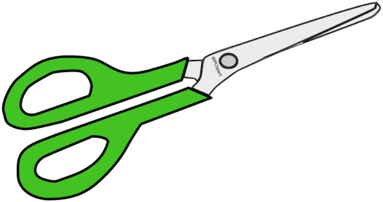
Truncated multicurrent algebras



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Outline

Goal: Describe the algebra of invariant polynomials for truncated multicurrent algebras.

Overview:

1. Motivation
   - Equivariant map algebras
   - Reduction to the truncated multicurrent case
2. Construct invariant polynomials
3. Algebraic independence
4. Transversal slices
5. Main result: description of algebra of invariant polynomials
6. Application to representation theory
7. Further directions

Map algebras

We work over the field $\mathbb{C}$ of complex numbers (for simplicity).

$\mathfrak{g}$ is a finite-dimensional Lie algebra.

A is a finitely-generated commutative associative unital algebra.

Map algebra

The Lie algebra $\mathfrak{g} \otimes A$ with bracket determined by

$$[x_1 \otimes a_1, x_2 \otimes a_2] = [x_1, x_2] \otimes a_1 a_2, \quad x_1, x_2 \in \mathfrak{g},\ a_1, a_2 \in A,$$

is a map algebra, or generalized current algebra.

We can view $\mathfrak{g} \otimes A$ as the Lie algebra of algebraic maps

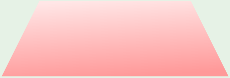
$$\operatorname{Spec} A \to \mathfrak{g}$$

with pointwise multiplication.

Examples of map algebras

Example (Current algebras)

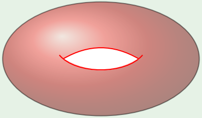
If $A = \mathbb{C}[t_1, \ldots, t_n]$, then $\operatorname{Spec} A$ is affine space $\mathbb{A}^n$ and $\mathfrak{g} \otimes \mathbb{C}[t_1, \ldots, t_n]$ is a multicurrent algebra.



$\longrightarrow \quad \mathfrak{g}$

Example (Multiloop algebras)

If $A = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, then $\operatorname{Spec} A$ is the n-dimensional torus and $\mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is a multiloop algebra.



$\longrightarrow \quad \mathfrak{g}$

Equivariant map algebras

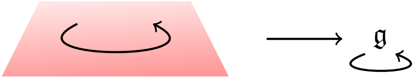
Now suppose a finite group Γ acts on $\mathfrak{g}$ and A via automorphisms.

Then Γ acts diagonally on $\mathfrak{g} \otimes A$.

Equivariant map algebras

The Lie subalgebra $(\mathfrak{g} \otimes A)^\Gamma$ of Γ-fixed points of $\mathfrak{g} \otimes A$ is an equivariant map algebra.

$(\mathfrak{g} \otimes A)^\Gamma$ is the Lie algebra of Γ-equivariant algebraic maps from $\operatorname{Spec} A$ to $\mathfrak{g}$, with pointwise multiplication.



Equivariant map algebras: Examples

Example

If $A = \mathbb{C}[t_1, \dots, t_n]$, then $(\mathfrak{g} \otimes A)^\Gamma$ is a twisted multicurrent alegbra

Example

If $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, then $(\mathfrak{g} \otimes A)^\Gamma$ is a twisted multiloop algebra.

Finite-dimensional representations

Evaluation representations

Given

- a maximal ideal m of A, and
- a f.d. representation $\rho\colon \mathfrak{g} \to \operatorname{End} V$,

we have the associated evaluation representation of $(\mathfrak{g} \otimes A)^\Gamma$:

$$(\mathfrak{g} \otimes A)^\Gamma \hookrightarrow \mathfrak{g} \otimes A \twoheadrightarrow \mathfrak{g} \otimes A/\mathsf{m} \cong \mathfrak{g} \xrightarrow{\rho} \operatorname{End} V.$$

Assume:

- Γ is abelian and acts freely on $\operatorname{maxSpec} A$.
- A is finitely generated
- $\mathfrak{g}$ is semisimple.

Theorem

All irreducible reps of $(\mathfrak{g} \otimes A)^\Gamma$ are tensor products of evaluation representations.

Twisting and untwisting

It follows that all f.d. reps of $(\mathfrak{g} \otimes A)^\Gamma$ are tensor products of ones factoring through

$$(\mathfrak{g} \otimes A)^\Gamma \hookrightarrow \mathfrak{g} \otimes A \twoheadrightarrow \mathfrak{g} \otimes A/\mathsf{m}^N$$

for some maximal ideal $\mathsf{m} \subseteq A$ and $N \in \mathbb{N}$. We say such a rep is supported at m.

Theorem (Fourier–Khandai–Kus–Savage 2012)

Give a maximal ideal m of A, we have twisting and untwisting functors yielding an equivalence of categories:

$$\begin{array}{c} \text{category of reps of } (\mathfrak{g} \otimes A)^\Gamma \text{ supported at m} \\ \text{untwist} \downarrow \quad \uparrow \text{twist} \\ \text{category of reps of } \mathfrak{g} \otimes A \text{ supported at m} \end{array}$$

Reduction to the truncated multicurrent case

Since A is finitely generated, we may assume

$$A = \mathbb{C}[t_1, \dots, t_\ell]/I \quad \text{for some ideal} \quad I \subseteq \mathbb{C}[t_1, \dots, t_\ell].$$

Maximal ideals of A correspond to maximal ideals of $\mathbb{C}[t_1, \dots, t_\ell]$ containing I.

By the above, we may restrict out attention to modules annihilated by

$$\mathfrak{g} \otimes (\mathsf{m}^N + I)$$

for some maximal ideal m of A containing I.

Translating if necessary, we may assume that

$$\mathsf{m} = (t_1, \dots, t_\ell)$$

is the maximal ideal corresponding to the origin.

Reduction to the truncated multicurrent case

If $\mathfrak{m}^N + I$ annihilates a module, so does $\mathfrak{m}^N$.

Therefore, we can focus on modules for the quotient

$$\mathfrak{g} \otimes \mathbb{C}[t_1, \ldots, t_n]/\mathfrak{m}^N.$$

Summary

We can "reduce" the study of f.d. $\mathfrak{g} \otimes A$-modules to the case

$$A = \mathbb{C}[t_1, \ldots, t_\ell]/I,$$

where I is an ideal of $\mathbb{C}[t_1, \ldots, t_\ell]$ of finite codimension, generated by monomials. We call such $\mathfrak{g} \otimes A$ truncated multicurrent algebras.

Invariant polynomials

Useful tool in rep theory of Lie algebras: Action of the center of the universal enveloping algebra.

Duflo isomorphism

Let $\mathfrak{a}$ be a f.d. algebra.

There is an algebra isomorphism between

- center of $U(\mathfrak{a})$ and
- the algebra $S(\mathfrak{a})^{\mathfrak{a}}$ of invariants in the symmetric algebra of $\mathfrak{a}$.

Goal of talk/paper

Describe the algebra of invariants for truncated multicurrent alegbras.

Overview

Step 1: Construct invariant polynomials

For any f.d. $\mathfrak{g}$, we will describe a procedure for producing invariant polys in $\mathfrak{g} \otimes A$ from invariant polys in $\mathfrak{g}$.

$$S(\mathfrak{g})^{\mathfrak{g}} \rightsquigarrow S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}.$$

Step 2: Algebraic independence

For any f.d. $\mathfrak{g}$, we show that a collection of invariant polys in $\mathfrak{g}$ is algebraically independent if and only if the corresponding collection of invariant polys in $\mathfrak{g} \otimes A$ is.

$$\text{alg. ind. for } \mathfrak{g} \iff \text{alg. ind. for } \mathfrak{g} \otimes A.$$

Step 3: Generating set

When $\mathfrak{g}$ is semisimple, we can find an algebraically independent system of invariant polys that also generates $S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}$.

Truncated multicurrent algebras

Assumption: $\mathfrak{g}$ is a f.d. Lie algebra.

Fix $\ell \geq 1$, and define a partial order on $\mathbb{N}^\ell$ by

$$(n_1, \ldots, n_\ell) \leq (m_1, \ldots, m_\ell) \iff n_i \leq m_i \text{ for all } i \in \{1, \ldots, \ell\}.$$

For $\omega = (\omega_1, \ldots, \omega_\ell) \in \mathbb{N}^\ell$, define

$$t^\omega = t_1^{\omega_1} t_2^{\omega_2} \cdots t_\ell^{\omega_\ell} \in \mathbb{C}[t_1, \ldots, t_\ell].$$

Fix $\mu \in \mathbb{N}^\ell$ and define

$$\begin{aligned} A &= \mathbb{C}[t_1, \ldots, t_\ell] / \operatorname{Span}_{\mathbb{C}}\{t^\omega \mid \omega \not\leq \mu\} \\ &= \mathbb{C}[t_1, \ldots, t_\ell] / (t_1^{\mu_1+1}, t_2^{\mu_2+1}, \ldots, t_\ell^{\mu_\ell+1}). \end{aligned}$$

We are interested in the truncated multicurrent algebra $\mathfrak{g} \otimes A$.

Adoint action

Suppose $\mathfrak{a}$ is a f.d. Lie algebra (e.g. $\mathfrak{g}$ or $\mathfrak{g} \otimes A$).

$\mathfrak{a}$ acts on itself via the adjoint action:

$$x \cdot y = [x, y], \quad x, y \in \mathfrak{a}.$$

Have induced action of $\mathfrak{a}$ on the symmetric algebra $S(\mathfrak{a})$ defined inductively by

$$x \cdot (p_1 p_2) = (x \cdot p_1) p_2 + p_1 (x \cdot p_2), \quad x \in \mathfrak{a},\ p_1, p_2 \in S(\mathfrak{a}).$$

Goal

Use knowledge of $S(\mathfrak{g})^{\mathfrak{g}}$ to describe $S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}$.

Gradings

Recall

$\mu \in \mathbb{N}^\ell$ and $A = \mathbb{C}[t_1, \ldots, t_\ell] / \operatorname{Span}_{\mathbb{C}}\{t^\omega \mid \omega \not\leq \mu\}$.

Note that A has a basis

$$t^\omega, \quad \omega \in \mathbb{N}^\ell,\ \omega \leq \mu.$$

(We denote the image of t^ω in A again by t^ω.)

A is naturally $\mathbb{N}^\ell$-graded, with

$$\deg t^\omega = \omega \quad \text{for } \omega \leq \mu.$$

This induces $\mathbb{N}^\ell$-gradings on

- $\mathfrak{g} \otimes A$, and
- $S(\mathfrak{g} \otimes A)$.

$S(\mathfrak{g}) \rightsquigarrow S(\mathfrak{g} \otimes A)$

Consider the linear map

$$\tau \colon \mathfrak{g} \to \mathfrak{g} \otimes A, \quad x \mapsto \sum_{\omega \le \mu} x \otimes t^\omega.$$

This induces an algebra homomorphism

$$\tau \colon S(\mathfrak{g}) \to S(\mathfrak{g} \otimes A), \quad \text{and } \tau\left(S^k(\mathfrak{g})\right) \subseteq S^k(\mathfrak{g} \otimes A).$$

Given $p \in S(\mathfrak{g})$, we can decompose $\tau(p)$ into a sum of homogeneous terms:

$$\tau(p) = \sum_{\omega \in \mathbb{N}^\ell} p_\omega, \quad \text{where } p_\omega \in S(\mathfrak{g} \otimes A)_\omega.$$

Note

For $p \in S^k(\mathfrak{g})$, we have $p_\omega = 0$ for $\omega \not\le k\mu = (k\mu_1, \ldots, k\mu_\ell)$.

Invariance

We have

$$\left(p \in S^k(\mathfrak{g}),\ \omega \in \mathbb{N}^\ell\right) \rightsquigarrow p_\omega \in S^k(\mathfrak{g} \otimes A)_\omega.$$

Question

If p is $\mathfrak{g}$-invariant, is p_ω invariant for all ω?

$$p \in S(\mathfrak{g})^{\mathfrak{g}},\ \omega \in \mathbb{N}^\ell \overset{?}{\Longrightarrow} p_\omega \in S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}$$

Answer: No. However...

Proposition (Macedo–S. 2016)

If $p \in S^k(\mathfrak{g})^{\mathfrak{g}}$, then

$$p_\omega \in S^k(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A} \quad \text{whenever} \quad (k-1)\mu \leq \omega \leq k\mu.$$

Note: For $\ell = 1$, this reduces to a result of Raïs–Tauvel.

Example

Suppose $\mathfrak{g} = \mathfrak{sl}_2$ and $A = \mathbb{C}[t]/(t^2)$. So $\mu = 1$.

Let $\{x_+, x_-, h\}$ be the standard Chevalley basis.

Then $S(\mathfrak{g})^{\mathfrak{g}}$ is generated by

$$p = h^2 + 4x_-x_+ \in S^2(\mathfrak{g})^{\mathfrak{g}}.$$

Then

$$p_2 = (h \otimes t)(h \otimes t) + 4(x_- \otimes t)(x_+ \otimes t) \in S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A},$$

$$p_1 = (h \otimes t)(h \otimes 1) + 4(x_- \otimes t)(x_+ \otimes 1) + 4(x_- \otimes 1)(x_+ \otimes t) \in S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A},$$

but

$$p_0 = (h \otimes 1)(h \otimes 1) + 4(x_- \otimes 1)(x_+ \otimes 1) \notin S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A},$$

since, for instance,

$$(h \otimes t) \cdot p_0 = -8(x_- \otimes t)(x_+ \otimes 1) + 8(x_- \otimes 1)(x_+ \otimes t) \neq 0.$$

What next?

Recap: We have a method for constructing elements of $S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}$ from elements of $S(\mathfrak{g})^{\mathfrak{g}}$.

$$S(\mathfrak{g})^{\mathfrak{g}} \rightsquigarrow S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}.$$

Ideally, we'd like to use this to get a presentation of $S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}$

Question 1

Do we get all of $S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}$ from the above construction?

Question 2

Can we deduce relations between elements of $S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}$ obtained from the above construction?

We first tackle Question 2.

Algebraic independence

Proposition (Macedo–S. 2016)

The collection

$$p^{(i)} \in S^{k_i}(\mathfrak{g}), \quad 1 \le i \le r,\ k_i \in \mathbb{N}_+,$$

is algebraically independent if and only if the collection

$$p_\omega^{(i)} \in S^{k_i}(\mathfrak{g} \otimes A), \quad 1 \le i \le r,\ (k_i - 1)\mu \le \omega \le k_i\mu,$$

is algebraically independent.

Note: When $\ell = 1$, this reduces to a result of Raïs–Tauvel.

For our purposes, this sufficiently answers Question 2 about relations.

Passing to the dual

From now on, we assume $\mathfrak{g}$ is semisimple.

Via the Killing form

$$\kappa \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$$

we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$.

The Killing form induces a symmetric invariant nondegenerate bilinear form

$$\kappa_A \colon (\mathfrak{g} \otimes A) \otimes (\mathfrak{g} \otimes A) \to \mathbb{C},$$

$$\kappa_A \left(\sum_{\omega \le \mu} x_\omega \otimes t^\omega, \sum_{\omega \le \mu} y_\omega \otimes t^\omega \right) = \sum_{\omega + \omega' = \mu} \kappa(x_\omega, y_{\omega'}).$$

Via κ_A we can identity elements of $(\mathfrak{g} \otimes A)^*_{\mu-\omega}$ with elements of $\mathfrak{g} \otimes A_\omega$.

Passing to the dual

Under the above identifications, our results so far become:

Suppose

$$p^{(i)} \in S^{k_i}(\mathfrak{g}^*)^{\mathfrak{g}} \quad \text{for } 1 \le i \le r.$$

Invariance

$p^{(i)}_\omega \in S^{k_i}\big((\mathfrak{g} \otimes A)^*\big)^{\mathfrak{g} \otimes A}$ for all $1 \le i \le r$, $0 \le \omega \le \mu$.

Independence

The $p^{(i)}$ are algebraically independent if and only if the $p^{(i)}_\omega$ are algebraically independent.

Transversal slices

We recall some well-known facts about transversal slices.

Definitions

- An element of $\mathfrak{g}$ is regular if its centralizer has minimal dimension.
- An $\mathfrak{sl}_2$-triple is principal if its elements are regular.

Fix a principal $\mathfrak{sl}_2$-triple (x_+, x_-, h) of $\mathfrak{g}$. (One always exists.)

Transversal slice

The associated transversal slice is

$$\mathfrak{t}_0 = x_+ + \mathfrak{g}^{x_-},$$

where

$$\mathfrak{g}^{x_-} = \{y \in \mathfrak{g} \mid [x_-, y] = 0\}$$

is the centralizer of x_- in $\mathfrak{g}$.

Transversal slices

$$\mathfrak{t}_0 = x_+ + \mathfrak{g}^{x_-}$$

Properties of transversal slices

1. Every element of $\mathfrak{t}_0$ is regular.
2. The orbit under the adjoint action of every regular element of $\mathfrak{g}$ intersects $\mathfrak{t}_0$ at a unique point, and this intersection is transversal. That is,

$$\mathfrak{g} = \mathfrak{g}^{x_-} \oplus [\mathfrak{g}, x] \quad \text{for all } x \in \mathfrak{t}_0.$$

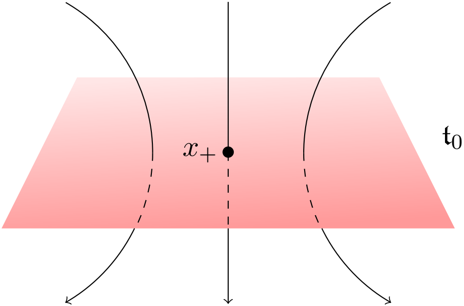
3. Consider the restriction of polynomial functions

$$R_0 \colon S(\mathfrak{g}^*) \to S(\mathfrak{t}_0^*) = \mathbb{C}[\mathfrak{t}_0].$$

   The restriction of R_0 to invariant polys yields an algebra isomorphism

$$S(\mathfrak{g}^*)^{\mathfrak{g}} \cong \mathbb{C}[\mathfrak{t}_0].$$

Transversal slices



$\mathfrak{t}_0 = x_+ + \mathfrak{g}^{x_-}$

Some ingredients

The affine space $\mathfrak{t}$ in $\mathfrak{g} \otimes A$

Easy: $\mathfrak{g}^{x_-} \otimes A$ is the centralizer of $x_- \otimes 1$ in $\mathfrak{g} \otimes A$.

We define the affine space

$$\mathfrak{t} = x_+ \otimes 1 + \mathfrak{g}^{x_-} \otimes A \subseteq \mathfrak{g} \otimes A.$$

Generators of $S(\mathfrak{g}^*)^{\mathfrak{g}}$

By a result of Kostant, $S(\mathfrak{g}^*)^{\mathfrak{g}}$ is a polynomial algebra.

So we can choose a system

$$p^{(i)} \in S^{k_i}(\mathfrak{g}^*)^{\mathfrak{g}}, \quad 1 \leq i \leq r,$$

of algebraically independent generators of $S(\mathfrak{g}^*)^{\mathfrak{g}}$.

Main result

Recall: $\mathfrak{g}$ is semisimple, $\mu \in \mathbb{N}^\ell$, $A = \mathbb{C}[t_1, \ldots, t_\ell]/(t_1^{\mu_1+1}, \ldots, t_\ell^{\mu_\ell+1})$.

Theorem (Macedo–S. 2016)

1. Restriction of poly functions yields an algebra isomorphism
$$S\big((\mathfrak{g} \otimes A)^*\big)^{\mathfrak{g}\otimes A} \to S(\mathfrak{t}^*) = \mathbb{C}[\mathfrak{t}].$$
2. The polys
$$p_\omega^{(i)}, \quad 1 \le i \le r,\ 0 \le \omega \le \mu$$
are algebraically independent generators of $S\big((\mathfrak{g} \otimes A)^*\big)^{\mathfrak{g}\otimes A}$.
3. The orbit under the adjoint action of every regular element of $\mathfrak{g} \otimes A$ intersects $\mathfrak{t}$ at a unique point, and this intersection is transversal. That is,
$$\mathfrak{g} \otimes A = \big(\mathfrak{g}^{x_-} \otimes A\big) \oplus [\mathfrak{g} \otimes A, X], \quad \text{for all } X \in \mathfrak{t}.$$

Note: When $\ell = 1$, this recovers results of Raïs–Tauvel.

Returning from the dual

Using the Killing form, for each $1 \le i \le r$, we can identify

$$S(\mathfrak{g}^*)^{\mathfrak{g}} \ni p^{(i)} \leftrightsquigarrow \rho^{(i)} \in S(\mathfrak{g})^{\mathfrak{g}}.$$

Corollary

The polynomials

$$\rho_\omega^{(i)} \in S^{k_i}(\mathfrak{g} \otimes A), \quad 1 \le i \le r,\ (k_i - 1)\mu \le \omega \le k_i\mu,$$

form a system of algebraically independent generators of $S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}$.

This answers our initial question.

Conclusion: We have a complete description of $S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}$.

Example

Suppose $\mathfrak{g} = \mathfrak{sl}_2$ and $A = \mathbb{C}[t]/(t^2)$. So $\mu = 1$.

Let $\{x_+, x_-, h\}$ be the standard Chevalley basis.

Then $S(\mathfrak{g})^{\mathfrak{g}}$ is generated by

$$p = h^2 + 4x_- x_+ \in S^2(\mathfrak{g})^{\mathfrak{g}}.$$

So

$$p_2 = (h \otimes t)(h \otimes t) + 4(x_- \otimes t)(x_+ \otimes t)$$

and

$$p_1 = 2(h \otimes t)(h \otimes 1) + 4(x_- \otimes t)(x_+ \otimes 1) + 4(x_- \otimes 1)(x_+ \otimes t)$$

are algebraically independent generators of $S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}$.

Application

Proposition (Macedo–S. 2016)

If $\mu > 0$ (i.e. $A \neq \mathbb{C}$), then the center of $U(\mathfrak{g} \otimes A)$ acts by the augmentation map $U(\mathfrak{g} \otimes A) \to \mathbb{C}$ on any irreducible f.d. $\mathfrak{g} \otimes A$-module.

In particular, all irreducible f.d. modules have the same central character.

Sketch of proof

The symmetrization map

$$S(\mathfrak{g} \otimes A) \to U(\mathfrak{g} \otimes A), \quad X_1 X_2 \cdots X_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} X_{\sigma(1)} \cdots X_{\sigma(n)},$$

restricts to a vector space isomorphism

$$S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A} \cong \text{center of } U(\mathfrak{g} \otimes A).$$

Application

Sketch of proof (cont.)

Then the corollary gives

$$\text{center of } U(\mathfrak{g} \otimes A) \subseteq U(\mathfrak{g} \otimes \mathsf{m}),$$

where m is the unique maximal ideal of A.

Then the result follows from the fact that all irreducible f.d. $(\mathfrak{g} \otimes A)$-modules are evaluation modules, so $\mathfrak{g} \otimes \mathsf{m}$ acts as zero.

Remark

The above does not imply that the center of $U(\mathfrak{g} \otimes A)$ acts by the augmentation map on all f.d. modules.

The category of $U(\mathfrak{g} \otimes A)$-modules is not semisimple in general.

Example: In the adjoint rep of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]/(t^2)$, the center acts nontrivially.

Further directions

Is $U(\mathfrak{g} \otimes A)$ free over its center?

- For $A = \mathbb{C}$, proved by Kostant.
- For $A = \mathbb{C}[t]/(t^2)$, proved by Geoffriau (1995).
- For $A = \mathbb{C}[t]/(t^n)$, proved by Mustaţă (2001).

Harish-Chandra homomorphism for $\mathfrak{g} \otimes A$

- For $A = \mathbb{C}$, we have the Harish-Chandra isomorphism between center of $U(\mathfrak{g})$ and $S(\mathfrak{h})^W$.
- An analogue for $A = \mathbb{C}[t]/(t^{n+1})$ was proved by Geoffriau.

Γ not acting freely

- In our discussion of reducing to the truncated multicurrent case, we assume that Γ acted freely on $\operatorname{maxSpec} A$.
- Can we describe $S\big((\mathfrak{g} \otimes A)^\Gamma\big)^{(\mathfrak{g}\otimes A)^\Gamma}$ more generally?