The twisted Heisenberg double and its categorification

Alistair Savage
University of Ottawa

Joint with: Daniele Rosso (UC Riverside)

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Outline

Previous work: Modules over towers of algebras categorify the Heisenberg double (S.–Yacobi 2013).

Summary of today's talk

- Graded supermodules over towers of graded superalgebras categorify the twisted Heisenberg double
- Examples: (Quantum) lattice Heisenberg algebras, quantum Weyl algebra

Overview

1. What is categorification?
2. Twisted Hopf algebras and the twisted Heisenberg double
3. Towers of graded superalgebras
4. Categorification of the twisted Heisenberg double
5. Examples

What is categorification?

Suppose M is a module for a ring R.

We would like to find an abelian category $\mathcal{M}$ such that

$$\mathcal{K}_0(\mathcal{M}) \xrightarrow[\cong]{\varphi} M \qquad \text{(as } \mathbb{Z}\text{-modules)},$$

where $\mathcal{K}_0(\mathcal{M})$ is the Grothendieck group of $\mathcal{M}$.

Then, for each $r \in R$ (or, for those r in a fixed generating set), we want an exact endofunctor F_r of $\mathcal{M}$ such that we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{K}_0(\mathcal{M}) & \xrightarrow{[F_r]} & \mathcal{K}_0(\mathcal{M}) \\ \varphi \downarrow & & \downarrow \varphi \\ M & \xrightarrow{r} & M \end{array}$$

Here $[F_r]$ denotes the map induced by F_r on $\mathcal{K}_0(\mathcal{M})$.

What is categorification?

We would also like isomorphisms of functors lifting the relations of R.

For example, suppose we have a relation in R:

$$rs = 2sr + 3.$$

Then we would like isomorphisms of functors

$$F_r \circ F_s \cong (F_s \circ F_r)^{\oplus 2} \oplus \mathrm{Id}^{\oplus 3}.$$

Fruits of categorification

- Classes of objects (simple, indecomposable projective) give distinguished bases with positivity and integrality properties.
- Uncovers hidden structure in the algebra and its representation.
- Provides tools for studying the category $\mathcal{M}$.
- Applications to topology.

Notation

Fix a commutative ring $\Bbbk$. Algebras, coalgebras, etc. are over this ring.

Grading: Let $\Lambda \cong \mathbb{N}^r$ be a commutative monoid, written additively.

Notation

- ∇ – multiplication
- Δ – comultiplication
- η – unit
- ε – counit

Twisted Hopf algebras

Twisted Hopf algebra data

Fix the following:

- $q \in \Bbbk^\times$
- A pair $\chi = (\chi', \chi'')$ of biadditive maps

$$\chi', \chi'' \colon \Lambda \times \Lambda \to \mathbb{Z}$$

Suppose H is a Λ-graded algebra.

Define a new associative multiplication $*_\chi$ on $H \otimes H$ by

$$(a_1 \otimes a_2) *_\chi (b_1 \otimes b_2) = q^{\chi'(|a_2|,|b_1|)+\chi''(|a_1|,|b_2|)} a_1 b_1 \otimes a_2 b_2,$$

where $|a|$ denotes the degree of a homogeneous element $a \in H$.

We denote this twisted multiplication structure by $(H \otimes H)_\chi$.

Twisted Hopf algebras

Suppose:

- (H, ∇, ε) is a Λ-graded algebra,
- (H, Δ, η) is a Λ-graded coalgebra.

Definition (Twisted Hopf algebra)

We say $(H, \nabla, \Delta, \varepsilon, \eta)$ is a Λ-graded connected twisted Hopf algebra or, more precisely, a (q, χ)-Hopf algebra if

- $H_0 = \Bbbk 1_H$,
- $\Delta \colon H \to (H \otimes H)_\chi$ is an algebra homomorphism.

When we wish to leave (q, χ) implied, we call it a twisted Hopf algebra.

Remark

The above axioms ensure the existence of an antipode, so we don't need it explicitly in the definition.

Twisted Hopf algebras

Remark

If

$$q^{\chi'(\lambda_1,\lambda_2)+\chi''(\mu_1,\mu_2)} = 1 \text{ for all } \lambda_1, \lambda_2, \mu_1, \mu_2 \in \Lambda$$

(for example, if $q = 1$ or $\chi' = \chi'' = 0$), then

$$(H \otimes H)_\chi = H \otimes H$$

with componentwise multiplication.

In this case, we recover the usual definition of a Hopf algebra.

Remark

For a given twisted Hopf algebra, the data (q, χ', χ'') is not unique.

Twisted Hopf pairings

Twisted pairing data

Fix the following:

- $c \in \Bbbk^\times$
- A pair $\gamma = (\gamma', \gamma'')$ of biadditive maps $\gamma', \gamma'' \colon \Lambda \times \Lambda \to \mathbb{Z}$

Definition (Twisted Hopf pairing)

Suppose H and H' are twisted Hopf algebras.

A (c, γ)-twisted Hopf pairing is a bilinear map $\langle -, - \rangle \colon H \times H' \to \Bbbk$ such that

- H_λ and H'_μ are orthogonal when $\lambda \neq \mu$,
- $\langle \nabla(x \otimes y), a \rangle = c^{\gamma'(|x|,|y|)} \langle x \otimes y, \Delta(a) \rangle$,
- $\langle x, \nabla(a \otimes b) \rangle = c^{\gamma''(|a|,|b|)} \langle \Delta(x), a \otimes b \rangle$,
- $\langle 1_H, a \rangle = \varepsilon(a)$, $\langle x, 1_{H'} \rangle = \varepsilon(x)$.

Dual pairs

Definition (Dual pair)

We (H^+, H^-) is a (c,γ)-dual pair if

- $H^\pm$ are twisted Hopf algebras,
- there exists a (c,γ)-twisted Hopf pairing $\langle -,-\rangle \colon H^- \times H^+ \to \Bbbk$ such that $\langle -,-\rangle|_{H^-_\lambda \times H^+_\lambda}$ is a perfect pairing for all $\lambda \in \Lambda$.

Lemma (Rosso–S. 2014)

Suppose (H^+, H^-) is a (q,γ)-dual pair and H^+ is a (q,χ)-Hopf algebra.

Then H^- is a (q,ξ)-Hopf algebra, where

$$\xi = (\xi', \xi''), \quad \xi' = (\chi')^T + \gamma' - (\gamma'')^T, \quad \xi'' = \chi'' + \gamma' - \gamma''.$$

Here, if $\zeta \colon \Lambda \times \Lambda \to \mathbb{Z}$ is a biadditive map, then

$$\zeta^T(\lambda,\mu) \stackrel{\text{def}}{=} \zeta(\mu,\lambda).$$

Compatibility

From now on, we will assume that

- (H^+, H^-) is a (q, γ)-dual pair,
- we have

$$\chi' = -(\gamma')^T. \tag{1}$$

Remark

We say that the dual pair (H^+, H^-) is compatible if there is a choice of χ satisfying (1).

Compatibility ensures nice commutation relations in what follows.

Since the twisting (q, χ) of H^+ is not unique, the issue of compatibility is subtle.

We are not aware of an example of a dual pair that is not compatible.

Twisted Heisenberg double: Definition (Rosso–S. 2014)

We define the twisted Heisenberg double $\mathfrak{h} = \mathfrak{h}(H^+, H^-)$ as follows.

We have $\mathfrak{h} = H^+ \otimes H^-$ as $\Bbbk$-modules. We write $a\#x$ for $a \otimes x$, viewed as an element of $\mathfrak{h}$.

Multiplication is given by (recall we use Sweedler notation)

$$(a\#x)(b\#y) \stackrel{\text{def}}{=} \sum_{(x),(b)} q^{\gamma''(|b|,|x_2|)+\xi''(|b|-|x_1|,|x_2|)+\gamma'(|b_1|,|b_2|)} \langle x_1, b_2\rangle ab_1\#x_2 y,$$

where we use Sweedler notation for coproducts:

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2$$

We can view $H^\pm$ as subalgebras of $\mathfrak{h}$ via the maps

$$a \mapsto a\#1, \quad x \mapsto 1\#x, \quad a \in H^+,\ x \in H^-.$$

Twisted Heisenberg double: Examples

Example (Heisenberg double)

If the twistings of $H^\pm$ and the pairing are trivial, $\mathfrak{h}$ is the usual Heisenberg double. Special cases include:

- Heisenberg algebra ($H^\pm = \mathrm{Sym}$)
- Weyl algebra ($H^\pm = \mathbb{Q}[x]$)
- Quantum Heisenberg algebras
- Lattice Heisenberg algebras

Example (Quantum Weyl algebra)

Take $\mathbb{k} = \mathbb{Q}[q]$, $H^+ = \mathbb{k}[x]$, $H^- = \mathbb{k}[\partial]$.

With appropriate choices of the twisting data, (H^+, H^-) is a dual pair and

$$\mathfrak{h} = \mathbb{k}\langle x, \partial \mid \partial x = qx\partial + 1\rangle$$

is the quantum Weyl algebra.

Fock space

H^+ acts on itself by left multiplication.

H^- acts on itself by right multiplication. Taking adjoints gives a (left) action of H^- on H^+.

Theorem (Rosso-S. 2014)

1. The twisted Heisenberg double $\mathfrak{h}$ is isomorphic to the subalgebra of $\mathrm{End}_{\Bbbk} H^+$ generated by the actions of $H^{\pm}$ described above.
2. Stone–von Neumann type thereom: Any $\mathfrak{h}$-module generated by a vector annihilated by H^- is isomorphic to H^+ with the above action.

We call H^+ with the action of $\mathfrak{h}$ described above Fock space.

Supermodules over graded superalgebras

Suppose we have an $\mathbb{N}$-graded superalgebra (over an alg. closed field)

$$B \cong \bigoplus_{i \in \mathbb{N},\ \epsilon \in \mathbb{Z}_2} B_{i,\epsilon}.$$

Let

- B-grmod = category of finitely generated graded B-supermodules,
- B-grpmod = category of finitely generated projective graded B-supermodules,

and

- $G_0(B)$ = Grothendieck group of B-grmod,
- $K_0(B)$ = Grothendieck group of B-grpmod.

Note that $G_0(B)$ and $K_0(B)$ are naturally $\mathbb{Z}[q, q^{-1}]$-modules, where q acts by the grading shift.

In this talk, for simplicity, we will extend scalars to consider $G_0(B)$ and $K_0(B)$ as $\mathbb{Q}[q, q^{-1}]$-modules.

Supermodules over graded superalgebras

For B-modules N and M, the space of B-supermodule maps

$$\mathrm{HOM}_B(N, M)$$

is naturally $\mathbb{Z}$-graded.

We have a perfect pairing

$$\langle -, - \rangle \colon K_0(B) \times G_0(B) \to \mathbb{Z}[q], \quad \langle [P], [M] \rangle = \operatorname{grdim} \mathrm{HOM}_B(P, M),$$

where

$$\operatorname{grdim} V = \bigoplus_{i \in \mathbb{Z}} q^i \dim V_i$$

is the graded dimension of a $\mathbb{Z}$-graded space V.

Towers of graded superalgebras

Suppose we have a Λ-graded superalgebra $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$.

Definition (Tower of graded superalgebras: Definition)

We call A a (strong) tower of graded superalgebras if

1. Each A_λ is a f.d. $\mathbb{N}$-graded superalgebra (with a different mult. than that of A) and A_0 is one-dimensional.
2. The "external" mult. $A_\lambda \otimes A_\mu \to A_{\lambda+\mu}$ is a homomorphism of graded superalgebras for all $\lambda, \mu \in \Lambda$.
3. For all $\lambda, \mu \in \Lambda$, $A_{\lambda+\mu}$ is a two-sided projective $(A_\lambda \otimes A_\mu)$-supermodule.
4. Induction is right adjoint to restriction up to conjugation by an automorphism and a grading shift.
5. We have a Mackey type isomorphism relating induction and restriction.

Note: Frobenius graded superalgebras always satisfy condition 4.

Towers of graded superalgebras

Assume from now on that A is a strong tower of algebras.

Let

$$\mathcal{G}(A) = \bigoplus_{\lambda \in \Lambda} G_0(A_\lambda) \quad \text{and} \quad \mathcal{K}(A) = \bigoplus_{\lambda \in \Lambda} K_0(A_\lambda).$$

We extend our previous pairing to a perfect pairing

$$\langle -, - \rangle \colon \mathcal{K}(A) \times \mathcal{G}(A) \to \mathbb{Q}[q, q^{-1}]$$

by declaring different summands to be orthogonal.

Define

$$A\text{-grmod}^{\otimes 2} = \bigoplus_{\lambda, \mu} (A_\lambda \otimes A_\mu)\text{-grmod}.$$

Hopf functors

Define the following functors:

$$\nabla\colon A\text{-grmod}^{\otimes 2} \to A\text{-grmod}, \quad \nabla|_{(A_\lambda \otimes A_\mu)\text{-grmod}} = \mathrm{Ind}_{A_\lambda \otimes A_\mu}^{A_{\lambda+\mu}},$$

$$\Delta\colon A\text{-grmod} \to A\text{-grmod}^{\otimes 2}, \quad \Delta|_{A_\lambda\text{-grmod}} = \bigoplus_{\mu+\nu=\lambda} \mathrm{Res}_{A_\mu \otimes A_\nu}^{A_\lambda},$$

$$\eta\colon \mathrm{grVect} \to A\text{-grmod}, \quad \eta(V) = V \in A_0\text{-grmod for } V \in \mathrm{grVect},$$

$$\varepsilon\colon A\text{-grmod} \to \mathrm{grVect}, \quad \varepsilon(V) = \begin{cases} V & \text{if } V \in A_0\text{-grmod}, \\ 0 & \text{otherwise.} \end{cases}$$

These functors are exact, and so induce operations on Grothendieck groups.

Definition (Compatible tower)

We say A is a compatible tower if, under the above operations, $(\mathcal{K}(A), \mathcal{G}(A))$ is a compatible dual pair of Hopf algebras.

Examples of strong compatible towers

Example (Hecke algebras of type A)

Hecke algebras of type A at generic q or at roots of unity (or the 0-Hecke algebra).

Special case $q = 1$ is the tower of groups algebras of symmetric groups.

Example (Wreath product algebras)

Suppose B is a Frobenius graded superalgebra.

Then the symmetric group S_n acts on $B^{\otimes n}$ by superpermutations.

Then we have the wreath product algebras $B^{\otimes n} \rtimes S_n$.

Example (Sergeev superalgebras)

If B is the rank one Clifford superalgebra, then $B^{\otimes n} \rtimes S_n$ is isomorphic to the Sergeev superalgebra.

Examples of strong compatible towers

Example (Nilcoxeter graded superalgebras)

Fix $d \in \mathbb{Z}$ and $\epsilon \in \mathbb{Z}_2$.

The nilcoxeter graded superalegbra $N_n^{d,\epsilon}$ is the graded unital superalegbra with:

Generators: $u_1, \dots, u_{n-1}$, such that each u_i has $\mathbb{Z}$-degree d and parity ϵ

Relations:

$$u_i^2 = 0 \quad \text{for all } i$$
$$u_i u_j = (-1)^\epsilon u_j u_i \quad \text{for } |i-j| > 1$$
$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \quad \text{for all } 1 \leq i \leq n-1$$

Note: If $d = 0$ and $\epsilon = 0$, these are just the usual nilcoxeter algebras.

The twisted Heisenberg double associated to a tower

Definition (Twisted Heisenberg double associated to a tower)

To a strong compatible tower of algebras A, we can now associate the twisted Heisenberg double

$$\mathfrak{h}(A) := \mathfrak{h}(\mathcal{G}(A), \mathcal{K}(A)) = \mathcal{G}(A) \# \mathcal{K}(A)$$

and its Fock space $\mathcal{F}(A) = \mathcal{G}(A)$.

We wish to categorify the Fock space representation of $\mathfrak{h}(A)$.

We have already categorified the underlying $\mathbb{Q}[q, q^{-1}]$-module:

$$\mathcal{G}(A) = \bigoplus_{n \in \mathbb{N}} G_0(A_n).$$

We now need define functors on $\mathcal{G}(A)$ that lift the action of $\mathfrak{h}(A)$ on Fock space and isomorphisms lifting the defining relations of the twisted Heisenberg double.

Categorification of the twisted Heisenberg double

Recall the direct sum of categories:

$$A\text{-grmod} = \bigoplus_{n \in \mathbb{N}} A_n\text{-grmod}.$$

For $M \in A_m$-grmod, we have the functor

$$\mathrm{Ind}_M \colon A\text{-grmod} \to A\text{-grmod}, \quad \mathrm{Ind}_M(N) = \mathrm{Ind}_{A_m \otimes A_n}^{A_{m+n}} (M \otimes N).$$

For $P \in A_p$-grpmod, we have the functor

$$\mathrm{Res}_P \colon A\text{-grmod} \to A\text{-grmod}, \quad \mathrm{Res}_P(N) = \mathrm{Hom}_{A_p}(P, \mathrm{Res}_{A_{n-p} \otimes A_p}^{A_n} N).$$

These functors are exact and so they induce endomorphisms

$$[\mathrm{Ind}_M] \quad \text{and} \quad [\mathrm{Res}_P]$$

of $\mathcal{G}(A)$.

Categorification of the twisted Heisenberg double

Theorem (Rosso–S. 2014)

The functors Ind_M and Res_P for $M \in A$-grmod and $P \in A$-grpmod categorify the Fock space representation $\mathcal{F}(A)$ of $\mathfrak{h}(A)$.

More precisely, for all $M, N \in A$-grmod and $P, Q \in A$-grpmod, we have isomorphisms of functors

$$\mathrm{Ind}_M \circ \mathrm{Ind}_N \cong \mathrm{Ind}_{\nabla(M \otimes N)},$$
$$\mathrm{Res}_P \circ \mathrm{Res}_Q \cong \mathrm{Res}_{\nabla(P \otimes Q)},$$
$$\mathrm{Res}_P \circ \mathrm{Ind}_M \cong \nabla \mathrm{Res}^{\text{twisted}}_{\Delta(P)}(M \otimes -).$$

Thus, on the level of Grothendieck groups, we have

$$([M]\#[P])([N]) = [\mathrm{Ind}_M] \circ [\mathrm{Res}_P]([N]) = [\mathrm{Ind}_M \circ \mathrm{Res}_P(N)] \in \mathcal{G}(A).$$

Note: In the non-graded non-super setting, we recover the result of S.–Yacobi (2013) on categorification of the (untwisted) Heisenberg double.

Examples

Example (Non-graded, non-super, S.–Yacobi 2014)

- Hecke algebras (generic q) $\rightsquigarrow$ (infinite-dim.) Heisenberg algebra
- nilcoxeter algebras $\rightsquigarrow$ Weyl algebra
- 0-Hecke algebras $\rightsquigarrow$ quasi-Heisenberg algebra

Example (Tower of wreath product algebras)

These towers yields categorifications of lattice Heisenberg algebras.

A special case of this categorification (certain choice of B) appears in work of Cautis–Licata.

Example (Tower of nilcoxeter graded superalgebras)

These towers yield categorifications of the quantum Weyl algebra.

Final remarks

The categorification discussed above does not depend on a generating set for the twisted Heisenberg double.

Our definition of the twisted Heisenberg double was motivated by the desire to extend work of S.–Yacobi to the graded superalgebra setting.

One can show that certain shifts in the twisting data do not affect the isomorphism type of the twisted Heisenberg double $\mathfrak{h}(H^+, H^-)$.

This allows us to conclude that certain degree shifts in the induction/restriction functors do not affect the algebra being categorified.