Advances in Heisenberg categorification

\[ \sum_{b \in B} b \uparrow \uparrow b^\vee \]

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Outline

Goals:

1. Describe a family of categories that categorify the Heisenberg algebra
2. Explain the relation to previous Heisenberg categories

Overview:

1. Strict monoidal categories and string diagrams
2. The Frobenius Heisenberg category
3. Actions on categories of modules
4. Work in progress: $q$-deformations
Strict monoidal categories

A strict monoidal category is a category $C$ equipped with

- a bifunctor (the tensor product) $\otimes: C \times C \to C$, and
- a unit object $1$,

such that, for objects $A$, $B$, $C$ and morphisms $f$, $g$, $h$,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $1 \otimes A = A = A \otimes 1$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_1 \otimes f = f = f \otimes 1_1$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.
A strict $\mathbb{k}$-linear monoidal category is a strict monoidal category such that
- each morphism space is a $\mathbb{k}$-module,
- composition of morphisms is $\mathbb{k}$-bilinear,
- tensor product of morphisms is $\mathbb{k}$-bilinear.

The interchange law

The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:
Categorification via split Grothendieck group

Suppose $\mathcal{C}$ is an additive category (i.e. have $\oplus$).

$Iso_{\mathbb{Z}}(\mathcal{C})$ = free abelian group generated by isom. classes of objects in $\mathcal{C}$.

The split Grothendieck group of $\mathcal{C}$ is

$$K_0(\mathcal{C}) = Iso_{\mathbb{Z}}(\mathcal{C}) / \langle [X \oplus Y] = [X] + [Y] \mid X, Y \in \mathcal{C} \rangle.$$ 

If $\mathcal{C}$ is monoidal, then $K_0(\mathcal{C})$ is a ring:

$$[X] \cdot [Y] = [X \otimes Y].$$

Categorification

For our purposes, to categorify a ring $R$ is to find an additive monoidal category $\mathcal{C}$ such that

$$K_0(\mathcal{C}) \cong R \text{ as rings.}$$
The Heisenberg algebra

Let \( \mathfrak{h} \) be the infinite-dimensional Heisenberg Lie algebra.

Thus, \( \mathfrak{h} \) is the complex Lie algebra with basis

\[ \{ c, q_n^\pm : n \geq 1 \} \]

and product

\[
[q_m^+, q_n^+] = [q_m^-, q_n^-] = [c, q_n^\pm] = 0, \quad [q_m^+, q_n^-] = \delta_{m,n} nc.
\]

The associative Heisenberg algebra at central charge \( \xi \in \mathbb{Z} \) is

\[ U(\mathfrak{h})/\langle c - \xi \rangle. \]

We will describe categories that categorify these algebras.
Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f : A \to B$ by:

![Diagram of a morphism $f : A \to B$]

The identity map $\text{id}_A : A \to A$ is a string with no label:

![Diagram of the identity map $\text{id}_A : A \to A$]

We sometimes omit the object labels when they are clear or unimportant.
String diagrams

Composition is vertical stacking and tensor product is horizontal juxtaposition:

\[
\begin{align*}
\begin{tikzpicture}
  
  \node (f) at (0,1) {$f$};
  \node (g) at (0,0) {$g$};

  \draw[->] (f) -- (g);
\end{tikzpicture}
\end{align*}
= \begin{align*}
\begin{tikzpicture}
  
  \node (fg) at (0,1) {$fg$};

\end{tikzpicture}
\end{align*}
\begin{align*}
\begin{tikzpicture}
  
  \node (f) at (0,1) {$f$};
  \node (g) at (1,1) {$g$};

  \draw[->] (f) -- (g);
\end{tikzpicture}
\end{align*}
= \begin{align*}
\begin{tikzpicture}
  
  \node (fg) at (0,1) {$fg$};
  \node (fg) at (1,1) {$fg$};

\end{tikzpicture}
\end{align*}
\begin{align*}
\begin{tikzpicture}
  
  \node (f) at (0,1) {$f$};
  \node (g) at (1,1) {$g$};

  \draw[->] (f) -- (g);
\end{tikzpicture}
\end{align*}
\]

The interchange law then becomes:

\[
\begin{align*}
\begin{tikzpicture}
  
  \node (f) at (0,1) {$f$};
  \node (g) at (0,0) {$g$};

  \draw[->] (f) -- (g);
\end{tikzpicture}
\end{align*}
= \begin{align*}
\begin{tikzpicture}
  
  \node (fg) at (0,1) {$fg$};

\end{tikzpicture}
\end{align*}
\begin{align*}
\begin{tikzpicture}
  
  \node (fg) at (0,1) {$fg$};
  \node (fg) at (0,0) {$fg$};

\end{tikzpicture}
\end{align*}
= \begin{align*}
\begin{tikzpicture}
  
  \node (fg) at (0,1) {$fg$};
  \node (fg) at (0,0) {$fg$};

\end{tikzpicture}
\end{align*}
\begin{align*}
\begin{tikzpicture}
  
  \node (f) at (0,1) {$f$};

\end{tikzpicture}
\end{align*}
\]

A morphism \( f : A_1 \otimes A_2 \rightarrow B_1 \otimes B_2 \) can be depicted:

\[
\begin{tikzpicture}
  
  \node (f) at (0,1) {$f$};
  \node (B1) at (1,2) {$B_1$};
  \node (B2) at (2,2) {$B_2$};
  \node (A1) at (2,1) {$A_1$};
  \node (A2) at (1,1) {$A_2$};

  \draw[->] (f) -- (B1);
  \draw[->] (f) -- (B2);
  \draw[->] (f) -- (A1);
  \draw[->] (f) -- (A2);
\end{tikzpicture}
\]
Presentations of strict monoidal categories

One can give presentations of some strict $\mathbb{k}$-linear monoidal categories, just as for monoids, groups, algebras, etc.

**Objects:** If the objects are generated by some collection $A_i$, $i \in I$, then we have all possible tensor products of these objects:

$$1, \ A_i, \ A_i \otimes A_j \otimes A_k \otimes A_\ell, \ etc.$$  

**Morphisms:** If the morphisms are generated by some collection $f_j$, $j \in J$, then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

$$\text{id}_{A_i}, \ f_j \otimes (f_i f_k) \otimes (f_\ell), \ etc.$$  

We then often impose some relations on these morphism spaces.

**String diagrams:** We can build complex diagrams out of our simple generating diagrams.
Define a strict $\mathbb{k}$-linear monoidal category $S$ with one generating object $\uparrow$ and denote

$$ \text{id}_{\uparrow} = \uparrow $$

We have one generating morphism

$$ \begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \\
\downarrow \uparrow
\end{array}
\end{array} : \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow. $$

We impose the relations:

$$ \begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \uparrow \downarrow = \uparrow \downarrow \uparrow \downarrow, \\
\begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \\
\downarrow \uparrow
\end{array}
\end{array}
\end{array}
\end{array} $$

Then

$$ \text{End}_S(\uparrow \otimes^n) = \mathbb{k}S_n $$

is the group algebra of the symmetric group on $n$ letters.
Start again with the strict $\mathbb{k}$-linear monoidal category $S$, but add a morphism:

\[
\begin{array}{c}
\circlearrowleft : \uparrow \rightarrow \uparrow
\end{array}
\]

We impose the additional relation:

\[
\begin{array}{c}
\begin{array}{c}
\circlearrowleft \quad - \quad \circlearrowright
\end{array}
\end{array}
\begin{array}{c}
\uparrow \quad \uparrow
\end{array}
\]

Now

\[
\text{End}(\uparrow \otimes n)
\]

is the degenerate affine Hecke algebra (of type $A$).
The wreath product category

Fix an associative \( \mathbb{k} \)-algebra \( F \). We add an endomorphism of \( \uparrow \) for each element of \( F \).

More precisely, let \( \mathcal{W}(F) \) be the strict \( \mathbb{k} \)-linear monoidal category obtained from \( S \) by adding morphisms such that we have an algebra homomorphism:

\[
F \to \text{End} \uparrow, \quad f \mapsto \uparrow f
\]

We impose the additional relations:

\[
f \bullet f = f \bullet f, \quad f \in F
\]
The wreath product category

\[
\text{End}_{\mathcal{W}(F)}(↑⊗^n) = F⊗^n \rtimes S_n
\]

is a wreath product algebra.

As a vector space,

\[
F⊗^n \rtimes S_n = F⊗^n \otimes_\mathbb{k} \mathbb{k}S_n.
\]

Multiplication is determined by

\[
(f_1 ⊗ π_1)(f_2 ⊗ π_2) = f_1(π_1 \cdot f_2) ⊗ π_1π_2, \quad f_1, f_2 ∈ F⊗^n, \quad π_1, π_2 ∈ S_n,
\]

where \(π_1 \cdot f_2\) denotes the natural action of \(S_n\) on \(F⊗^n\) by permutation of the factors.

Note: \(\mathcal{W}(\mathbb{k}) = S\), the symmetric group category.

Want: An affine version of the wreath product category. \(F = \mathbb{k}\) should recover the degenerate affine Hecke category.
A Frobenius algebra is a f.d. associative algebra $F$ together with a linear trace map
\[ \text{tr}: F \to k \]
such that the induced map
\[ F \to \text{Hom}_k(F, k), \quad f \mapsto (g \mapsto \text{tr}(gf)) , \]
is an isomorphism.

For simplicity, we assume that the trace is symmetric:
\[ \text{tr}(fg) = \text{tr}(gf), \quad \text{for all } f, g \in F. \]
Example ($\mathbb{k}$)

$\mathbb{k}$ is a Frobenius algebra with $\text{tr} = \text{id}_\mathbb{k}$.

Example (Matrix algebra)

Any matrix algebra over a field is a Frobenius algebra with the usual trace.

Example ($\mathbb{k}[x]/(x^k)$)

$\mathbb{k}[x]/(x^k)$ is a Frobenius algebra with

$$\text{tr}(x^\ell) = \delta_{\ell,k-1}.$$
Frobenius algebras: Examples

Example (Group algebra)
Suppose $G$ is a finite group.
The group algebra $kG$ is a Frobenius algebra with

$$\text{tr}(g) = \delta_{g,1_G}, \quad g \in G.$$ 

Example (Zigzag algebra)
Associated to every quiver is a zigzag algebra. These are Frobenius algebras.

Example (Hopf algebras)
Every f.d. Hopf algebra is a Frobenius algebra.

From now on: $F$ is a Frobenius algebra with trace $\text{tr}$. 
Fix a basis $B$ of $F$. The dual basis is

$$B^\vee = \{b^\vee \mid b \in B\}$$

defined by

$$\text{tr} \left( b^\vee c \right) = \delta_{b,c}, \quad b, c \in B.$$ 

It is easy to check that

$$\sum_{b \in B} b \otimes b^\vee \in F \otimes F$$

is independent of the basis $B$. 
Affine wreath product category

Start with the wreath product category $\mathcal{W}(F)$, but add a morphism:

$$
\begin{array}{c}
\uparrow \\
\circ \\
\uparrow \\
\end{array}
\begin{array}{c}
\uparrow \\
\circ \\
\uparrow \\
\end{array}
$$

We impose the additional relations:

$$
\begin{array}{c}
\begin{array}{c}
\circ \\
\rightarrow \\
\circ \\
\end{array}
\begin{array}{c}
\uparrow \\
\circ \\
\uparrow \\
\end{array}

- \begin{array}{c}
\begin{array}{c}
\circ \\
\rightarrow \\
\circ \\
\end{array}
\begin{array}{c}
\uparrow \\
\circ \\
\uparrow \\
\end{array}

= \sum_{b \in B} b \begin{array}{c}
\uparrow \\
\circ \\
\uparrow \\
\end{array} b^\vee \begin{array}{c}
\uparrow \\
\circ \\
\uparrow \\
\end{array}

, \quad f \begin{array}{c}
\uparrow \\
\circ \\
\uparrow \\
\end{array} = \begin{array}{c}
\uparrow \\
\circ \\
\uparrow \\
\end{array}

, \quad f \in F
\end{array}
$$

Call the resulting category $\mathcal{AW}(F)$ the affine wreath product category.

Now

$$\text{End}_{\mathcal{AW}(F)}(\uparrow \otimes n)$$

is an affine wreath product algebra.

Note: $\mathcal{AW}(\mathbb{K})$ is the degenerate affine Hecke category.
Suppose a strict monoidal category $C$ has two objects $\uparrow$ and $\downarrow$, with

$$\text{id}_\uparrow = \uparrow, \quad \text{id}_\downarrow = \downarrow.$$ 

A morphism $\mathbf{1} \rightarrow \downarrow \otimes \uparrow$ would have string diagram

$$\begin{array}{c}
\text{Diagram Here}
\end{array}, \quad \text{where} \quad | = \text{id}_\mathbf{1}.$$

We typically omit the dotted line and draw:

$$\begin{array}{c}
\text{Diagram Here}
\end{array} : \mathbf{1} \rightarrow \downarrow \otimes \uparrow.$$

Similarly, we can have

$$\begin{array}{c}
\text{Diagram Here}
\end{array} : \uparrow \otimes \downarrow \rightarrow \mathbf{1}.$$
Adjunction

We say that $\downarrow$ is right adjoint to $\uparrow$ (and $\uparrow$ is left adjoint to $\downarrow$) if there exist morphisms

$$ \mu : \mathbf{1} \to \downarrow \otimes \uparrow. \quad \text{and} \quad \eta : \uparrow \otimes \downarrow \to \mathbf{1}. $$

such that

$$ \mu \circ \eta = \mathbf{1} \quad \text{and} \quad \eta \circ \mu = \mathbf{1}. $$

(This is analogous to the unit-counit formulation of adjunction of functors.)

We say $\uparrow$ and $\downarrow$ are biadjoint if they are both left and right adjoint to each other. So we also have

$$ \mu : \mathbf{1} \to \uparrow \otimes \downarrow \quad \text{and} \quad \eta : \downarrow \otimes \uparrow \to \mathbf{1} $$

such that

$$ \mu \circ \eta = \mathbf{1} \quad \text{and} \quad \eta \circ \mu = \mathbf{1}. $$
The Frobenius Heisenberg category

Recall the affine wreath product category $\mathcal{AW}(F)$. It is the strict $k$-linear monoidal category with:

**Objects:** Generated by object $\uparrow$.

**Morphisms:** Generated by

\[
\begin{align*}
\begin{tikzpicture}[baseline=-0.65ex]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture} & : \uparrow \otimes \uparrow \rightarrow \uparrow \otimes \uparrow, \\
\begin{tikzpicture}[baseline=-0.65ex]
\draw (0,0) -- (0,1);
\end{tikzpicture} & : \uparrow \rightarrow \uparrow, \\
\begin{tikzpicture}[baseline=-0.65ex]
\filldraw[blue] (0,0) circle (0.1cm);
\draw (0,0) -- (0,1);
\end{tikzpicture} f : \uparrow \rightarrow \uparrow, \quad f \in F,
\end{align*}
\]

with relations

\[
\begin{align*}
\begin{tikzpicture}[baseline=-0.65ex]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture} & = \begin{tikzpicture}[baseline=-0.65ex]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}, \\
\begin{tikzpicture}[baseline=-0.65ex]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture} & = \begin{tikzpicture}[baseline=-0.65ex]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}, \\
\begin{tikzpicture}[baseline=-0.65ex]
\filldraw[blue] (0,0) circle (0.1cm);
\draw (0,0) -- (0,1);
\end{tikzpicture} f & = \begin{tikzpicture}[baseline=-0.65ex]
\filldraw[blue] (0,0) circle (0.1cm);
\draw (0,0) -- (0,1);
\end{tikzpicture} f, \quad f \in F,
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}[baseline=-0.65ex]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture} - \begin{tikzpicture}[baseline=-0.65ex]
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture} & = \sum_{b \in B} \begin{tikzpicture}[baseline=-0.65ex]
\filldraw[blue] (0,0) circle (0.1cm);
\filldraw[red] (1,0) circle (0.1cm);
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}, \\
\begin{tikzpicture}[baseline=-0.65ex]
\filldraw[blue] (0,0) circle (0.1cm);
\draw (0,0) -- (0,1);
\end{tikzpicture} f & = \begin{tikzpicture}[baseline=-0.65ex]
\filldraw[blue] (0,0) circle (0.1cm);
\draw (0,0) -- (0,1);
\end{tikzpicture} f, \quad f \in F.
\end{align*}
\]

For $n \in \mathbb{N}$, define

\[
\begin{tikzpicture}[baseline=-0.65ex]
\filldraw[blue] (0,0) circle (0.1cm);
\draw (0,0) -- (0,1);
\filldraw[red] (0,-1) circle (0.1cm);
\draw (0,-1) -- (0,-2);
\end{tikzpicture} n = \left\{ \begin{array}{c}
\begin{tikzpicture}[baseline=-0.65ex]
\filldraw[blue] (0,0) circle (0.1cm);
\draw (0,0) -- (0,1);
\end{tikzpicture} \\
\begin{tikzpicture}[baseline=-0.65ex]
\filldraw[red] (0,0) circle (0.1cm);
\draw (0,0) -- (0,1);
\end{tikzpicture}
\end{array} \right\} n \text{ dots}.
\]
The Frobenius Heisenberg category

Fix a central charge $\xi \in \mathbb{Z}$, $\xi \leq 0$.

(Actually, we can take any $\xi \in \mathbb{Z}$, but we choose $\xi \leq 0$ for simplicity of exposition.)

To $A\mathcal{W}(F)$ we add another object $\downarrow$ that is right adjoint to $\uparrow$:

\[
\begin{align*}
\begin{array}{c}
\downarrow \\
\downarrow
\end{array}
&= \\
\begin{array}{c}
\downarrow \\
\downarrow
\end{array}
\quad \text{and} \quad
\begin{aligned}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{aligned}
&= \\
\begin{array}{c}
\downarrow \\
\uparrow
\end{array}
.
\end{align*}
\]

We can then define right crossings:

\[
\begin{aligned}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{aligned}
:= \\
\begin{aligned}
\begin{array}{c}
\downarrow \\
\uparrow
\end{array}
\end{aligned}
: \begin{array}{c}
\downarrow \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
\downarrow \\
\uparrow
\end{array}
.
\end{aligned}
\]

(We start denoting tensor product by juxtaposition: $\begin{array}{c}
\uparrow \\
\downarrow
\end{array}$ := $\begin{array}{c}
\uparrow \\
\downarrow
\end{array}$ $\otimes$ $\begin{array}{c}
\downarrow \\
\uparrow
\end{array}$.)
The Frobenius Heisenberg category

We then impose the crucial inversion relation:

The following matrix of morphisms is an isomorphism in the additive envelope:

$$ \begin{bmatrix} \begin{array}{c} \vdots \\ b \end{array} \end{bmatrix}, \quad 0 \leq r \leq -\xi - 1, \quad b \in B \right] : (\uparrow \otimes \downarrow) \oplus \mathbf{1} \oplus (-\xi \dim F) \rightarrow \downarrow \otimes \uparrow. $$

More precisely, we add in some other morphisms that are the matrix components of an inverse to the above morphism.

We call the resulting category $\mathcal{H}_{\text{Heis}}_{F,\xi}$ the Frobenius Heisenberg category.
The Frobenius Heisenberg category

**Theorem (S. 2018)**

There are unique morphisms

\[
\begin{align*}
\uparrow \cup & : 1 \to \uparrow \downarrow, \\
\downarrow \cup & : \downarrow \uparrow \to 1
\end{align*}
\]  

such that the following relations hold:

\[
\begin{align*}
\uparrow \downarrow & = \uparrow \downarrow \\
\downarrow \uparrow & = \downarrow \uparrow \\
\sum_{k,s\geq 0} \sum_{a,b \in B} & a \lor b - k - s - 2 = \delta_{\xi,0} \\
\delta_{r,-\xi-1} \text{tr}(f) & \quad \text{if } 0 \leq r < -\xi.
\end{align*}
\]

In addition \( \text{Heis}_{F,\xi} \) can be presented equivalently by replacing the inversion relation with the existence of morphisms (1) and above relations.
The Frobenius Heisenberg category

The previous theorem involves left crossings

\[ \begin{array}{c}
\begin{array}{c}
\circlearrowleft \\
\uparrow \\
\downarrow \\
\circlearrowright
\end{array}
\end{array}\quad :=
\begin{array}{c}
\begin{array}{c}
\circlearrowleft \\
\uparrow \\
\downarrow \\
\circlearrowright
\end{array}
\begin{array}{c}
\begin{array}{c}
\circlearrowleft \\
\uparrow \\
\downarrow \\
\circlearrowright
\end{array}
\end{array}
\end{array}\]

and negatively dotted bubbles: for \( r \leq -\xi \),

\[
r + \xi - 1 \quad \begin{array}{c}
\begin{array}{c}
\circlearrowleft \\
\uparrow \\
\downarrow \\
\circlearrowright
\end{array}
\end{array} f := (-1)^{r+1} \sum_{b_1, \ldots, b_{r-1} \in B} \det \left( b_{j-1}^\vee b_j \begin{array}{c}
\begin{array}{c}
\circlearrowleft \\
\uparrow \\
\downarrow \\
\circlearrowright
\end{array}
\end{array} i-j-\xi \right)^r_{i,j=1}.
\]

Theorem (S. 2018)

1. The objects \( \downarrow \) and \( \uparrow \) are biadjoint.
2. The category \( \mathcal{H}eis_{F,\xi} \) is strictly pivotal (isotopy invariance for morphisms).
3. One can compute an infinite grassmannian relation, curl relations, bubble slide relations, and an alternating braid relation (omitted here).
4. Under a mild assumption on \( F \), the category \( \mathcal{H}eis_{F,\xi} \) categorifies the Heisenberg algebra at central charge \( \xi \dim F \).
Suppose $\xi \leq -1$.

The category $\mathcal{H}eis_{F,\xi}$ acts naturally on modules for cyclotomic wreath product algebras. We have a chain of algebras

$$k = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots .$$

Then

- $\uparrow$ acts by induction from $A_n$-mod to $A_{n+1}$-mod,
- $\downarrow$ acts by restriction from $A_n$-mod to $A_{n-1}$-mod.

The morphisms (diagrams) act by certain natural transformations.

Fact that $\uparrow$ and $\downarrow$ are biadjoint corresponds to fact that induction and restriction are biadjoint.

In other words $A_n$ is a Frobenius extension of $A_{n-1}$. 
Heisenberg categorification and actions ($\xi = 0$)

**$F = k$ case**

$\mathcal{Heis}_{k,0}$ is the affine oriented Brauer category of Brundan–Comes–Nash–Reynolds.

$\mathcal{Heis}_{k,0}$ acts naturally on $\mathfrak{gl}_n(k)$-mod: If $V$ is the natural rep, then

- $\uparrow \mapsto V \otimes -$  
- $\downarrow \mapsto V^* \otimes -$ 

**General case: open problem**

What does $\mathcal{Heis}_{F,0}$ act naturally on for a general Frobenius algebra $F$?
Historical remarks

Original Heisenberg category (Khovanov)
- Morphisms were planar diagrams up to isotopy, so strictly pivotal property was part of the definition.
- Central charge $\xi = -1$ and $F = k$.

Frobenius modification (central charge $-1$)
- For $F$ the zigzag algebra, defined by Cautis–Licata and studied in relation to geometry of the Hilbert scheme.
- General definition given in joint work with Rosso.
- Still have central charge $\xi = -1$.

Higher central charge (Mackaay–S.)
- Generalized to higher central charge (with $F = k$).
- Again, pivotal property part of the definition.
Historical remarks

**Inversion relation approach, $F = k$ (Brundan)**

- New approach to the definition of higher charge category (Mackaay-S.) using the inversion relation.
- Now, pivotal property is a consequence of the definition.
- **Advantage**: proof that category acts on modules over degenerate (cyclotomic) affine Hecke algebras is much easier. Uses a well-known Mackey-type theorem.

**Current work**

- Follows inversion relation approach of Brundan.
- Defines a Frobenius algebra version of higher charge category (Mackaay–S.).
- Defines a higher charge version of previous Frobenius Heisenberg category (Rosso–S.).
Summarizing the relationship between the Heisenberg categories appearing in the literature, we have:

\[
\begin{align*}
\mathcal{H}eis_{F,\xi} & \\
\xi = -1 & \\
F = k & \\
\text{Rosso–S. Cautis–Licata (} F = \text{ zigzag) } & \text{Mackaay–S. Brundan (inversion)} \\
F = k & \\
\xi = -1 & \\
\text{Khovanov} & \\
\end{align*}
\]
Some remarks

One can actually work in a more general setting than the one described here:

1. $F$ can be a graded Frobenius superalgebra. Then $\mathcal{H}_{\text{eis}} F, \xi$ is a strict $k$-linear graded monoidal supercategory.

2. The trace need not be symmetric. In general, there exists a Nakayama automorphism $\psi: F \to F$ such that

$$\text{tr}(fg) = (-1)^{\bar{f}\bar{g}} \text{tr}(g\psi(f)) \quad \text{for all } f, g \in F.$$ 

Then, for instance,

$$f \uparrow = \uparrow_{\psi(f)}, \; f \in F,$$

3. Above remarks mean we can take $F$ to be the Clifford superalgebra. Then $\mathcal{H}_{\text{eis}} F, \xi$ acts on modules for affine Sergeev algebras (a.k.a. degenerate affine Hecke–Clifford algebras).
One can \textit{q-deform} the Frobenius Heisenberg category. When $F = \mathbb{k}$, this corresponds to

\text{deg. affine Hecke algebra} \sim \text{affine Hecke algebra}.

\textbf{Generating objects:} $\uparrow$ and $\downarrow$

\textbf{Generating morphisms:}

\[
\begin{array}{ccc}
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture}
\end{array}, \quad
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture}, \quad
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture}.
\]

\textbf{Relations:} Fix $z \in \mathbb{k}$.

\[
\begin{array}{ccc}
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture} = \begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture},
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture} = \begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture},
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture} = \begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture},
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture} = \begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture},
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture} = \begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture},
\end{array}
\]

\[
\begin{array}{ccc}
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture} - \begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture} = z \begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture},
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture} = \begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture},
\end{array}
\]

+ inversion relation.
Case: $\xi = 0$

When $\xi = 0$, one obtains the affine oriented skein category, an affinization of the HOMFLY-PT skein category.

This category acts on modules for $U_q(\mathfrak{gl}_n)$.

Certain closed diagrams correspond to the Casimir elements in $U_q(\mathfrak{gl}_n)$.

Relation to previous constructions

When $\xi = -1$, the category contains the previously defined $q$-deformed Heisenberg category (Licata–S. 2013).

Main difference between two constructions is that, in the previous $q$-deformed Heisenberg category, the dot was not invertible.

Case $\xi \neq 0$: Action on modules for cyclotomic Hecke algebras.
Generally, one can again incorporate a graded Frobenius superalgebra to get a more general quantum Frobenius Heisenberg category.

When $\xi \neq 0$, category should act on cyclotomic quotients of quantum affine wreath product algebras. The theory of these algebras is yet to be developed.

When $\xi = 0$, the natural action is an open question for general $F$. Should be some $F$-deformation of $U_q(\mathfrak{gl}_n)$. 
Happy 60th Birthday!

WHY I COULD NEVER BE A MATH TEACHER:

TEACHER! WILL WE EVER USE ANY OF THIS ALGEBRA?

YOU WON’T, BUT ONE OF THE SMART KIDS MIGHT.