Twisted Frobenius extensions

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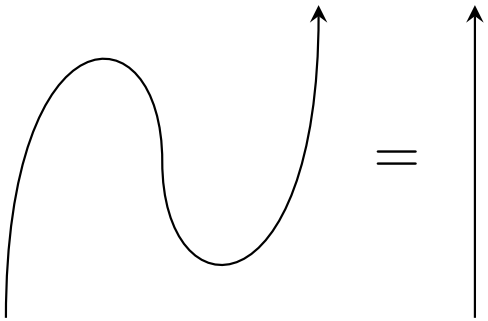
Motivation I

In categorification, a crucial ingredient is often a pair of adjoint functors.

Common setup: We have a ring A with a subring B, and we want induction and restriction to be biadjoint functors.

Induction is always left adjoint to restriction.

So the real work is in showing that induction is also right adjoint to restriction. Sometimes this only holds up to grading shift or automorphism.



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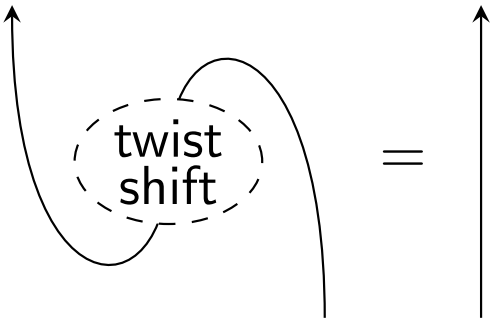

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Motivation I

Many proofs in the literature look similar.

- We use a trace map and dual bases coming from some Frobenius structure to define unit and counit maps.
- Then we show by direct computation that these satisfy the definition of adjunction data.

Goal

Develop a general theory with a theorem that

1. tells us exactly when induction is (twisted shifted) right adjoint to restriction, and
2. gives us the adjunction (unit and counit maps) explicitly.

Such a theory should save time and effort in developing new categorifications.

We want the theory to be as general as possible. So we work in the setting of graded superalgebras.

Graded superrings

Fix an abelian group Λ. By graded, we mean Λ-graded.

Definition (Graded superring)

A graded supering ring A is ring such that

$$A = \bigoplus_{\mu \in \Lambda, \gamma \in \mathbb{Z}_2} A_{\mu,\gamma}, \quad A_{\mu,\gamma} A_{\nu,\delta} \subseteq A_{\mu+\nu,\gamma+\delta}.$$

From now on, we will simply use the term ring to mean graded superring.

Modules

A left A-module for a ring A satisfies

$$M = \bigoplus_{\mu \in \Lambda, \gamma \in \mathbb{Z}_2} M_{\mu,\gamma}, \quad A_{\mu,\gamma} M_{\nu,\delta} \subseteq M_{\mu+\nu,\gamma+\delta}.$$

Similarly for right modules.

Notation

- $\mathbb{F}$ is a field,
- A is ring, B is a subring,
- α and β are ring automorphisms of A and B,
- $\lambda \in \Lambda$, $\pi \in \mathbb{Z}_2$.

Notation

- $\mathrm{HOM}_A^{\mathrm{R}}/\mathrm{HOM}_A^{\mathrm{L}}$ denotes homs of right/left A-modules (arbitrary degree)
- ${}_A A_B$ denotes A, considered as an (A, B)-bimodule (similarly for ${}_B A_A$)
- ${}_B^{\beta} A_A^{\alpha}$ denotes ${}_B A_A$ with twisted action

$$b \cdot x \cdot a = \beta(b) x \alpha(a), \quad a \in A,\ b \in B,\ x \in {}_B^{\beta} A_A^{\alpha}.$$

- $\{\lambda, \pi\}$ denotes a grading shift (the parity shift by π involves some signs in general)

Motivation II

Definition (Frobenius algebra)

A Frobenius algebra over $\mathbb{F}$ is a finite-dimensional $\mathbb{F}$-algebra such that

$$A \cong \mathrm{HOM}_{\mathbb{F}}(A, \mathbb{F}) \quad \text{(as } A\text{-modules)}$$

Suppose A is an $\mathbb{F}$-algebra and B is a subalgebra.

Recall induction from B-modules to A-modules is left adjoint to restriction.

Fact

If A and B are both Frobenius algebras, then induction is twisted right adjoint to restriction (precise definition later).

However, the converse is false.

Motivating question

What condition on A and B is equivalent to induction being twisted right adjoint to restriction?

Twisted Frobenius extensions

Proposition (Pike–S.)

The set of conditions

(L1) A is finitely generated and projective as a left B-module,

(L2) ${}_A A_B \cong \mathrm{HOM}^{\mathrm{L}}_B \left({}^{\beta}_{B} A^{\alpha}_{A}, \{\lambda, \pi\} \, {}_B B_B \right)$ as (A, B)-bimodules,

is equivalent to the set of conditions

(R1) A is finitely generated and projective as a right B-module,

(R2) ${}_B A_A \cong \mathrm{HOM}^{\mathrm{R}}_B \left({}^{\alpha^{-1}}_{A} A^{\beta^{-1}}_{B}, \{\lambda, \pi\} \, {}_B B_B \right)$ as (B, A)-bimodules.

Definition (Twisted Frobenius extension)

If these conditions are satisfied, we say A is an (α, β)-Frobenius extension of B of degree $(-\lambda, \pi)$.

If we don't wish to specify all the data, we say A is a twisted Frobenius extension of B.

Special cases

Frobenius algebra

If $B = \mathbb{F}$ is a field, then an $(\mathrm{id}_A, \mathrm{id}_{\mathbb{F}})$-Frobenius extension is a Frobenius (super)algebra.

Frobenius extension of the second kind

If the gradings are trivial, then an (id_A, β)-Frobenius extension is a Frobenius extension of the second kind (Nakayama–Tuszuku 1960).

Morita

If the gradings are trivial, then (α, β)-Frobenius extensions were considered by Morita (1965).

Trace characterization

Definition (Left trace map)

A homomorphism of (B, B)-bimodules

$$\mathrm{tr}\colon \ {}^{\beta}_{B}A^{\alpha}_{B} \to \{\lambda, \pi\}\, {}_{B}B_{B}$$

is a left trace map if it satisfies:

- $\mathrm{tr}(Aa) = 0 \implies a = 0$
- for every $\varphi \in \mathrm{HOM}^{\mathrm{L}}_{B}({}^{\beta}_{B}A, \{\lambda, \pi\}\, {}_{B}B)$, there exists $a \in A$ such that $\varphi(x) = (-1)^{\bar{x}\bar{a}}\,\mathrm{tr}(xa)$ for all $x \in A$.

Proposition (Pike–S.)

A is an (α, β)-Frobenius extension of B of degree $(-\lambda, \pi)$ if and only if

1. A is finitely generated and projective as a left B-module, and
2. there is a left trace map as above.

There exist right versions of the above statements.

Other characterizations

Bilinear form

The trace map yields a bilinear form:

$$\langle -, - \rangle \colon A \to \{\lambda, \pi\} B, \quad \langle x, y \rangle = \mathrm{tr}(xy).$$

One can give an equivalent definition of twisted Frobenius algebra in terms of this form.

Dual sets of generators

One can also give a characterization in terms of dual sets of generators of A over B.

Uniqueness of the data

The data (α, β, trace map, etc.) are not unique.

However, one can precisely quantify the non-uniqueness.

Nakayama automorphism

Let A be an (α, β)-Frobenius extension of B of degree $(-\lambda, \pi)$ with left trace map tr. Consider the centralizer of B in A:

$$C_A(B) = \{a \in A \mid ab = (-1)^{\bar{a}\bar{b}} ba \text{ for all } b \in B\}$$

Proposition (Pike–S.)

For any $c \in C_A(B)$, there is a unique element $\psi(c)$ of A such that

$$\operatorname{tr}(ca) = (-1)^{\bar{a}\bar{c}} \operatorname{tr}(a\psi(c)) \quad \text{for all } a \in A.$$

This yields an isomorphism of rings

$$\psi \colon C_A(B) \cong C_A(\alpha(B)),$$

which we call ψ the Nakayama automorphism associated to tr.

If the twistings and gradings are trivial, this is the usual Nakayama automorphism of a Frobenius extension.

Adjointness properties

Recall that, for any subring B of a ring A, induction is always left adjoint to restriction. That is,

$$ {}_A A_B \otimes_B - \quad \text{is left adjoint to} \quad {}_B A_A \otimes_A -. $$

Theorem (Pike–S.)

$$ {}_A A_B \otimes_B - \quad \text{is right adjoint to} \quad \{-\lambda, \pi\} \, {}^{\beta}_{B} A^{\alpha}_{A} \otimes_A - $$

if and only if

$$ A \text{ is an } (\alpha, \beta)\text{-Frobenius extension of } B \text{ of degree } (-\lambda, \pi). $$

In other words induction is twisted shifted right adjoint to restriction if and only if A is a twisted Frobenius extension of B.

Remark

One can write down explicit unit and conunit maps giving the adjunction (very useful in categorification).

A useful class of examples

Theorem (Pike–S.)

Suppose

- A is a Frobenius graded superalgebra of degree $(-\lambda_A, \pi_A)$ (the degree of its trace map),
- B is a graded subalgebra of A that is itself a Frobenius graded superalgebra of degree $(-\lambda_B, \pi_B)$, and
- A is projective as a left B-module.

Then A is a (ψ_A, ψ_B)-Frobenius extension of B of degree $(\lambda_B - \lambda_A, \pi_B + \pi_A)$. In particular, induction is twisted shifted right adjoint to restriction.

Remarks

- This explains the motivating fact mentioned earlier (with false converse).
- This theorem recovers many results in the categorification literature.

Example: NilCoxeter algebras

The nilCoxeter algebra N_n, $n \in \mathbb{N}$, is the unital $\mathbb{Z}$-graded $\mathbb{F}$-superalgebra with generators $u_1, \dots, u_{n-1}$ in degree $(1,1)$, and relations

$$u_i^2 = 0 \text{ for } i = 1, 2, \dots, n-1,$$
$$u_i u_j = u_j u_i \text{ for } i, j = 1, \dots, n-1 \text{ with } |i-j| > 1,$$
$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \text{ for } i = 1, 2, \dots, n-2.$$

N_n is a Frobenius graded superalgebra of degree $\left(\binom{n}{2}, \binom{n}{2}\right)$.

N_n is free (hence projective) as a left N_m-module for any $m < n$.

By the above theorem, N_n is a twisted Frobenius extension of N_m.

However, it is not an (id_A, β)-Frobenius extension for any automorphism β of N_m.

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