Affine wreath product algebras

\[ \sum_{b \in B} b \]

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Goals: Unify and generalize the theory of algebras defined by “degenerate affine Hecke algebra” type relations.

Overview:
1. Background and motivation
2. Frobenius algebras
3. Affine wreath product algebras
4. Structure theory
5. Representation theory
6. Cyclotomic quotients
7. Further directions
Background and motivation
Degenerate affine Hecke algebras

Fix a commutative ring $\mathbb{k}$.

The degenerate affine Hecke algebra $\mathcal{H}_n$ of type $A$ is

$$\mathbb{k}[x_1, \ldots, x_n] \otimes \mathbb{k}S_n$$

as a $\mathbb{k}$-module.

The factors $\mathbb{k}[x_1, \ldots, x_n]$ and $\mathbb{k}S_n$ are subalgebras, and

$$s_ix_j = x_js_i, \quad j \neq i, i + 1,$$

$$s_ix_i = x_{i+1}s_i - 1,$$

where $s_i = (i, i + 1)$ is the simple transposition.

$\mathcal{H}_n$ is a “degeneration” ($q \to 1$) of the affine Hecke algebra.
Surprisingly, modular branching rules for $H_n$ and their cyclotomic quotients are related to affine Lie algebras of type $A$ (Ariki, Grojnowski, Vazirani, Mathas, Lascoux, Leclerc, Thibon, ...).

Brundan and Kleshchev have related cyclotomic quotients to quiver Hecke algebras (Khovanov–Lauda–Rouquier algebras).

Algebras related to $H_n$ have appeared in other places:
- affine Sergeev algebra (degenerate affine Hecke–Clifford algebra),
- wreath Hecke algebra,
- affine zigzag algebra.

It would be nice to unify (and generalize) the treatments of these algebras.
The $\mathcal{H}_n$ appeared in endomorphism spaces of categories introduced by Khovanov to categorify the Heisenberg algebra.

Khovanov’s categories have been generalized (Rosso–S.). One obtains a Heisenberg category for every $\mathbb{Z}$-graded Frobenius superalgebra.

In the endomorphism spaces of these more general categories, one finds a large family of algebras that specialize to many well-studied analogues of $\mathcal{H}_n$.

**Goal**

Study the structure and representation theory of this large family of algebras.

**Terminology:** In this talk, all algebras and modules are super and $\mathbb{Z}$-graded.
Frobenius algebras
Frobenius algebras: Definition

**Definition 1 (trace map)**

A Frobenius algebra is a f.d. associative algebra $F$ together with a linear trace map

$$\text{tr}: F \to k$$

such that $\ker \text{tr}$ contains no nonzero left ideals.

**Definition 2 (bilinear form)**

A Frobenius algebra is a f.d. associative algebra $F$ together with a nondegenerate bilinear form satisfying

$$\langle fg, h \rangle = \langle f, gh \rangle \quad \text{for all } f, g, h \in F.$$

The connection between the two definitions is given by

$$\langle f, g \rangle = \text{tr}(fg).$$

For simplicity, we assume $\text{tr}$ is an even map.
Frobenius algebras: Examples

Example ($\mathbb{k}$)

$\mathbb{k}$ is a Frobenius algebra with $\text{tr} = \text{id}_\mathbb{k}$.

Example (Clifford algebra)

The Clifford algebra

$$\text{Cl} = \mathbb{k}[c]/(c^2 - 1), \quad \bar{c} = 1,$$

is a Frobenius algebra with $\text{tr}(1) = 1$, $\text{tr}(c) = 0$.

Example ($\mathbb{k}[x]/(x^k)$)

$\mathbb{k}[x]/(x^k)$ is a Frobenius algebra with

$$\text{tr}(x^\ell) = \delta_{\ell,k-1},$$

We can give it nontrivial $\mathbb{Z}$-grading by setting $|x| = 1$. 
Frobenius algebras: Examples

Example (Matrix algebra)
Any matrix algebra over a field is a Frobenius algebra with the usual trace.

Example (Group algebra)
Suppose $G$ is a finite group and fix $h \in G$.

The group algebra $\mathbb{k}G$ is a Frobenius algebra with

$$\text{tr}(g) = \delta_{g,h}, \quad g \in G.$$ 

Standard choice: $h = 1_G$.

Example (Hopf algebras)
Every f.d. Hopf algebra is a Frobenius algebra.

From now on: $F$ is a Frobenius algebra with trace $\text{tr}$. 
Frobenius algebras: Nakayama automorphism

The **Nakayama automorphism** is the algebra automorphism \( \psi : F \to F \) defined by

\[
\text{tr}(fg) = (-1)^{\bar{f} \bar{g}} \text{tr}(g\psi(f)), \quad f, g \in F.
\]

We say \( F \) is **symmetric** if \( \psi = \text{id}_F \).

**Examples**

<table>
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<th>( \text{tr} )</th>
<th>( \psi )</th>
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<tr>
<td>( k )</td>
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<td>( Cl )</td>
<td>\text{tr}(1) = 1, \text{tr}(c) = 0</td>
<td>( c \mapsto -c )</td>
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<td>( k[x]/(x^k) )</td>
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<td>matrix algebra</td>
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<td>( kG )</td>
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<td>( g \mapsto h^{-1}gh )</td>
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**Assumption:** the characteristic of \( k \) does not divide the order of \( \psi \).
Frobenius algebras: dual bases

Fix a basis $B$ of $F$. The left dual basis is

$$B^\vee = \{ b^\vee \mid b \in B \}$$

defined by

$$\text{tr} (b^\vee c) = \delta_{b,c}, \quad b, c \in B.$$ 

It is easy to check that

$$\sum_{b \in B} b \otimes b^\vee \in F \otimes F$$

is independent of the basis $B$.

Let $\delta$ be the top nonzero degree of $F$. Then

- $\text{tr}$ has degree $-\delta$,
- $\sum_{b \in B} b \otimes b^\vee$ is even of degree $\delta$. 

Affine wreath product algebras
The symmetric group $S_n$ acts on $F^\otimes n$ by superpermutations:

$$s_i \cdot (f_1 \otimes \cdots \otimes f_n) = (-1)^{\bar{f}_i \bar{f}_{i+1}} f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \otimes \cdots \otimes f_n,$$

We let $\pi f$ denote the action of $\pi \in S_n$ on $f \in F^\otimes n$.

The wreath product algebra is

$$F^\otimes n \rtimes S_n = F^\otimes n \otimes kS_n$$

as $k$-modules. Multiplication is determined by

$$(f_1 \otimes \pi_1)(f_2 \otimes \pi_2) = f_1 \pi_1 f_2 \otimes \pi_1 \pi_2.$$
Example ($F = \mathbb{k}$)

$\mathbb{k}^{\otimes n} \rtimes S_n \cong \mathbb{k}S_n$

Example ($F = \mathbb{Cl}$)

$\mathbb{Cl}^{\otimes n} \rtimes S_n$ is the **Sergeev algebra**, which plays an important role in the projective representation theory of the symmetric group.

Example ($F = \mathbb{k}G$, $G = \mathbb{Z}/2\mathbb{Z}$)

When $G$ is a cyclic group of order 2, $(\mathbb{k}G)^{\otimes n} \rtimes S_n$ is the group algebra of the **hyperoctahedral group**, the Weyl group of type $B$.

Example ($F = \mathbb{k}G$, $G = \mathbb{Z}/r\mathbb{Z}$)

When $G$ is a cyclic group of order $r$, $(\mathbb{k}G)^{\otimes n} \rtimes S_n$ is the group algebra of the **complex reflection group** $G(r, 1, n)$. 
Affine wreath product algebras: Definition

Fix $n \in \mathbb{N}$. For $f \in F$ and $1 \leq i \leq n$, define

$$f_i = 1 \otimes (i-1) \otimes f \otimes 1 \otimes (n-i) \in F \otimes n,$$

$$\psi_i = \text{id} \otimes (i-1) \otimes \psi \otimes \text{id} \otimes (n-i) : F \otimes n \to F \otimes n.$$  

We define the affine wreath product algebra $A_n(F)$ to be the $\mathbb{Z}$-graded superalgebra that is the free product of $k$-algebras

$$k[x_1, \ldots, x_n] \star F \otimes n \star kS_n,$$

modulo the relations

$$fx_i = x_i \psi_i(f), \quad 1 \leq i \leq n, \ f \in F \otimes n,$$

$$s_ix_j = x_js_i, \quad 1 \leq i \leq n-1, \ 1 \leq j \leq n, \ j \neq i, i+1,$$

$$s_ix_i = x_{i+1}s_i - t_{i,i+1}, \quad 1 \leq i \leq n-1,$$

$$\pi f = \pi f \pi, \quad \pi \in S_n, \ f \in F \otimes n,$$

where

$$t_{i,j} := \sum_{b \in B} b_i b_j^\vee \quad \text{for} \quad 1 \leq i, j \leq n, \ i \neq j.$$
The degree and parity on $A_n(F)$ are determined by

$$|x_i| = \delta, \quad \bar{x}_i = 0, \quad 1 \leq i \leq n,$$

$$|\pi| = 0, \quad \bar{\pi} = 0, \quad \pi \in S_n,$$

while degree and parity for elements of $F^\otimes n$ are as they are in $F^\otimes n$.

Lemma (S. 2017)

Up to isomorphism, $A_n(F)$ depends only on the underlying algebra $F$, and not on the trace map $tr$.

Proof: Uses fact that different trace maps differ by multiplication by an invertible element of $F$. 
Example ($F = k$)

$A_n(k)$ is the degenerate affine Hecke algebra.

Example ($F = Cl$)

$A_n(Cl)$ is the affine Sergeev algebra, otherwise known as the degenerate affine Hecke–Clifford algebra.

This algebra was introduced by Nazarov in his study of the projective representation theory of the symmetric group.

Example ($F = kG$)

$A_n(kG)$ is the wreath Hecke algebra studied by Wan and Wang.

Example (Affine zigzag algebras)

When $F$ is a certain skew-zigzag algebra, $A_n(F)$ is related to imaginary strata for quiver Hecke algebras by work of Kleshchev and Muth.
Deformed divided difference operators

Let

$$P_n = \mathbb{k}[x_1, \ldots, x_n],$$

and let

$$P_n(F) = P_n \otimes F \otimes^n,$$

where the two factors are subalgebras and

$$f x_i = x_i \psi_i(f), \quad 1 \leq i \leq n, \quad f \in F \otimes^n.$$

**Definition**

Define a skew derivation $\Delta_i : P_n(F) \rightarrow P_n(F)$ inductively, by

$$\Delta_i(F \otimes^n) = 0,$$

$$\Delta_i(x_i) = t_{i,i+1}, \quad \Delta_i(x_{i+1}) = -t_{i+1,i}, \quad \Delta_i(x_j) = 0, \quad j \neq i, i + 1,$$

and

$$\Delta_i(a_1 a_2) = \Delta_i(a_1) a_2 + s_i a_1 \Delta_i(a_2), \quad a_1, a_2 \in P_n(F).$$
Lemma (S. 2017)

For all $a \in P_n(F)$ and $1 \leq i \leq n - 1$, in $A_n(F)$ we have

\[ s_i a = s_i a s_i - \Delta_i(a). \]

The $\Delta_i$ are $F$-deformations of divided difference operators. In particular, if

\[ \partial_i(p) = \frac{p - s_i p}{x_i - x_{i+1}}, \quad p \in P_n, \]

is the usual divided difference operator, then

\[ \Delta_i \left( x_i^k \right) = \sum_{b \in B} b_i \partial_i \left( x_i^k \right) b_{i+1}^\vee, \]

\[ \Delta_i \left( x_{i+1}^k \right) = \sum_{b \in B} b_{i+1} \partial_i \left( x_{i+1}^k \right) b_i^\vee. \]
Deformed divided difference operators

The $\Delta_i$ have other properties analogous to those of divided difference operators.

**Proposition (S. 2017)**

We have

\[
\begin{align*}
\Delta_i (s_j a) &= s_j \Delta_i (a), & 1 \leq i, j \leq n - 1, \ |i - j| > 1, \ a \in P_n(F), \\
\Delta_i (s_i a) &= -s_i \Delta_i (a), & 1 \leq i \leq n - 1, \ a \in P_n(F), \\
\Delta_i \Delta_j &= \Delta_j \Delta_i, & 1 \leq i, j \leq n - 1, \ |i - j| > 1, \\
\Delta_i^2 &= 0, & 1 \leq i \leq n - 1.
\end{align*}
\]
Basis Theorem

**Theorem (S. 2017)**

The map

\[ k[x_1, \ldots, x_n] \otimes F^\otimes n \otimes kS_n \to A_n(F'), \]

\[ p \otimes f \otimes \pi \mapsto pf\pi, \]

is an isomorphism of \( \mathbb{Z} \times \mathbb{Z}_2 \)-graded \( k \)-modules.

For special choices of \( F \), recovers known results, but with a uniform proof.

**Corollary**

As \( \mathbb{Z} \times \mathbb{Z}_2 \)-graded \( k \)-modules, we have

\[ A_n(F') = k[x_1, \ldots, x_n] \otimes (F^\otimes n \rtimes S_n), \]

and the two factors are subalgebras.

This is the motivation for the name **affine wreath product algebra**.
For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, let

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \alpha \in \mathbb{N}^n,$$

$$F^{(\alpha)} = \left\{ f \in F^\otimes n \mid gf = (-1)^{\overline{\alpha} \cdot \overline{\phi} f_1^{\psi_1} \cdots \psi_n^{\alpha_n} (g)} \text{ for all } g \in F^\otimes n \right\},$$

$$F^{(\alpha)}_\psi = \left\{ f \in F^{(\alpha)} \mid \psi_i(f) = f, \ 1 \leq i \leq n \right\}.$$
The center: Examples

Example ($F = \mathbb{k}$)

We recover the well-known result that the center of the degenerate affine Hecke algebra consists of all symmetric polynomials in the $x_i$:

$$Z(\mathcal{A}_n(\mathbb{k})) = \mathbb{k}[x_1, \ldots, x_n]^{S_n}.$$ 

Example ($F = \mathbb{k}G$)

We recover a description of the center of wreath Hecke algebras given by Wan and Wang.

Example ($F = \text{Cl}$)

We recover the known result (due to Nazarov) that the center of the affine Sergeev algebra is

$$Z(\mathcal{A}_n(\text{Cl})) = \mathbb{k} \left[ x_1^2, \ldots, x_n^2 \right]^{S_n}.$$
Corollary
The center contains
\[ k \left[ x_1^\theta, \ldots, x_n^\theta \right]^{S_n}, \]
where \( \theta \) is the order of the Nakayama automorphism \( \psi \).
In particular, \( \mathcal{A}_n(F) \) is finitely generated as a module over its center.

Corollary
If \( k \) is an algebraically closed field, all simple \( \mathcal{A}_n(F) \)-modules are finite-dimensional.
Jucys–Murphy elements: The classical case ($\mathbb{k}S_n$)

In $\mathbb{k}S_n$, the Jucys–Murphy elements are

$$J_1 = 0, \quad J_k = \sum_{i=1}^{k-1} (i, k), \quad 2 \leq k \leq n,$$

where $(i, k) \in S_n$ is the transposition of $i$ and $k$.

**Properties**

1. We have a surjective algebra homomorphism

   $$\text{deg. aff. Hecke alg. } \mathcal{H}_n \twoheadrightarrow \mathbb{k}S_n$$

   that is the identity on $\mathbb{k}S_n$ and maps $x_k$ to $J_k$.

2. $J_n$ generates the centralizer of $\mathbb{k}S_{n-1}$ in $\mathbb{k}S_n$.

3. The center of $\mathbb{k}S_n$ consists of symmetric polynomials in the $J_k$.

4. Irreducible reps of $S_n$ have bases given by standard tableaux of a given shape. Entries in the tableaux correspond to eigenvalues of the JM elements.
Define the Jucys–Murphy elements

\[ J_1 = 0, \quad J_k = \sum_{i=1}^{k-1} t_{i,k}(i, k) = \sum_{i=1}^{k-1} \sum_{b \in B} b_i b_k^\vee(i, k), \quad 2 \leq k \leq n. \]

**Proposition (S. 2017)**

There is a surjective algebra homomorphism

\[ \mathcal{A}_n(F) \to F \otimes^n \rtimes S_n, \]

\[ x_k \mapsto J_k, \quad 1 \leq k \leq n, \quad a \mapsto a, \quad a \in F \otimes^n \rtimes S_n. \]

**Examples**

1. \( F = \mathbb{k} \): the \( J_k \) are the usual JM elements for \( \mathbb{k}S_n \).
2. \( F = \text{Cl} \): the \( J_k \) are the JM elements of the Sergeev algebra.
3. \( F = \mathbb{k}G \): the \( J_k \) are the JM elements def. by Pushkarev and Wang.
Representation theory
Some associated algebras

Twisting by the Nakayama automorphism \( \psi \) yields a permutation of the isomorphism classes of irreducible representations of \( F \).

Fix a list

\[
L_1, L_2, \ldots, L_N
\]

of representatives of the orbits of this permutation (up to degree shift).

For \( 1 \leq \ell \leq N \) and \( n \in \mathbb{N} \), we let

\[
H^{\ell}_n = \begin{cases} 
A_n(\mathbb{k}), \\
A_n(\text{Cl}), \\
(\mathbb{k}[y] \rtimes A)^{\otimes n} \rtimes S_n.
\end{cases}
\]

where the particular case depends on the representation \( L_\ell \), and \( A \) is a certain f.d. algebra related to \( L_\ell \).

Important: Simple modules of the \( H^{\ell}_n \) are classified.
An equivalence of categories

Let

\[ \mathcal{R}_n = \bigoplus_{\mu} (\mathcal{H}_{\mu_1}^1 \otimes \cdots \otimes \mathcal{H}_{\mu_N}^N), \]

where the sum is over all compositions of \( n \) of length at most \( N \):

\[ \mu = (\mu_1, \ldots, \mu_N), \quad \sum \mu_i = n. \]

**Theorem (S. 2017)**

The category \( \mathcal{R}_n \)-mod is equivalent to the category of \( \mathcal{A}_n(F) \)-modules that are semisimple as \( F^\otimes n \)-modules.

**Proposition (S. 2017)**

Every simple \( \mathcal{A}_n(F) \)-module is semisimple as an \( F^\otimes n \)-module.
Classification of simple $\mathcal{A}_n(F)$-modules

Define the parabolic subalgebra

$$\mathcal{A}_\mu(F') = \mathcal{A}_{\mu_1}(F') \otimes \cdots \otimes \mathcal{A}_{\mu_N}(F') \subseteq \mathcal{A}_n(F).$$

**Theorem (S. 2017)**

Every simple $\mathcal{A}_n(F')$-module is isomorphic one of the form

$$\text{Ind}_{\mathcal{A}_\mu(F')}^{\mathcal{A}_n(F')} \left( L(\mu) \otimes_{E(\mu)} (V_1 \star \cdots \star V_N) \right),$$

where

- $\mu$ is a composition of $n$ of length at most $N$,
- $V_j$ is a simple $\mathcal{H}_{\mu_\ell}$-module for $1 \leq \ell \leq N$,
- $\star$ denotes the simple tensor product of supermodules (simple summand of the usual tensor product),
- $E(\mu)$ is a fixed subalgebra of $\mathcal{R}_\mu$,
- $L(\mu)$ is a fixed bimodule.
Cyclotomic quotients
Cyclotomic quotients: The classical case

Any f.d. representation of the degenerate affine Hecke algebra factors through a f.d. cyclotomic quotient

$$\mathcal{H}_n^f := \mathcal{H}_n / (f(x_1)),$$

where $f$ is a monic polynomial.

Induction/restriction functors on the categories $\mathcal{H}_n^f$-mod, $n \in \mathbb{N}$, relate these categories to the irreducible highest weight representation of an affine Lie algebra of type $A$.

The highest weight of the Lie algebra representation is encoded in the polynomial $f$.

This gives a powerful technique for studying the representation theory of $\mathcal{H}_n$ and its cyclotomic quotients.
Cyclotomic quotients for $\mathcal{A}_n(F)$

Recall that $\theta$ is the order of the Nakayama automorphism $\psi$ and $\delta$ is the top degree of $F$.

For $1 \leq k \leq \theta$, let $\mathbf{F}_1^{(k)} \subseteq F^\otimes n$ consist of all elements $\mathbf{f} \in F^\otimes n$ such that

- $\psi_i(\mathbf{f}) = \mathbf{f}$ for all $1 \leq i \leq n$,
- $g\mathbf{f} = \mathbf{f}\psi_1^k(g)$ for all $g \in F^\otimes n$,
- $\pi\mathbf{f} = \mathbf{f}$ for all $\pi \in S_n$ such that $\pi(1) = 1$.

Intuitively: $\mathbf{F}_1^{(k)}$ is the subspace of $F^\otimes n$ consisting of those elements that commute with elements of $\mathcal{A}_n(F)$ such as $x_1^k$ does.

For $1 \leq k \leq \theta$, choose $e_k \in \mathbb{N}$ and degree $k\delta$ elements

$$\mathbf{c}^{(k,1)}, \ldots, \mathbf{c}^{(k,e_k)} \in \mathbf{F}_1^{(k)}.$$ 

Define

$$\mathbf{C} = \left(\mathbf{c}^{(1,1)}, \ldots, \mathbf{c}^{(1,e_1)}, \ldots, \mathbf{c}^{(\theta,1)}, \ldots, \mathbf{c}^{(\theta,e_\theta)}\right).$$
Cyclotomic quotients for $\mathcal{A}_n(F)$

Let $J_C$ be the two-sided ideal in $\mathcal{A}_n(F)$ generated by

$$\chi_C = \prod_{\theta} \prod_{k=1}^{e_k} \prod_{j=1} \left(x_1^k - c^{(k,j)} \right).$$

The cyclotomic wreath product algebra is

$$\mathcal{A}_n^C(F) = \mathcal{A}_n(F)/J_C.$$

The level $d_C$ of $C$ and $\mathcal{A}_n^C(F)$ is the degree of $\chi_C$ as a polynomial in $x_1$.

Note that the $\mathcal{A}_n^C(F)$ are finite-dimensional.
Cyclotomic basis theorem

**Theorem (S. 2017)**

The canonical images of the elements

\[ x^\alpha b \pi, \quad \alpha_1, \ldots, \alpha_n \prec d_C, \quad b \in B \otimes^n, \quad \pi \in S_n, \]

form a basis for \( \mathcal{A}_C^n(F) \).

**Corollaries**

1. \( F = k \) or \( F = Cl \): Theorem recovers known results.
2. \( F = kG \): Theorem recovers result of Wan and Wang.
3. \( F \) even and symmetric: Theorem proves an open conjecture of Kleshchev and Muth.

**Corollary**

Every level one cyclotomic wreath product algebra is isomorphic to the wreath product algebra \( F \otimes^n \rtimes S_n \).
Frobenius algebra structure on $\mathcal{A}_n^C(F)$

Define an even linear map $\text{tr}_C : \mathcal{A}_n^C(F) \to \mathbb{k}$ by

$$\text{tr}_C (x^\alpha f \pi) = \delta_{\alpha,(d-1,\ldots,d-1)} \text{tr} \otimes^n (f) \delta_{\pi,1},$$

for $f \in F \otimes^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\alpha_1, \ldots, \alpha_n < d_C$.

**Theorem (S. 2017)**

$\mathcal{A}_n^C(F)$ is an $\mathbb{N}$-graded Frobenius superalgebra with trace map $\text{tr}_C$ and Nakayama automorphism given by

$$x_i \mapsto x_i, \quad f \mapsto \left(\psi^{d_C}\right) \otimes^n (f), \quad \pi \mapsto \pi, \quad 1 \leq i \leq n, \; f \in F \otimes^n, \; \pi \in S_n.$$

**Corollary**

*F even and symmetric*: Theorem gives a proof of another open conjecture of Kleshchev and Muth.
Future directions
**Future directions**

### $q$-deformations

Many special cases of affine wreath product algebras are degenerations of $q$-deformed versions:

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<th>Degeneration</th>
<th>$q$-deformation</th>
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<td>affine Hecke algebra</td>
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<td>affine Sergeev algebra</td>
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<tr>
<td>wreath Hecke algebra</td>
<td>affine Yokonuma–Hecke algebra</td>
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It would be interesting to construct a $q$-deformation of affine wreath product algebras.

### Double affine versions

Are there natural double affine versions of wreath product algebras generalizing double affine Hecke algebras (Cherednik algebras) and their various degenerations?
Khovanov conjecturally categorified the Heisenberg algebra via a graphical category based on the rep theory of $\mathbb{k}S_n$. The degenerate affine Hecke algebra appears naturally in endomorphism spaces.

This was generalized by replacing $\mathbb{k}S_n$ by wreath product algebras (Rosso–S.). Affine wreath product algebras appear naturally in endomorphism spaces.

Khovanov’s cat. was also generalized to higher level by replacing $\mathbb{k}S_n$ with degenerate cyclotomic Hecke algebras (Mackaay–S.).

There should be a graphical category based on $A_n^C(F)$ generalizing/unifying all of the above.
Future directions

Branching rules

- **Branching rules** involve explicit descriptions of restriction/induction functors acting on irreducible representations.
- For $F = \mathbb{k}$, $\text{Cl}$, or $\mathbb{k}G$, branching rules are related to representation theory of affine Lie algebras.
- For $F$ semisimple, previous methods should work.
- For general $F$, situation would be more involved.

Other types, formal group laws

- The $\mathcal{A}_n(F)$ are, in many ways, **type A** objects.
- It would be nice to generalize to other types.
- One can associate a **formal affine Hecke algebra** to any **formal group law** (Malagón-Lopez–Hoffnung–Zainouilline–S.).
- Can one define “Frobenius algebra deformations” of formal affine Hecke algebras?