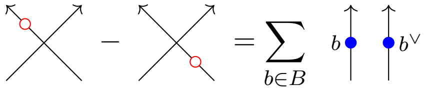
Frobenius Heisenberg categorification



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Slides available online: alistairsavage.ca/talks

Preprint (first part): arXiv:1802.01626
Second part: Work in progress with J. Brundan

Outline

Goal:

1. Unify, generalize, and simplify theory of Heisenberg categorification
2. Study categorical actions

Overview:

1. Monoidally generated algebras
2. The Frobenius Heisenberg category
3. Categorification and categorical actions
4. Quantum Heisenberg categories (work in progress with J. Brundan)

Monoidally generated symmetric groups

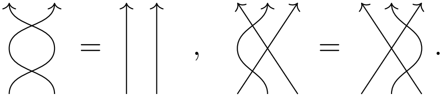
Define a strict $\Bbbk$-linear monoidal category $\mathcal{S}$ with one generating object $\uparrow$ and denote

$$\mathrm{id}_{\uparrow} = \uparrow$$

We have one generating morphism

$$: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow .$$

We impose the relations:



Then

$$\mathrm{End}_{\mathcal{S}}(\uparrow^{\otimes n}) = \Bbbk S_n$$

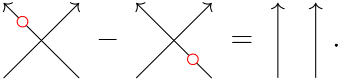
is the group algebra of the symmetric group on n letters.

The degenerate affine Hecke category

Start again with the strict $\Bbbk$-linear monoidal category $\mathcal{S}$, but add a morphism:

 $: \uparrow \to \uparrow$

We impose the additional relation:



Now

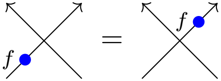
$$\mathrm{End}(\uparrow^{\otimes n}) = \text{degenerate affine Hecke algebra (of type } A\text{)}.$$

The wreath product category

Fix an associative $\Bbbk$-algebra F. Let $\mathcal{W}(F)$ be the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{S}$ by adding morphisms such that we have an algebra homomorphism:

$$F \to \operatorname{End} \uparrow, \qquad f \mapsto$$ 

We impose the additional relations:



$$, \quad f \in F$$

Now

$$\operatorname{End}_{\mathcal{W}(F)}(\uparrow^{\otimes n}) = F^{\otimes n} \rtimes S_n = \text{wreath product algebra}.$$

Note: $\mathcal{W}(\Bbbk) = \mathcal{S}$, the symmetric group category.

Want

An affine version of the wreath product category. $F = \Bbbk$ should recover the degenerate affine Hecke category.

Frobenius algebras: Definition

Frobenius algebra

A Frobenius algebra is a f.d. associative algebra F together with a linear trace map

$$\mathrm{tr}\colon F \to \Bbbk$$

such that the induced map

$$F \to \mathrm{Hom}_{\Bbbk}(F, \Bbbk), \quad f \mapsto (g \mapsto \mathrm{tr}(gf))\,,$$

is an isomorphism.

For simplicity, we assume that the trace is symmetric:

$$\mathrm{tr}(fg) = \mathrm{tr}(gf), \quad \text{for all } f, g \in F.$$

Frobenius algebras: Examples

Example (Frobenius $\Bbbk$-algebras)

1. $\Bbbk$
2. Matrix algebra over a field
3. $\Bbbk[x]/(x^k)$
4. Group algebra of a finite group
5. Zigzag algebra
6. Finite-dimensional Hopf algebra
7. Clifford algebra (Frobenius superalgebra)

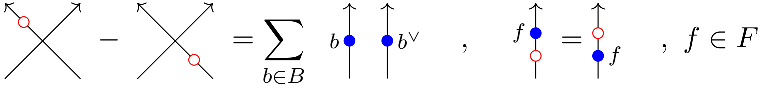
From now on: F is a Frobenius algebra with trace tr.

Affine wreath product category

Start with the wreath product category $\mathcal{W}(F)$, but add a morphism:

 $: \uparrow \to \uparrow$

We impose the additional relations:



(B is a basis of F, $\{b^\vee : b \in B\}$ is dual basis.)

Call the resulting category $\mathcal{AW}(F)$ the affine wreath product category:

$$\mathrm{End}_{\mathcal{AW}(F)}(\uparrow^{\otimes n}) = \text{affine wreath product algebra.}$$

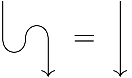
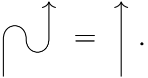
Note: $\mathcal{AW}(\mathbb{k})$ is the degenerate affine Hecke category.

Adjunction

We say that $\downarrow$ is right adjoint to $\uparrow$ (and $\uparrow$ is left adjoint to $\downarrow$) if there exist morphisms

 $: \mathbf{1} \to \downarrow \otimes \uparrow.$ and  $: \uparrow \otimes \downarrow \to \mathbf{1}.$

such that

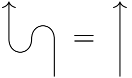
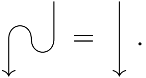
 and 

(This is analogous to the unit-counit formulation of adjunction of functors.)

We say $\uparrow$ and $\downarrow$ are biadjoint if they are both left and right adjoint to each other. So we also have

 $: \mathbf{1} \to \uparrow \otimes \downarrow$ and  $: \downarrow \otimes \uparrow \to \mathbf{1}$

such that

 and 

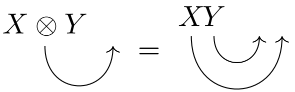
Pivotal categories

A strict monoidal category $\mathcal{C}$ is a strict pivotal category if every object X has a right dual X^* and the following conditions are satisfied:

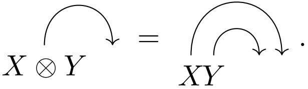
1. For all objects X and Y in $\mathcal{C}$,

$$(X^*)^* = X, \quad (X \otimes Y)^* = Y^* \otimes X^*, \quad \mathbf{1}^* = \mathbf{1}.$$

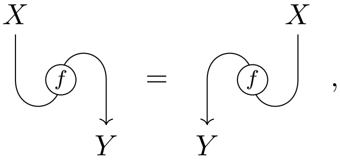
2. For all objects X and Y in $\mathcal{C}$, we have



and



3. For every $f \colon X \to Y$ in $\mathcal{C}$, its right and left mates are equal:



where left caps/cups are defined as right cup/caps for the dual object.

Pivotal categories

Isotopy invariance

In a strict pivotal category, isotopic string diagrams represent the same morphism!

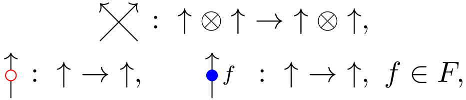
In some categories appearing in the categorification literature, morphism spaces are defined in terms of planar diagrams modulo isotopy. Here the categories are pivotal by assumption.

The Frobenius Heisenberg category

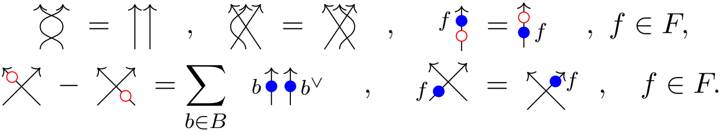
Recall the affine wreath product category $\mathcal{AW}(F)$. It is the strict $\Bbbk$-linear monoidal category with:

Objects: Generated by object $\uparrow$.

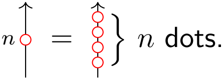
Morphisms: Generated by



with relations



For $n \in \mathbb{N}$, define



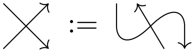
The Frobenius Heisenberg category

Fix a central charge $\xi \in \mathbb{Z}$, $\xi \leq 0$.

(Actually, we can take any $\xi \in \mathbb{Z}$, but we choose $\xi \leq 0$ for simplicity of exposition.)

To $\mathcal{AW}(F)$ we add another object $\downarrow$ that is right adjoint to $\uparrow$:

We can then define right crossings:

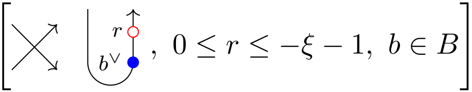
 $: \uparrow\downarrow \to \downarrow\uparrow$.

(We start denoting tensor product by juxtaposition: $\uparrow\downarrow := \uparrow \otimes \downarrow$.)

The Frobenius Heisenberg category

We then impose the crucial inversion relation:

The following matrix of morphisms is an isomorphism in the additive envelope:



$$: (\uparrow\downarrow) \oplus \mathbf{1}^{\oplus(-\xi \dim F)} \to \downarrow\uparrow .$$

More precisely, we add in some other morphisms that are the matrix components of an inverse to the above morphism.

We call the resulting category $\mathcal{H}eis_{F,\xi}$ the Frobenius Heisenberg category.

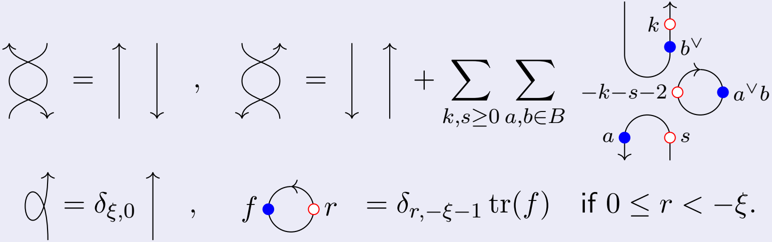
The Frobenius Heisenberg category

Theorem (S. 2018)

There are unique morphisms

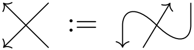
 $: \mathbf{1} \to \uparrow\downarrow,$  $: \downarrow\uparrow \to \mathbf{1}$ (1)

such that the following relations hold:

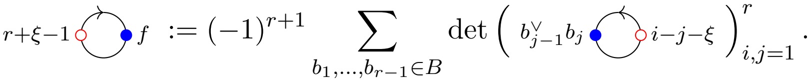


In addition $\mathcal{H}eis_{F,\xi}$ can be presented equivalently by replacing the inversion relation with the existence of morphisms (1) and above relations.

The previous theorem involves left crossings



and negatively dotted bubbles: for $r \leq -\xi$,



The Frobenius Heisenberg category

Theorem (S. 2018)

1. The objects $\downarrow$ and $\uparrow$ are biadjoint.
2. The category $\mathcal{H}eis_{F,\xi}$ is strictly pivotal (isotopy invariance for morphisms).

Important note

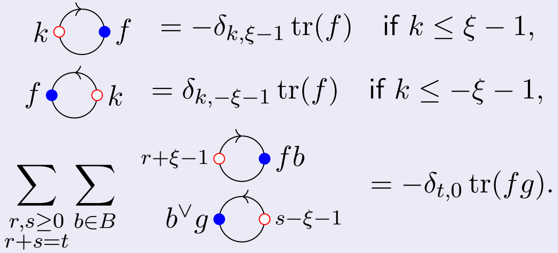
In the original Heisenberg categories, morphisms consisted of planar diagrams up to isotopy. So pivotal was built into the definition.

In the inversion relation approach (following Brundan), one deduces that the category is pivotal.

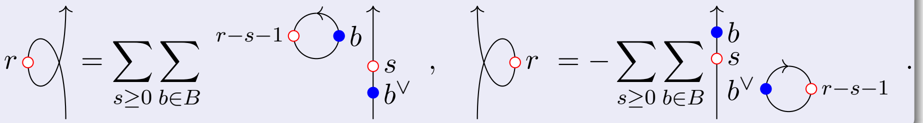
The Frobenius Heisenberg category

Theorem (S. 2018)

Infinite grassmannian relations: For $f, g \in F$, we have



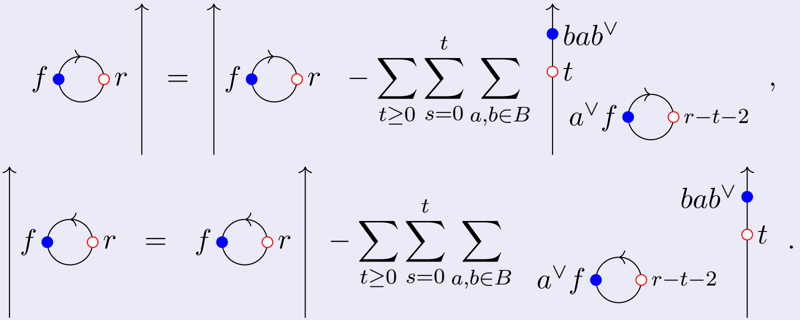
Curl relations: For all $r \geq 0$,



The Frobenius Heisenberg category

Theorem (S. 2018)

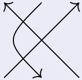
Bubble slides: For all $f \in F$ and $r \geq 0$,



The Frobenius Heisenberg category

Theorem (S. 2018)

Alternating braid relation: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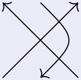
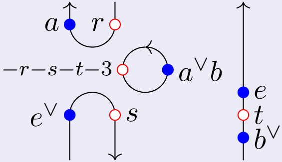
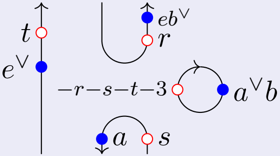


$=$

- $\sum_{r,s,t\geq 0}\sum_{a,b,e\in B}$ 

if $\xi \geq 2$,
- 0 if $-1 \leq \xi \leq 1$,
- $\sum_{r,s,t\geq 0}\sum_{a,b,e\in B}$ 

if $\xi \leq -2$.

Categorification of the Heisenberg algebra

The infinite-dimensional Heisenberg Lie algebra $\mathfrak{h}$ is the complex Lie algebra with basis

$$\{c, q_n^\pm : n \geq 1\}$$

and product

$$[q_m^+, q_n^+] = [q_m^-, q_n^-] = [c, q_n^\pm] = 0, \qquad [q_m^+, q_n^-] = \delta_{m,n} nc.$$

The associative Heisenberg algebra at central charge $k \in \mathbb{Z}$ is

$$U(\mathfrak{h})/\langle c - k \rangle.$$

Theorem (S. 2018)

Under some mild assumptions, the category $\mathcal{H}eis_{F,\xi}$ categorifies the Heisenberg algebra at central charge $k = \xi \dim F$.

This is a Grothendieck ring categorification.

Categorical actions ($\xi \leq -1$)

Suppose $\xi \leq -1$.

The category $\mathcal{H}eis_{F,\xi}$ acts naturally on modules for cyclotomic wreath product algebras. We have a chain of algebras

$$\Bbbk = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots .$$

Then

- $\uparrow$ acts by induction from A_n-mod to A_{n+1}-mod,
- $\downarrow$ acts by restriction from A_n-mod to A_{n-1}-mod.

The morphisms (diagrams) act by certain natural transformations.

Fact that $\uparrow$ and $\downarrow$ are biadjoint corresponds to fact that induction and restriction are biadjoint.

In other words A_n is a Frobenius extension of A_{n-1}.

Categorical actions ($\xi = 0$)

$F = \Bbbk$ case

$\mathcal{H}eis_{\Bbbk,0}$ is the affine oriented Brauer category of Brundan–Comes–Nash–Reynolds.

$\mathcal{H}eis_{\Bbbk,0}$ acts naturally on $\mathfrak{gl}_n(\Bbbk)$-mod: If V is the natural rep, then

- $\uparrow \mapsto V \otimes -$
- $\downarrow \mapsto V^* \otimes -$

The dot acts by the Casimir tensor:

$$\mapsto \sum_{i,j=1}^{n} e_{i,j} \otimes e_{j,i}$$

General case: work in progress

Action of $\mathcal{H}eis_{F,0}$ for a general Frobenius algebra F.

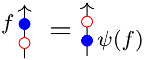
More general setting

One can actually work in a more general setting than the one described here:

1. F can be a graded Frobenius superalgebra. Then $\mathcal{H}eis_{F,\xi}$ is a strict $\Bbbk$-linear graded monoidal supercategory.
2. The trace need not be symmetric. In general, there exists a Nakayama automorphism $\psi \colon F \to F$ such that

   $$\operatorname{tr}(fg) = (-1)^{\bar{f}\bar{g}} \operatorname{tr}(g\psi(f)) \quad \text{for all } f, g \in F.$$

   Then, for instance,

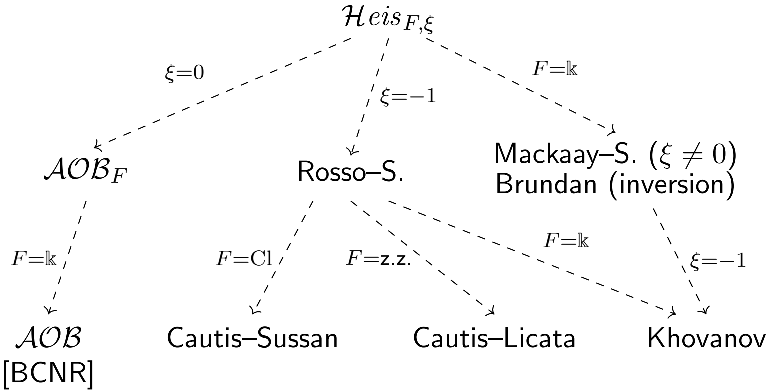
   

   , $f \in F$,

3. Above remarks mean we can take F to be the Clifford superalgebra. Then $\mathcal{H}eis_{F,\xi}$ acts on modules for affine Sergeev algebras (a.k.a. degenerate affine Hecke–Clifford algebras).

Heisenberg categories

Summarizing the relationship between the (non-quantum) Heisenberg categories appearing in the literature, we have:



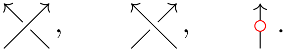
Quantum Heisenberg category (with J. Brundan)

One can q-deform the Frobenius Heisenberg category. When $F = \Bbbk$, this corresponds to

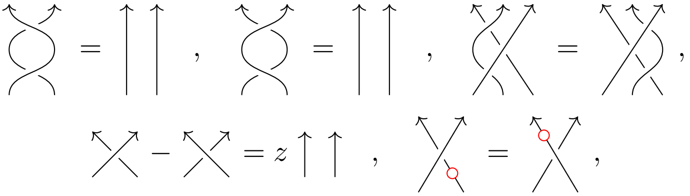
deg. affine Hecke algebra $\rightsquigarrow$ affine Hecke algebra.

Generating objects: $\uparrow$ and a right dual $\downarrow$

Generating morphisms:



Relations: Fix $z, t \in \Bbbk^\times$.

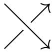


$+$ inversion relation $+$ one bubble relation.

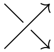
Quantum Heisenberg category (with J. Brundan)

Now have three equivalent approaches:

1. Inversion relation involving



2. Inversion relation involving



3. Adding left cups/caps + additional relations:
    - Double crossing relations
    - (Undotted) curl relations
    - Some bubble relations

In addition:

- The category is pivotal
- Infinite grassmannian relations
- Curl relations
- Bubble slides
- Alternating braid relation

Quantum Heisenberg category (with J. Brundan)

Case: $\xi = 0$

When $\xi = 0$, one obtains the affine oriented skein category, an affinization of the HOMFLY-PT skein category.

This category acts on modules for $U_q(\mathfrak{gl}_n)$.

Closed diagrams correspond to central elements in $U_q(\mathfrak{gl}_n)$.

Case: $\xi \neq 0$

Acts on modules for cyclotomic Hecke algebras.

When $\xi = -1$, the category contains the previously defined q-deformed Heisenberg category (Licata–S. 2013).

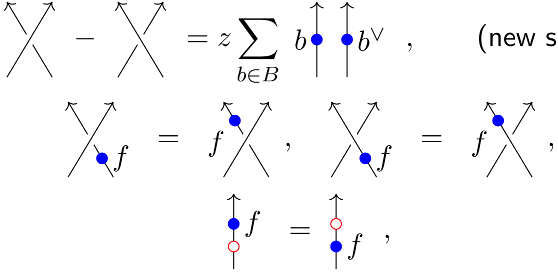
Main difference between two constructions is that, in the previous q-deformed Heisenberg category, the dot was not invertible.

Generally, one can again incorporate a graded Frobenius superalgebra F to get a more general quantum Frobenius Heisenberg category.

Strand can now carry tokens:

$$, \quad f \in F.$$

We have additional relations:



(new skein)

+ inversion, etc.

Categorical actions (with J. Brundan)

Categorical actions: Largely unexplored.

Case: $\xi = 0$

Obtain a "Frobenius deformation" of the affine oriented skein category.

Natural action is an open question for general F.

Should act on some F-deformation of $U_q(\mathfrak{gl}_n)$.

Case: $\xi \neq 0$

Should act on cyclotomic quotients of quantum affine wreath product algebras (or affine Frobenius Hecke algebras).

The structure theory and rep theory of these algebras is work in progress.