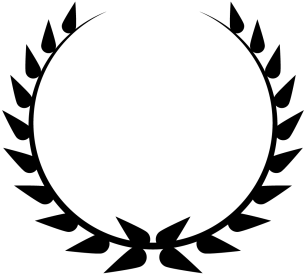
Heisenberg categorification and wreath product algebras



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Slides available online: alistairsavage.ca/talks

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Outline

Summary: We develop a graphical categorification of the Heisenberg algebra that depends on a Frobenius algebra. The graphical category acts naturally on modules over wreath product algebras.

Overview:

1. Background: Heisenberg categorification
2. Frobenius algebras and wreath product algebras
3. Definition of the graphical category
4. Action of the category on modules for wreath product algebras
5. Main theorem: categorification of the Heisenberg algebra
6. Extra algebraic structure arising from the category
7. Remarks and further directions

The Heisenberg algebra

Definition

The rank one Heisenberg algebra has generators p, q and relation

$$pq - qp = 1.$$

It is the algebra of operators in the quantization of the harmonic oscillator.

Definition

The (infinite rank) Heisenberg algebra $\mathfrak{h}$ is the algebra with generators p_i, q_i, $i = 1, 2, 3, \dots$, and relations

$$p_i q_j - q_j p_i = \delta_{i,j} 1, \quad p_i p_j - p_j p_i = 0, \quad q_i q_j - q_j q_i = 0.$$

Plays a fundamental role in QFT and the theory of affine Lie algebras.

Categorification

Suppose $\mathcal{C}$ is an additive category.

$\mathrm{Iso}(\mathcal{C})$ = free abelian group generated by isom. classes of objects in $\mathcal{C}$.

The split Grothendieck group of $\mathcal{C}$ is

$$K_0(\mathcal{C}) = \mathrm{Iso}(\mathcal{C}) / \langle [X \oplus Y] = [X] + [Y] \mid X, Y \in \mathcal{C} \rangle.$$

If $\mathcal{C}$ is monoidal, then $K_0(\mathcal{C})$ is a ring:

$$[X] \cdot [Y] = [X \otimes Y].$$

For our purposes, to categorify a ring R is to find an additive monoidal category $\mathcal{C}$ such that

$$K_0(\mathcal{C}) \cong R \quad \text{(as rings)}.$$

Categorification of the Heisenberg algebra: Background

Geissinger (1970s)

The bialgebra of symmetric functions can be realized as

$$\mathrm{Sym} \cong \bigoplus_{n=0}^{\infty} K_0(\mathbb{C}[S_n]\text{-mod}).$$

Algebra structure given by induction functor:

$$[\mathrm{Ind}]\colon K_0(\mathbb{C}[S_n]\text{-mod}) \otimes K_0(\mathbb{C}[S_m]\text{-mod}) \to K_0(\mathbb{C}[S_{n+m}]\text{-mod}).$$

Coalgebra structure given by restriction.

Multiplication by classes, together with the adjoint operations, define an action of the (infinite rank) Heisenberg algebra $\mathfrak{h}$.

One obtains the Fock space representation.

Categorification of the Heisenberg algebra: Background

Khovanov (2010)

Khovanov defined a monoidal category $\mathcal{H}$ in terms of planar diagrams.

$\mathcal{H}$ acts naturally on the category of modules for symmetric groups.

One has an injective map $\mathfrak{h} \hookrightarrow K_0(\mathcal{H})$.

The map is conjectured to be surjective. If true, this yields a categorification of $\mathfrak{h}$.

Cautis–Licata (2010)

Khovanov's construction can be modified, depending on a finite subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{C})$.

Obtain a categorification (not just conjectural) of the Heisenberg algebra.

Category acts on derived categories of coherent sheaves on Hilbert schemes of points on minimal resolution $\widetilde{\mathbb{C}^2/\Gamma}$.

Current Goal

Goal

- Define a graphical category depending on a Frobenius algebra B.
- Should recover Khovanov and Cautis–Licata constructions as special cases.
- Under a mild assumption on B, we should obtain a categorification of the Heisenberg algebra.

Advantages

- We obtain a large class of categorifications of the Heisenberg algebras, each adapted to a different presentation. Useful for categorifying further constructions (vertex algebras, etc.).
- Should be possible to generalize many of the existing results concerning Heisenberg categorification.

Frobenius algebras

Fix an algebraically closed field $\mathbb{F}$.

Definition (Frobenius algebra)

A Frobenius (graded super) algebra B over $\mathbb{F}$ is a finite-dimensional unital associative $\mathbb{Z}$-graded superalgebra with a homogeneous linear trace map

$$\mathrm{tr}_B \colon B \to \mathbb{F}$$

such that $\ker \mathrm{tr}_B$ contains no nonzero left ideal of B.

From now on, we fix a Frobenius algebra B.
For homogeneous $b \in B$, $\bar{b} \in \mathbb{Z}_2$ denotes the parity.

Definition (Nakayama automorphism)

There exists a Nakayama automorphism ψ of B with the property that

$$\mathrm{tr}_B(ab) = (-1)^{\bar{a}\bar{b}} \mathrm{tr}_B(b\psi(a)), \quad a, b \in B.$$

Dual bases

We fix a basis $\mathcal{B}$ of B.

The (right) dual basis $\mathcal{B}^\vee = \{b^\vee \mid b \in \mathcal{B}\}$ is defined by

$$\operatorname{tr}_B(b_1 b_2^\vee) = \begin{cases} 0, & \text{if } b_1 \neq b_2, \\ 1, & \text{if } b_1 = b_2. \end{cases}$$

Lemma

The elements

$$\sum_{b \in \mathcal{B}} b \otimes b^\vee \qquad \text{and} \qquad \sum_{b \in \mathcal{B}} b^\vee \otimes b$$

are independent of the basis $\mathcal{B}$.

Superpermutations

The tensor product $B^{\otimes n}$ is a Frobenius algebra with multiplication

$$(b_1 \otimes \cdots \otimes b_n)(b_1' \otimes \cdots \otimes b_n') = (-1)^{\sum_{i<j} \bar{b}_i \bar{b}_j'} b_1 b_1' \otimes \cdots \otimes b_n b_n'.$$

Superpermutations

The symmetric group S_n acts on $B^{\otimes n}$ by superpermutations.

More precisely, if $s_k = (k, k+1)$ is the simple transposition, then

$$s_k \cdot (b_1 \otimes \cdots \otimes b_n) = (-1)^{\bar{b}_k \bar{b}_{k+1}} b_1 \otimes \cdots \otimes b_{k-1} \otimes b_{k+1} \otimes b_k \otimes b_{k+2} \otimes \cdots \otimes b_n.$$

Wreath product algebras

Definition (Wreath product algebra)

The wreath product algebra is

$$A_n := B^{\otimes n} \rtimes S_n = B^{\otimes n} \otimes \mathbb{F}[S_n] \quad \text{as graded } \mathbb{F}\text{-vector space}$$

where $\mathbb{F}[S_n]$ is in degree zero.

Multiplication: $B^{\otimes n}$ and $\mathbb{F}[S_n]$ are subalgebras, and

$$\tau \beta \tau^{-1} = \tau \cdot \beta, \quad \beta \in B^{\otimes n},\ \tau \in S_n.$$

Lemma

The wreath product algebra A_n is a Frobenius algebra with trace map

$$\operatorname{tr}_n \colon A_n \to \mathbb{F}, \quad \operatorname{tr}_n(b_1 \otimes \cdots \otimes b_n \otimes \tau) = \operatorname{tr}_B(b_1) \cdots \operatorname{tr}_B(b_n) \delta_{\tau,1}.$$

Wreath product algebras: examples

Example (Group algebra of the symmetric group)

If $B = \mathbb{F}$, then $A_n = B^{\otimes n} \rtimes \mathbb{F}[S_n] \cong \mathbb{F}[S_n]$.

Example (Sergeev superalgebra)

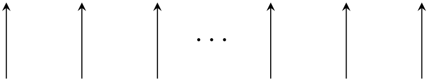
If B is the rank one Clifford superalgebra, then $A_n = B^{\otimes n} \rtimes \mathbb{F}[S_n]$ is isomorphic to the Sergeev superalgebra.

Example (Cautis–Licata)

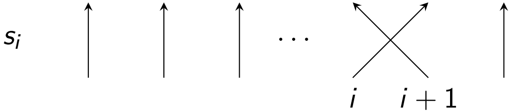
Certain choice of B yields the algebras appearing in Heisenberg categorification of Cautis–Licata.

Wreath product algebras: graphical notation

Consider n (upward pointing) strands.



Simple transposition s_i is a crossing of strands i and $i+1$.

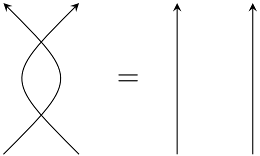


The algebra $\mathbb{F}[S_n]$ is then

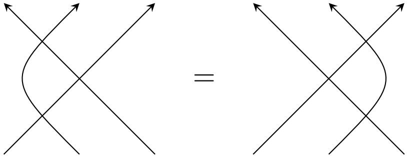
- the vector space spanned by all diagrams consisting of n upward pointing strands with crossings,
- with no maxima/minima (with respect to vertical coordinate)—i.e. strands keep "heading up",
- up to isotopy preserving the boundary (the endpoints of the strands),
- modulo the relations defining the symmetric group...

Wreath product algebras: graphical notation

1. $s_i^2 = 1$



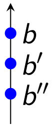
2. $s_i s_j = s_j s_i$ if $|j-i| > 1$ becomes "distant crossings commute" (isotopy).
3. $s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i$ (braid relation)



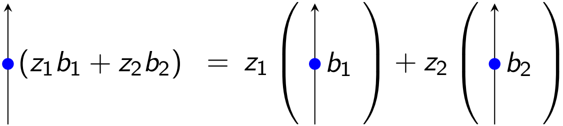
Multiplication in $\mathbb{F}[S_n]$ is given by vertical composition (we read from bottom to top).

Wreath product algebras: graphical notation

Next we introduce the factor $B^{\otimes n}$. We allow strands to carry dots labeled by elements of B:

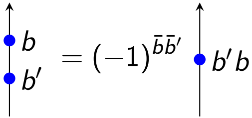


Diagrams are linear in the dots:



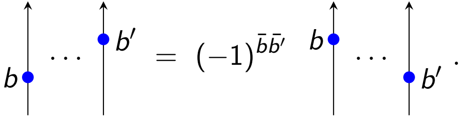
for $z_1, z_2 \in \mathbb{F}$, $b_1, b_2 \in B$.

Collision of dots is controlled by multiplication in the algebra B (actually, B^{op}):



Wreath product algebras: graphical notation

Dots on distinct strands supercommute when they move past each another:



The relation $\tau\beta\tau^{-1} = \tau \cdot \beta$, $\beta \in B^{\otimes n}$, $\tau \in S_n$, becomes

The algebra $A_n = (B^{\mathrm{op}})^{\otimes n} \rtimes \mathbb{F}[S_n]$ is then

- the vector space spanned by all diagrams consisting of n upward pointing strands with crossings and dots,
- with no maxima/minima (with respect to vertical coordinate)—i.e. strands keep "heading up",
- up to "superisotopy" preserving the endpoints of the strands,
- modulo the local relations.

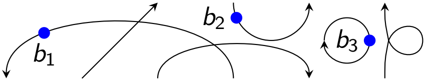
A graphical category $\mathcal{H}'$

Motivated by our graphical interpretation of the wreath product algebra, we define a category $\mathcal{H}'$ as follows.

- $\mathcal{H}'$ is a strict monodial category with two generating objects: $\uparrow$, $\downarrow$. In other words, objects of $\mathcal{H}'$ are sequences (tensor products) of $\uparrow$'s and $\downarrow$'s.

$$\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow, \quad \downarrow\uparrow\downarrow, \quad \varnothing, \quad \text{etc.}$$

- Morphisms of $\mathcal{H}'$ between two objects are $\mathbb{F}$-linear combinations of planar diagrams, carrying dots labeled by elements of B, agreeing with the two sequences at the boundary, modulo super isotopy preserving the boundary



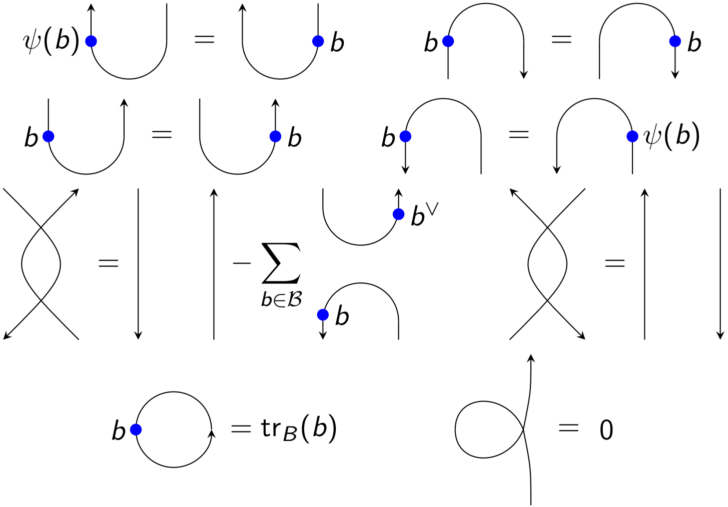
$$: \downarrow\uparrow\uparrow\uparrow\downarrow\uparrow \longrightarrow \uparrow\downarrow\uparrow\uparrow$$

and modulo some local relations...

- Composition of morphisms is given by vertical composition of diagrams.

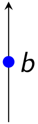
A graphical category $\mathcal{H}'$: Local relations

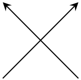
In addition to the local relations we had before (for the wreath product algebra) we impose:



A graphical category $\mathcal{H}'$: Grading

The morphism spaces are naturally graded:

deg  $= \deg b$

deg  $= (0,0)$,

deg  $= (0,0)$, deg  $= (0,0)$,

deg  $= (\delta,\sigma)$, deg  $= (-\delta,\sigma)$,

where $(-\delta,\sigma)$ is the $\mathbb{Z}\times\mathbb{Z}_2$-degree of the trace map tr_B.

A graphical category $\mathcal{H}'$: Isotopy invariance

We consider diagrams up to super isotopy:

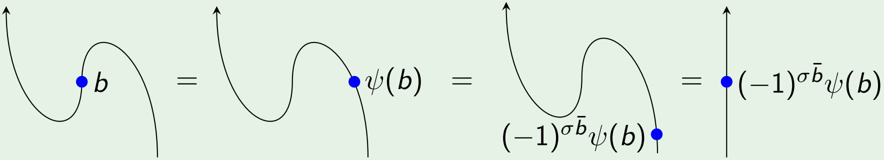
- We allow isotopies involving horizontal strips not containing dots, except that we introduce a sign when odd cups/caps change heights.
- Interchanging heights of odd dots and odd cups/caps also introduces a sign.

In general, to perform an isotopy involving dots, we should

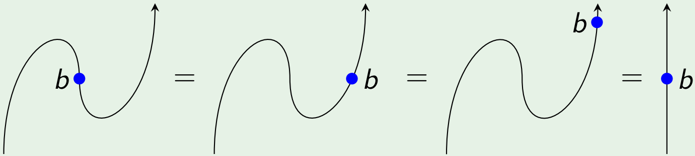
- slide dots above or below the strip to be isotoped,
- perform the isotopy (keeping track of signs),
- slide dots back into position.

A graphical category $\mathcal{H}'$: Isotopy invariance

Example



Example



A graphical category $\mathcal{H}'$

Remarks

1. The relations for upward pointing strands are simply the wreath product algebra relations.
2. Other relations are motivated by certain functors on the category of modules for wreath product algebras (as we shall see).
3. If $B = \mathbb{F}$, the category $\mathcal{H}'$ reduces to category studied by Khovanov.
4. Certain choice of B recovers category defined by Cautis–Licata.

Properties of our category

1. Acts on the category of modules for wreath product algebras.
2. Yields a categorification of the Heisenberg algebra (when $\delta \neq 0$).
3. Morphism spaces contain interesting generalizations of the degenerate affine Hecke algebra.

Notation for bimodules for wreath product algebras

We view A_{n-1} as a subalgebra of A_n via the natural map

$$A_{n-1} = B^{\otimes(n-1)} \rtimes S_{n-1} \cong \left(B^{\otimes(n-1)} \otimes \mathbb{F}\right) \rtimes S_{n-1} \hookrightarrow B^{\otimes n} \rtimes S_n = A_n.$$

Then

- (n) denotes A_n as an (A_n, A_n)-bimodule.
- $(n)_{n-1}$ denotes A_n as an (A_n, A_{n-1})-bimodule.
- ${}_{n-1}(n)$ denotes A_n as an (A_{n-1}, A_n)-bimodule.
- ${}_{n-1}(n)_{n-1}$ denotes A_n as an (A_{n-1}, A_{n-1})-bimodule.

Tensor product of bimodules is denoted by juxtaposition. E.g.

$(n)_{n-1}(n-1)_{n-2}(n-2)_{n-2}(n-1) := A_n \otimes_{A_{n-1}} A_{n-1} \otimes_{A_{n-2}} A_{n-2} \otimes_{A_{n-2}} A_{n-1}.$

Bimodules for wreath product algebras

For $k, \ell \geq 0$, let ${}_\ell\mathrm{Bimod}_k$ be the category of (A_ℓ, A_k)-bimodules.

Let $\mathrm{Bimod}_k = \bigoplus_{\ell \geq 0} {}_\ell\mathrm{Bimod}_k$.

Relation to induction/restriction

- The functor ${}_{n-1}(n) \otimes -$ is restriction from the category of A_n-modules to the category of A_{n-1}-modules.
- The functor $(n)_{n-1} \otimes -$ is induction from the category of A_{n-1}-modules to the category of A_n-modules.
- Similarly, $(n)_{n-1}(n-1)_{n-2}(n-2)_{n-3}(n-2) \otimes -$ is a composition of induction and restriction functors.
- Morphisms of bimodules yield natural transformations of functors.

Action of $\mathcal{H}'$ on modules for wreath product algebras

Fix $k \geq 0$. We define a functor from our graphical category to Bimod_k.

Functor on objects

$\uparrow$ is sent to the induction bimodule.

$\downarrow$ is sent to the restriction bimodule.

Easiest to see if we label the regions between $\uparrow$'s and $\downarrow$'s by

- labeling the rightmost region by k,
- increasing labels by one as we move left across $\uparrow$, and
- decreasing labels by one as we move left across $\downarrow$.

Example

$$\uparrow\downarrow\downarrow\uparrow\uparrow \quad \rightsquigarrow \quad k+1 \uparrow k \downarrow k+1 \downarrow k+2 \uparrow k+1 \uparrow k$$

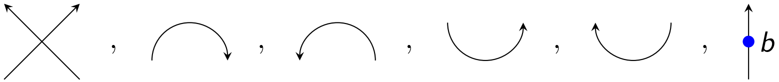
So our functor maps $\uparrow\downarrow\downarrow\uparrow\uparrow$ to the bimodule

$$(k+1)_k (k+1)_{k+1} (k+2)_{k+2} (k+2)_{k+1} (k+1)_k.$$

Action of $\mathcal{H}'$ on modules for wreath product algebras

It remains to define our functor on morphisms.

Our planar diagrams are built from compositions of crossings, cups, caps, and dots:

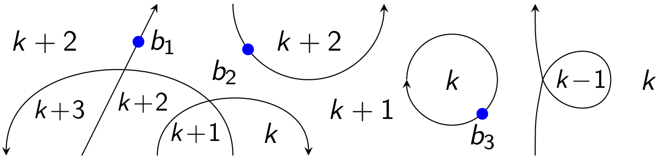


So we define our functor on these building blocks.

Action of $\mathcal{H}'$ on modules for wreath product algebras

Give a planar diagram (a morphism in $\mathcal{H}'$), label the regions by

- labeling the rightmost region by k,
- increasing labels by one as we move left across $\uparrow$, and
- decreasing labels by one as we move left across $\downarrow$.



We then assign a bimodule morphism to each of our building blocks.

Action of our functor on generating morphisms

We associate to our diagram building blocks, the following bimodule maps:

$$n \quad : \quad (n)_{n-1}(n) \to (n), \qquad a \otimes a' \mapsto aa' \qquad \text{(multipication)}$$

$$n \quad : \quad (n) \hookrightarrow {}_n(n+1)_n, \qquad \text{(natural inclusion)}$$

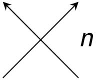
$$n \quad : \quad {}_n(n+1)_n \twoheadrightarrow (n),$$

$$(b_1 \otimes \cdots \otimes b_{n+1})\tau = \begin{cases} 0 & \text{if } \tau \notin S_n, \\ (-1)^{\sigma(\bar{b}_1 + \cdots + \bar{b}_n)} \operatorname{tr}_B(b_{n+1})(b_1 \otimes \cdots \otimes b_n)\tau & \text{if } \tau \in S_n, \end{cases}$$

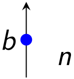
$$n \quad : \quad (n) \to (n)_{n-1}(n),$$

$$a \mapsto a \sum_{\substack{b \in \mathcal{B} \\ i \in \{1,\dots,n+1\}}} s_i \cdots s_n (1_B^{\otimes n} \otimes b^\vee) \otimes (1_B^{\otimes n} \otimes b) s_n \cdots s_i.$$

Action of our functor on generating morphisms



$$: \ (n+2)_n \to (n+2)_n, \quad a \mapsto a s_{n+1},$$



$$: \ (n+1)_n \to (n+1)_n, \quad a \mapsto (-1)^{\bar{a}\bar{b}} a \left(1_B^{\otimes n} \otimes b\right).$$

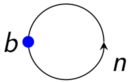
Action of $\mathcal{H}'$ on modules for wreath product algebras

Question: Have we really defined a functor from $\mathcal{H}'$ to the category of bimodules?

One needs to check:

1. the bimodule map associated to a diagram is invariant under super isotopy (preserving the boundary) on horizontal strips not involving dots, and
2. the defining local relations of the graphical category are satisfied.

Checking the defining relations is just a matter of computation. E.g. for



$$= \mathrm{tr}_B(b),$$

the LHS is mapped to the composition

$$(n) \longrightarrow {}_n(n+1)_n \longrightarrow {}_n(n+1)_n \longrightarrow (n),$$
$$a \mapsto a \mapsto (-1)^{\bar{a}\bar{b}} a \otimes (1_B^{\otimes n} \otimes b) \mapsto \mathrm{tr}_B(b)a.$$

Isotopy: cups and caps

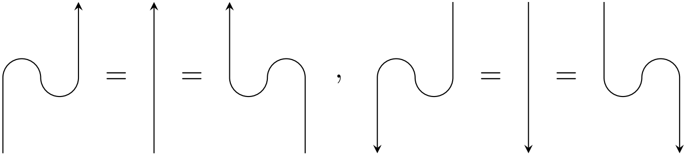
The cups and caps

correspond to natural transformations

$$\begin{aligned} &\text{induction} \circ \text{restriction} \Rightarrow \text{id}, && \text{restriction} \circ \text{induction} \Rightarrow \text{id}, \\ &\text{id} \Rightarrow \text{restriction} \circ \text{induction}, && \text{id} \Rightarrow \text{induction} \circ \text{restriction}. \end{aligned}$$

These natural transformations give adjunction data implying that induction and restriction are biadjoint up to a grading shift.

In particular, this means that they satisfy the following relations:



Action of $\mathcal{H}'$ on modules for wreath product algebras

Therefore, our functor is well-defined.

So

- our category maps (via our functor) to the category of bimodules for the wreath product algebras, and
- the category of bimodules acts on the category of modules for the wreath product algebras (via tensoring on the left).

Thus, our category acts on the category of modules of wreath product algebras.

A categorification of the Heisenberg algebra

Definition: Karoubi envelope

The Karoubi envelope of a category $\mathcal{C}$ is the category whose

- objects are pairs (A, e) where $A \in \operatorname{Ob} \mathcal{C}$ and $e \in \operatorname{Mor}_{\mathcal{C}}(A, A)$ is an idempotent ($e^2 = e$), and
- morphisms from (A, e) to (B, f) are $\varphi \in \operatorname{Mor}_{\mathcal{C}}(A, B)$ such that

$$f\varphi = \varphi = \varphi e.$$

Intuition: One thinks of passing to the Karoubi envelope as adding in objects such that the idempotents correspond to projections onto direct summands.

$$\begin{tikzcd}
A \cong X \oplus Y \arrow[r, two heads] \arrow[rr, bend left, "e"] & X \arrow[r, hook] & X \oplus Y \cong A
\end{tikzcd}$$

A categorification of the Heisenberg algebra

Let $\mathcal{H}$ be the Karoubi envelope of our graphical category $\mathcal{H}'$.

Recall that $(-\delta, \sigma)$ is the $\mathbb{Z} \times \mathbb{Z}_2$-degree of the trace map tr_B of B.

Theorem (Rosso-S.)

There is a natural injective homomorphism

$$\mathfrak{h} \to \text{(split) Grothendieck group of } \mathcal{H}.$$

When $\delta > 0$ this map is surjective. Hence $\mathcal{H}$ categorifies $\mathfrak{h}$.

Remarks

1. The condition $\delta > 0$ allows one to give an explicit description of all idempotents in $\mathcal{H}'$, hence describe all objects $\mathcal{H}$.
2. We expect that the map is surjective even if $\delta = 0$.

Categorification of bosonic Fock space

Bosonic Fock space is the representation of $\mathfrak{h}$ on

$$\mathbb{C}[x_1, x_2, \ldots]$$

given by

$$p_i \mapsto \frac{\partial}{\partial x_i}, \qquad q_i \mapsto x_i.$$

The action of $\mathcal{H}$ on the category of modules for wreath product algebras is a categorification of this important representation.

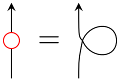
Faithfulness of this representation is a crucial ingredient in proving that the map

$$\mathfrak{h} \to \text{(split) Grothendieck group of } \mathcal{H}.$$

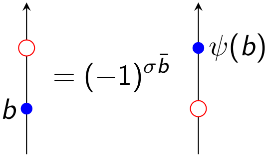
is injective.

Right curls

Let an open dot denote a right curl:



Then one can check that

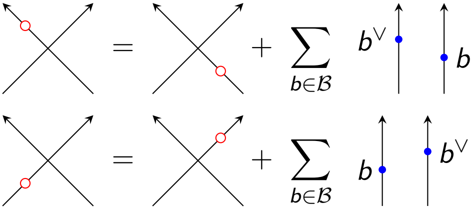


So passing through open dots corresponds to the Nakayama automorphism.

In our action on modules of wreath product algebras, open dots correspond to multiplication by Jucys-Murphy type elements.

Generalization of the degenerate affine Hecke algebra

One can check from our defining relations that we have the following equalities:



Remark

When $B = \mathbb{F}$, the above reduce to the relations of the degenerate affine Hecke algebra.

Thus, our morphism spaces naturally contain a generalization of the degenerate affine Hecke algebra.

Remarks

Special cases

1. When $B = \mathbb{F}$, we recover Khovanov's construction.
2. For certain B, we recover construction of Cautis–Licata.
3. For another choice of B, we recover recent work of Hill–Sussan on categorification of twisted Heisenberg algebras.

Presentations and integral forms

Although all choices of B correspond to categorifications of the infinite rank Heisenberg algebra,

- different B yield different presentations of $\mathfrak{h}$ (so-called "lattice Heisenberg algebras"),
- taking the Grothendieck group over $\mathbb{Z}$, one obtains different integral forms of the Heisenberg algebra.

Further directions

- For some choices of B, we should get a geometric action of our category, generalizing work of Cautis–Licata.
- Trace decategorification of Khovanov's Heisenberg category has been related to W-algebras. One should be able to generalize these results.
- Cautis and Sussan have developed braid group actions from categorified Heisenberg complexes. This could be generalized.
- Vertex algebra categorifications (Cautis–Licata) could also be generalized. The various presentations obtained from the current work are ideally suited for this.