\xi}$ is strictly pivotal.
3. One can compute an infinite grassmannian relation, curl relations, bubble slide relations, and an alternating braid relation (omitted here).
Action

The category $\mathcal{H}eis_{F,\xi}$ acts naturally on modules for cyclotomic wreath product algebras. We have a chain of algebras

$$k = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots.$$  

Then

- $Q_+$ acts by induction from $A_n$-mod to $A_{n+1}$-mod,
- $Q_-$ acts by restriction from $A_n$-mod to $A_{n-1}$-mod.

The morphisms (diagrams) act by certain natural transformations.

Categorification Theorem (S. 2018)

Under a mild assumption on $F$, the category $\mathcal{H}eis_{F,\xi}$ categorifies the Heisenberg algebra at central charge $\xi \dim F$. 
### Original Heisenberg category (Khovanov)
- Morphisms were planar diagrams up to isotopy, so strictly pivotal property was part of the definition.
- Central charge $\xi = -1$ and $F = k$.

### Frobenius modification (central charge $-1$)
- For $F$ the zigzag algebra, defined by Cautis–Licata and studied in relation to geometry of the Hilbert scheme.
- General definition given in joint work with Rosso.
- Still have central charge $\xi = -1$.

### Higher central charge (Mackaay–S.)
- Generalized to higher central charge (with $F = k$).
- Again, pivotal property part of the definition.
Historical remarks

Inversion relation approach, $F = \mathbb{k}$ (Brundan)

- New approach to the definition of higher charge category (Mackaay-S.) using the inversion relation.
- Now, pivotal property is a consequence of the definition.
- Advantage: proof that category acts on modules over degenerate (cyclotomic) affine Hecke algebras is much easier. Uses a well-known Mackey-type theorem.

Current work

- Follows inversion relation approach of Brundan.
- Defines a Frobenius algebra version of higher charge category (Mackaay–S.).
- Defines a higher charge version of previous Frobenius Heisenberg category (Rosso–S.).
Summarizing the relationship between the Heisenberg categories appearing in the literature, we have:

\[ H_{\text{Heis}} F, \xi \]

- Rosso–S. Cautis–Licata \((F = \text{zigzag})\)
- Mackaay–S. Brundan (inversion)
- Khovanov

\[ \xi = -1 \]
\[ F = k \]
\[ F = k \]
\[ \xi = -1 \]
Final remarks

One can actually work in a more general setting than the one described here:

1. $F$ can be a graded Frobenius superalgebra. Then $\mathcal{H}eis_{F,\xi}$ is a strict $k$-linear graded monoidal supercategory.

2. The trace need not be symmetric. In general, there exists a Nakayama automorphism $\psi: F \to F$ such that

$$\text{tr}(fg) = (-1)^{\bar{f}\bar{g}} \text{tr}(g\psi(f)) \quad \text{for all } f, g \in F.$$ 

Then, for instance,

$$f \quad \psi(f) \quad , \quad f \in F,$$

3. Above remarks mean we can take $F$ to be the Clifford superalgebra. Then $\mathcal{H}eis_{F,\xi}$ acts on modules for affine Sergeev algebras (a.k.a. degenerate affine Hecke–Clifford algebras).