Adjunction
uOttawa Math Club

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Suppose $V$ and $W$ are finite-dimensional complex vector spaces.

The dual of $V$ is the space of linear functionals on $V$:

$$V^* := \{ f : V \to \mathbb{C} : f \text{ is linear} \}.$$ 

It is itself a vector space under

- pointwise addition:
  $$ (f + g)(v) := f(v) + g(v), \quad f, g \in V^*, \quad v \in V, $$

- scalar multiplication:
  $$ (\alpha f)(v) := \alpha f(v), \quad \alpha \in \mathbb{C}, \quad f \in V^*, \quad v \in V.$$
Transposes

Suppose we have a linear map

\[ A: V \rightarrow W. \]

Then, given \( f \in W^* \), we can produce an element of \( V^* \):

\[ V \xrightarrow{A} W \]

\[ f \circ A \]

\[ f \]

\[ C \]

The transpose of \( A: V \rightarrow W \) is the linear map

\[ A^T: W^* \rightarrow V^*, \]

\[ A^T f := f \circ A, \quad f \in W^*. \]
Inner product spaces

An inner product on $V$ is a map

$$\langle \cdot, \cdot \rangle_V : V \times V \to \mathbb{C},$$

satisfying the following axioms for all $u, v, w \in V$ and all $\alpha \in \mathbb{C}$:

- **conjugate symmetry:**
  $$\langle u, v \rangle_V = \overline{\langle v, u \rangle_V}$$

- **linearity in the first argument:**
  $$\langle \alpha v, w \rangle_V = \alpha \langle v, w \rangle_V$$
  $$\langle u + v, w \rangle_V = \langle u, w \rangle_V + \langle v, w \rangle_V$$

- **positive-definiteness:**
  $$\langle v, v \rangle_V \geq 0$$
  $$\langle v, v \rangle_V = 0 \iff v = 0$$
Inner product spaces

Example

If $V = \mathbb{C}^n$ (column vectors), then we have the standard inner product
\[
\langle u, v \rangle = u^T \bar{v}.
\]

Suppose $V$ has an inner product. For each $v \in V$, we can define a linear map
\[
f_v : V \to \mathbb{C}, \quad f_v(u) = \langle u, v \rangle_V, \quad u \in V.
\]
So $f_v \in V^*$. One can check that this defines an isomorphism of vector spaces
\[
V \cong V^*, \quad v \mapsto f_v.
\]

Summary: Inner products allow us to identify a f.d. vector space with its dual.
Suppose $V$ and $W$ both have inner products, and that we have a linear map

$$A : V \to W.$$ 

Then we have the following:

$$W^* \xrightarrow{A^T} V^* \xrightarrow{\mathbb{R}} V^* \xrightarrow{\mathbb{R}} V$$

We can complete the square to a commutative diagram. The bottom map is the **adjoint** $A^*$ of $A$.

The adjoint is **uniquely determined** by the fact that

$$\langle Av, w \rangle_W = \langle v, A^* w \rangle_V, \quad v \in V, \; w \in W.$$
Suppose $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$, with the standard inner products.

Every linear map $A : V \rightarrow W$ corresponds to a matrix.

The defining property of the adjoint is that

$$(Av)^T \bar{w} = \langle Av, w \rangle = \langle v, A^* w \rangle = v^T \overline{A^*} w, \quad v \in V, \ w \in W.$$ 

So, for all $v \in V$ and $w \in W$, we have

$$v^T A^T \bar{w} = v^T \overline{A^*} w = v^T \overline{A^*} \bar{w}.$$ 

This implies that $A^T = \overline{A^*}$, which is equivalent to

$$A^* = \overline{A^T}.$$ 

So the adjoint $A^*$ is the conjugate transpose of the matrix $A$. 
A category $\mathcal{C}$ consists of

- a class of objects $\text{Ob}\mathcal{C}$,
- a class of morphisms $\text{hom}_\mathcal{C}(X, Y)$ for all $X, Y \in \text{Ob}\mathcal{C}$,

together with a composition

$$\text{hom}_\mathcal{C}(Y, Z) \times \text{hom}_\mathcal{C}(X, Y) \to \text{hom}_\mathcal{C}(X, Z), \quad (f, g) \mapsto f \circ g,$$

and an identity morphism $1_X \in \text{hom}_\mathcal{C}(X, X)$ for all objects $X \in \text{Ob}\mathcal{C}$.

The composition must be associative:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

whenever $f \circ g$ and $g \circ h$ are defined.

The identity morphism has the property that

$$1_Y \circ f = f = f \circ 1_X \quad \text{for all} \ f \in \text{hom}_\mathcal{C}(X, Y).$$
Categories (Examples)

Example (Sets)
- **Objects**: sets
- **Morphisms**: set maps

Example (Vector spaces)
- **Objects**: vector spaces over a fixed field
- **Morphisms**: linear maps

Example (Groups)
- **Objects**: groups
- **Morphisms**: group homomorphisms
Categories (Examples)

Example (Rings)
- Objects: rings
- Morphisms: ring homomorphisms

Example (Topological spaces)
- Objects: topological spaces
- Morphisms: continuous maps

Other examples
- modules over a fixed ring
- smooth manifolds
- algebraic varieties
- ...
Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories.

A functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ consists of

- a map $F: \text{Ob} \mathcal{C} \to \text{Ob} \mathcal{D}$,
- for all $X, Y \in \text{Ob} \mathcal{C}$, a map $F: \text{hom}_\mathcal{C}(X, Y) \to \text{hom}_\mathcal{D}(F(X), F(Y))$.

We require that the map on morphisms respects composition:

$$F(f \circ g) = F(f) \circ F(g),$$

whenever the composition of $f, g \in \text{hom} \mathcal{C}$ is defined.

We also require it to preserve identities:

$$F(1_X) = 1_{F(X)}.$$
Example (Forgetful functors)

We can define a functor

\[ F : \text{category of groups} \to \text{category of sets} \]

as follows:

- for a group \( G \), we define \( F(G) \) to be the underlying set of \( G \),
- for a group homomorphism \( f : G_1 \to G_2 \), we define \( F(f) \) to be the underlying set map.

So \( F \) just forgets the group structure.

There are many other examples of forgetful functors:

- category of rings \( \to \) category of abelian groups, \((R, +, \cdot) \mapsto (R, +)\)
- category of rings \( \to \) category of sets
- category of vector spaces \( \to \) category of sets
Functors

Example (Double dual)

There is a functor from the category of complex vector spaces to itself that

- maps any vector space to its double dual (the dual of its dual space),
- maps any linear map to its double transpose.

Example (Fundamental group)

Suppose

- \textbf{Top} is the category of pointed topological spaces (topological spaces together with a distinguished point),
- \textbf{Group} is the category of groups.

We have a functor \textbf{Top} \rightarrow \textbf{Group} that maps a pointed topological space to its fundamental group.
Suppose $C$ and $D$ are categories. Let $\textbf{Set}$ be the category of sets.

**General philosophy**

We will think of $\hom_C(\cdot, \cdot) : C \times C \to \textbf{Set}$ as a categorical analogue of an inner product on the category $C$.

**Goal**

Define an appropriate notion of adjoint functor.

**First guess**

The adjoint of a functor $F : C \to D$ is a functor $G : D \to C$ such that

$$\hom_D(FX, Y) = \hom_C(X, GY), \quad X \in \text{Ob} C, \ Y \in \text{Ob} D.$$
Problem 1: Lack of symmetry

For inner products, we had conjugate symmetry:

\[ \langle u, v \rangle_V = \overline{\langle v, u \rangle_V}, \quad u, v \in V. \]

It follows that \((A^*)^* = A\).

There is no such symmetry (in general) in the setting of categories.

Solution: Break the symmetry!

Second guess

A functor \(F : \mathcal{C} \to \mathcal{D}\) is left adjoint to a functor \(G : \mathcal{D} \to \mathcal{C}\) (and \(G\) is right adjoint to \(F\)) if

\[ \text{hom}_\mathcal{D}(FX, Y) = \text{hom}_\mathcal{C}(X, GY), \quad X \in \text{Ob} \mathcal{C}, \ Y \in \text{Ob} \mathcal{D}. \]
Example

Define two groups as follows:

\[ G = \{a, b\}, \quad a^2 = b^2 = 1, \quad ab = ba = b, \]
\[ H = \{x, y\}, \quad x^2 = y^2 = 1, \quad xy = yx = y. \]

Are these the same group (i.e. are they equal)? Technically, no. However, they are isomorphic via the map \( a \mapsto x, \ b \mapsto y \).

In fact, both are isomorphic to the cyclic group of order 2.

Observation: In category theory, the concept of isomorphism is more fundamental than the notion of equality.
From equality to isomorphism

Our observation suggests that the condition

$$\text{hom}_C(FX, Y) = \text{hom}_D(X, GY), \quad X \in \text{Ob} \, C, \; Y \in \text{Ob} \, D,$$

is not very natural.

Solution: Change from equality to isomorphism!

Third guess

A functor $F : C \to D$ is left adjoint to a functor $G : D \to C$ (and $G$ is right adjoint to $F$) if

$$\text{hom}_D(FX, Y) \cong \text{hom}_C(X, GY), \quad X \in \text{Ob} \, C, \; Y \in \text{Ob} \, D.$$

(Here $\cong$ is isomorphism of sets.)
Problem 3: Naturality

Generally, in mathematics, natural constructions should respect the structure that is present:

1. **vector space** homomorphisms should be linear,
2. **group** homomorphisms should commute with the group operation,
3. **ring** homomorphisms should commute with multiplication and addition.

Our current definition doesn’t really fully respect all of the structure of the categories \( \mathcal{C} \) and \( \mathcal{D} \) coming from morphisms.

Suppose

\[ f \in \text{hom}_\mathcal{C}(X_1, X_2), \quad g \in \text{hom}_\mathcal{C}(Y_1, Y_2). \]

Then we have a map

\[ \mu_{f,g} : \text{hom}_\mathcal{C}(X_2, Y_1) \to \text{hom}_\mathcal{C}(X_1, Y_2), \]

\[ A \mapsto g \circ A \circ f. \]
Naturality

Suppose

$$\text{hom}_C(FX, Y) \cong \text{hom}_D(X, GY), \quad X \in \text{Ob} \, C, \ Y \in \text{Ob} \, D.$$  

We say these isomorphisms are natural if the diagram

\[
\begin{array}{ccc}
\text{hom}_C(FX_2, Y_1) & \xrightarrow{\text{R}} & \text{hom}_D(X_2, GY_1) \\
\downarrow{\mu_{F(f), g}} & & \downarrow{\mu_{f, G(g)}} \\
\text{hom}_C(FX_1, Y_2) & \xrightarrow{\text{R}} & \text{hom}_D(X_1, GY_2)
\end{array}
\]

commutes for all

$$f \in \text{hom}_C(X_1, X_2), \quad g \in \text{hom}_D(Y_1, Y_2).$$
Definition (Adjoint functors)

A functor $F: C \to D$ is left adjoint to a functor $G: D \to C$ (and $G$ is right adjoint to $F$) if we have natural isomorphisms

$$\text{hom}_D(FX, Y) \cong \text{hom}_C(X, GY), \quad X \in \text{Ob} C, \ Y \in \text{Ob} D.$$ 

Now that we’ve formulated the correct notion of adjoint functors, there is an obvious question:

**Question:** Should we care? Are there nice/familiar examples of adjoint functors?

**Answer:** YES!
Let $\textbf{Vect}$ denote the category of complex vector spaces.

Recall that we have the forgetful functor

$$\text{forget} : \textbf{Vect} \rightarrow \textbf{Set}.$$ 

Let's try to find a left adjoint to $\text{forget}$.

So we want a functor $F : \textbf{Set} \rightarrow \textbf{Vect}$ such that we have natural isomorphisms

$$\text{hom}_{\textbf{Vect}}(FX, Y) \cong \text{hom}_{\textbf{Set}}(X, \text{forget}(Y)),$$

for $X$ a set and $Y$ a vector space.

So, if $X$ is a set and $Y$ is a vector space, we want to associate to every set map $f : X \rightarrow Y$ a vector space map $Ff : FX \rightarrow Y$. 
Recall from linear algebra

Suppose $V$ and $W$ are vector spaces. If

- $v_i, i \in I$, is a basis of $V$, and
- $w_i, i \in I$, are arbitrary elements of $W$,

then there is a unique linear map

$$A: V \to W \text{ such that } Av_i = w_i.$$ 

Put another way: If

- $B$ is a basis of $V$, and
- $f: B \to W$ is a map of sets,

then there is a unique way to extend $f$ to a linear map

$$A: V \to W \text{ such that } Av = f v \forall v \in B.$$
Recall: We want a functor $F: \textbf{Set} \to \textbf{Vect}$ such that we have natural isomorphisms

$$\text{hom}_{\textbf{Vect}}(FX, Y) \cong \text{hom}_{\textbf{Set}}(X, \text{forget}(Y)),$$

for $X$ a set and $Y$ a vector space.

So $FX$ should be a vector space with basis $X$.  

Free vector spaces

Question: Given a set $X$, how do we form a vector space with basis $X$?

For a set $X$, let

$$FX = \{ f: X \to \mathbb{C} : f(x) = 0 \text{ for all but finitely many } x \in X \}.$$ 

For $x \in X$, we have the Dirac delta function:

$$\delta_x: X \to \mathbb{C}, \quad \delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

Exercise: The $\delta_x, x \in X$, are a basis for $FX$.

So if we identify $x$ with $\delta_x$, then $FX$ is a vector space with basis $X$.

$FX$ is called the free vector space on the set $X$. 
Example of adjoint functors in linear algebra

To every set map $f : X \to Y$, we can associate a natural linear map $Ff : FX \to FY$, and show that we have a functor $\textbf{Set} \to \textbf{ Vect}$.

**Theorem**

The free vector space functor

$$F : \textbf{Set} \to \textbf{ Vect}$$

is left adjoint to the forgetful functor

$$\text{forget} : \textbf{ Vect} \to \textbf{ Set}.$$ 

We’ve done most of the work in proving this. One just needs to verify that the isomorphisms

$$\text{hom}_{\textbf{ Vect}}(FX, Y) \cong \text{hom}_{\textbf{Set}}(X, \text{forget}(Y)),$$

are natural (exercise).
Let \textbf{Group} be the category of groups.

We have a forgetful functor

\[ \text{forget} : \text{Group} \to \text{Set}. \]

Just as for vector spaces, its left adjoint is the \textit{free} group functor.

Suppose \( X \) is a set. The \textit{free group} on \( X \) is the group \( FX \) with elements that are \textbf{words} in the alphabet \( X \) and formal inverses of elements of \( X \):

\[ a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}, \quad a_1, a_2, \ldots, a_n \in X. \]

Multiplication is given by concatenation and cancelling inverses: e.g.

\[ (aba^{-1}bc^{-1}ab)(b^{-1}ac) = aba^{-1}bc^{-1}aac. \]
Application: group theory

If \( f : X \to Y \) is a map of sets, then we have the corresponding map of free groups:

\[
FX \to FY, \quad a_1 \pm 1 a_2 \pm 1 \cdots a_n \pm 1 \mapsto f(a_1) \pm 1 f(a_2) \pm 1 \cdots f(a_n) \pm 1.
\]

So we have a free group functor

\[
F : \text{Set} \to \text{Group}.
\]

Theorem

The free group functor

\[
F : \text{Set} \to \text{Group}
\]

is left adjoint to the forgetful functor

\[
\text{forget} : \text{Group} \to \text{Set}.
\]

Note: Can replace \textbf{Group} by the category of abelian groups to get the free abelian group on a set.
Application: adjoining an identity to a ring

Let \textbf{Ring} be the category of rings.

Let \textbf{Rng} be the category of \textit{general rings} (rings without the multiplicative identity axiom).

The left adjoint to the forgetful functor

\[
\text{forget} : \textbf{Ring} \rightarrow \textbf{Rng}
\]

is a functor that formally adjoins an identity to a ring. It maps a general ring \( R \) to \( R \times \mathbb{Z} \) with multiplication determined by

\[
(r, 0)(0, 1) = (r, 0) = (0, 1)(r, 0),
\]

\[
(r, 0)(s, 0) = (rs, 0),
\]

\[
(0, 1)(0, 1) = (0, 1).
\]

This is a ring with identity \((0, 1)\).
Consider the inclusion functor

\[
\text{category of abelian groups} \to \text{Group}.
\]

It has a left adjoint called \textit{abelianization}, which assigns to every group \( G \) the quotient group

\[
G^{ab} = G/\![G, G],
\]

where \([G, G]\) is the subgroup of \( G \) generated by

\[
g^{-1}h^{-1}gh, \quad g, h \in G.
\]
Let $\textbf{Top}$ be the category of topological spaces.

Consider the forgetful functor

$$\text{forget} : \textbf{Top} \rightarrow \textbf{Set}.$$ 

**Left adjoint:** The functor taking a set to that same set with the \text{discrete topology} is left adjoint to $\text{forget}$.

**Right adjoint:** The functor taking a set to that same set with the \text{trivial topology} is right adjoint to $\text{forget}$. 
Adjoint functors are a natural concept in mathematics that lead to many useful constructions:

- free vector spaces
- free (abelian) groups
- abelianization
- discrete/trivial topology

The idea of adjoint functors can be generalized further, to the setting of

- monoidal categories,
- 2-categories.

This is related to higher representation theory.