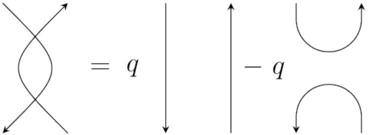
A gentle introduction to categorification



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Outline

1. Refresher on basic category theory
2. Common algebraic objects as categories
3. Categorification
4. Simple examples
5. Advanced example: quantum groups
6. Advanced example/application: TQFTs and knot invariants
7. The future

Categories (Definition)

A (small) category $\mathcal{C}$ consists of

- a set of objects $\mathrm{Ob}\,\mathcal{C}$,
- a set of morphisms $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$,

together with a composition

$$\mathrm{Mor}_{\mathcal{C}}(Y, Z) \times \mathrm{Mor}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}_{\mathcal{C}}(X, Z), \quad (f, g) \mapsto f \circ g,$$

and an identity morphism $1_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$ for all objects $X \in \mathrm{Ob}\,\mathcal{C}$.

The composition must be associative:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

whenever $f \circ g$ and $g \circ h$ are defined.

The identity morphism has the property that

$$1_Y \circ f = f = f \circ 1_X \quad \text{for all } f \in \mathrm{Mor}_{\mathcal{C}}(X, Y).$$

Categories (Examples)

Example (Sets)

- Objects: sets
- Morphisms: set maps

Example (Vector spaces)

- Objects: vector spaces over a fixed field
- Morphisms: linear maps

Example (Groups)

- Objects: groups
- Morphisms: group homomorphisms

Categories (Examples)

Example (Rings)

- Objects: rings
- Morphisms: ring homomorphisms

Example (Topological spaces)

- Objects: topological spaces
- Morphisms: continuous maps

Other examples

- modules over a fixed ring
- smooth manifolds
- algebraic varieties
- ...

Categories with one object

Suppose $\mathcal{C}$ is a category with one object X.

Then we only have one set of morphisms:

$$\mathrm{Mor}_{\mathcal{C}}(X, X)$$

Composition gives an associative operation on $\mathrm{Mor}_{\mathcal{C}}(X, X)$.

The identity 1_X is a identity element for this operation.

So $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is a monoid!

Conclusion

Monoids are one-object categories.

Categories with one object

Definition (Isomorphism)

An isomorphism in a category $\mathcal{C}$ is an element $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ such that there exists a $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ satisfying

$$f \circ g = 1_Y, \qquad g \circ f = 1_X.$$

Suppose $\mathcal{C}$ is a category with one object X and such that all morphisms are isomorphisms.

As before, $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is a monoid. But now all elements are invertible! So it is a group!

Conclusion

Groups are one-object categories in which all morphisms are isomorphisms.

Preadditive categories

A category is preadditive if

- $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is an abelian group for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$ (we write the group operation as addition), and
- composition is distributive:

$$f \circ (g + h) = f \circ g + f \circ h, \qquad (g + h) \circ f = g \circ f + h \circ f,$$

whenever the above compositions are defined.

Example (Category of abelian groups)

The category of abelian groups is preadditive.

We can add two group homomorphisms pointwise and composition is distributive over this addition.

Preadditive categories with one object

Suppose $\mathcal{C}$ is a preadditive category with one object X.

Then

$$\mathrm{Mor}_{\mathcal{C}}(X, X)$$

has

- an associative operation (composition),
- an abelian group structure (addition), and
- composition is distributive over addition.

Thus, $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is a ring, with multiplication given by composition!

Conclusion

Rings are one-object preadditive categories.

$\mathbb{C}$-linear categories

A category is $\mathbb{C}$-linear if

- $\operatorname{Mor}_{\mathcal{C}}(X, Y)$ is a complex vector space for all $X, Y \in \operatorname{Ob} \mathcal{C}$, and
- composition is bilinear: for all $f, g, h \in \operatorname{Mor} \mathcal{C}$, $\alpha, \beta \in \mathbb{C}$,

$$f \circ (\alpha g + \beta h) = \alpha(f \circ g) + \beta(f \circ h),$$
$$(\alpha g + \beta h) \circ f = \alpha(g \circ f) + \beta(h \circ f),$$

  whenever the above compositions are defined.

Example (Category of vector spaces)

The category of complex vector spaces is $\mathbb{C}$-linear.

The space of linear maps between two vector spaces is itself a vector space, and composition of linear maps is bilinear.

$\mathbb{C}$-linear categories with one object

Suppose $\mathcal{C}$ is a $\mathbb{C}$-linear category with one object X.

Then

$$\mathrm{Mor}_{\mathcal{C}}(X, X)$$

has

- an associative operation (composition),
- a $\mathbb{C}$-vector space structure, and
- composition is bilinear.

Thus, $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is a $\mathbb{C}$-algebra, with multiplication given by composition!

Conclusion

$\mathbb{C}$-algebras are one-object $\mathbb{C}$-linear categories.

Rings with idempotents

The above observations can be generalized to categories with several objects. Here's one example:

Suppose R is a ring.

An element $e \in R$ is called an idempotent if $e^2 = e$.

Let's call a set $\{e_1, \dots, e_n\}$ of idempotents of R a system of idempotents if

- $e_i e_j = 0$ when $i \neq j$,
- $e_1 + e_2 + \cdots + e_n = 1$.

Example: Ring of matrices

Let $R = M_{n\times n}(\mathbb{C})$.

Let E_{ij} be the matrix with a 1 in position (i,j) and a 0 in every other position.

Then

- $E_{ii}E_{jj} = \delta_{ij}E_{ii}$, and
- $1 = E_{11} + E_{22} + \cdots + E_{nn}$, where $1 \in R$ is the identity matrix.

So $\{E_{11}, E_{22}, \dots, E_{nn}\}$ is a system of idempotents.

It is easy to check that

- $E_{ii}RE_{jj} = \mathbb{C}E_{ij}$ is the set of matrices that are zero outside of position (i,j),
- $R = \bigoplus_{i,j} E_{ii}RE_{jj}$, and
- $(E_{ii}RE_{jj})(E_{kk}RE_{\ell\ell}) \subseteq \delta_{jk}\, E_{ii}RE_{\ell\ell}$.

Example: Ring of matrices

Since

$$(E_{ii}RE_{jj})(E_{kk}RE_{\ell\ell}) \subseteq \delta_{jk} E_{ii}RE_{\ell\ell},$$

the only "interesting" multiplication is when $j = k$.

So we can think of $R = M_{n\times n}(\mathbb{C})$ as a category with

- objects $\{E_{11}, \dots, E_{nn}\}$,
- $\mathrm{Mor}(E_{ii}, E_{jj}) = E_{jj}RE_{ii} = \mathbb{C}E_{ji}$,
- composition given by matrix multiplication.

Generalization

A preadditive category with finitely many objects is equivalent to a ring together with a system of idempotents.

Groupoids

For preadditive categories, we had:

$$\begin{array}{ccc} \text{one object category} & \rightsquigarrow & \text{multiple object category} \\ \| & & \| \\ \text{ring} & \rightsquigarrow & \text{ring with idempotents} \end{array}$$

For categories in which all morphisms are isomorphisms, we have:

$$\begin{array}{ccc} \text{one object category} & \rightsquigarrow & \text{multiple object category} \\ \| & & \| \\ \text{group} & \rightsquigarrow & \text{groupoid} \end{array}$$

Functors

Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories.

A functor F from $\mathcal{C}$ to $\mathcal{D}$ consists of

- a map $F\colon \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$,
- for all $X, Y \in \operatorname{Ob}\mathcal{C}$, a map $F\colon \operatorname{Mor}_{\mathcal{C}}(X, Y) \to \operatorname{Mor}_{\mathcal{D}}(F(X), F(Y))$.

We require that the map on morphisms respects composition:

$$F(f \circ g) = F(f) \circ F(g),$$

whenever the composition of $f, g \in \operatorname{Mor}\mathcal{C}$ is defined.

We also require it to preserve identities:

$$F(1_X) = 1_{F(X)}.$$

Functors

Example (Forgetful functors)

We can define a functor

$$F\colon \text{category of groups} \to \text{category of sets}$$

as follows:

- for a group G, we define $F(G)$ to be the underlying set of G,
- for a group homomorphism $f\colon G_1 \to G_2$, we define $F(f)$ to be the underlying set map.

So F just forgets the group structure.

There are many other examples of forgetful functors.

Functors

Example (Double dual)

There is a functor from the category of complex vector spaces to itself that

- maps any vector space to its double dual (the dual of its dual space),
- maps any linear map to its double dual.

Example (Fundamental group)

Suppose

- **Top** is the category of pointed topological spaces (topological spaces together with a distinguished point),
- **Group** is the category of groups.

We have a functor **Top** $\to$ **Group** that maps a pointed topological space to its fundamental group.

Group actions

Recall, from group theory, that one is often interested in actions of groups on sets.

Example: The group S_n acts on a set with n elements by permuting the elements.

Suppose $\mathcal{C}$ is a group, thought of as a one-element category.

Question

What is a functor $\mathcal{C} \to \mathbf{Set}$? ($\mathbf{Set}$ = category of sets)

The single object of $\mathcal{C}$ must be sent to an object of **Set**, i.e. to a set A.

Each morphism of $\mathcal{C}$ (i.e. element of the group) must be sent to a map from A to itself in a way that respects composition.

Answer

So a functor $\mathcal{C} \to \mathbf{Set}$ is just an action of the group on a set!

Other examples

As before, suppose $\mathcal{C}$ is a group, thought of as a one-element category.

Example

A functor from $\mathcal{C}$ to the category of topological spaces is an action of a group on a topological space.

Example

A functor from $\mathcal{C}$ to the category of vector spaces is a representation of the group.

Using this idea, one can define representations of various algebraic objects (monoids, groups, rings, algebras) in any appropriate category.

Structure on sets of isomorphism classes

Given a category $\mathcal{C}$, define

$$\operatorname{Iso}\mathcal{C} = \text{set of isomorphism classes of objects of } \mathcal{C}.$$

Extra structure on the category (e.g. direct sums, tensor products) becomes extra structure on $\operatorname{Iso}\mathcal{C}$.

Example (Finite sets)

Let $\mathcal{C}$ be the category of finite sets.

- $\operatorname{Iso}\mathcal{C} \cong \mathbb{N}$ since every finite set is determined, up to isomorphism, by its cardinality,
- disjoint union on $\mathcal{C}$ becomes addition on $\mathbb{N}$ since $|A \sqcup B| = |A| + |B|$,
- cartesian product in $\mathcal{C}$ becomes multiplication on $\mathbb{N}$ since $|A \times B| = |A| \cdot |B|$,

Structure on sets of isomorphism classes

Example (Vector spaces)

Let $\mathcal{C}$ be the category of finite dimensional vector spaces over $\mathbb{C}$.

- $\operatorname{Iso}\mathcal{C} \cong \mathbb{N}$ since every f.d. vector space is determined, up to isomorphism by its dimension,
- direct sum on $\mathcal{C}$ becomes addition on $\mathbb{N}$ since

$$\dim(V \oplus W) = \dim V + \dim W,$$

- tensor product on $\mathcal{C}$ becomes multiplication on $\mathbb{N}$ since

$$\dim(V \otimes W) = \dim V \cdot \dim W.$$

Grothendieck groups

Typically, one is interested in a more sophisticated procedure.

Split Grothendieck group

If $\mathcal{C}$ is an additive category (have direct sums), then the split Grothendieck group $K^{\text{split}}(\mathcal{C})$ of $\mathcal{C}$ is

- the free abelian group generated by isomorphism classes of objects,
- modulo the relations

$$[M \oplus N] = [M] + [N], \quad M, N \text{ objects of } \mathcal{C}.$$

Grothendieck group

If $\mathcal{C}$ is an abelian category, the Grothendieck group $K_0(\mathcal{C})$ of $\mathcal{C}$ is

- the free abelian group generated by isomorphism classes of objects,
- modulo the relations $[M_2] = [M_1] + [M_3]$ for every short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Example: Distinguished bases

Suppose A is a finite-dimensional algebra over a field. Let

- A-mod be the category of finite-dimensional A-modules, and
- A-pmod be the category of finitely-generated projective A-modules.

$K_0(A\text{-mod})$ is a free abelian group generated by the classes of the (finitely many) f.d. irreducible A-modules.

In $K_0(A\text{-mod})$, the class of any f.d. module is equal to the sum of the classes (with multiplicity) appearing in a Jordan-Hölder series.

$K^{\text{split}}(A\text{-pmod})$ is a free abelian group generated by the classes of the indecomposable projective modules.

Decategorification

The process

$$\mathcal{C} \rightsquigarrow \operatorname{Iso}\mathcal{C} \text{ or } K^{\text{split}}(\mathcal{C}) \text{ or } K_0(\mathcal{C})$$

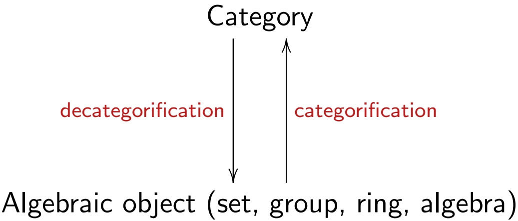
is called decategorification.

In decategorification, one loses information:

category $\rightsquigarrow$ "algebraic object" (set, group, ring, algebra)

Categorification

Categorification is the process that is the reverse of decategorification:



In other words, to categorify a set/group/ring/algebra A means to come up with a category $\mathcal{C}$ and some operations on that category such that

$$\operatorname{Iso}\mathcal{C} \text{ or } K^{\text{split}}(\mathcal{C}) \text{ or } K_0(\mathcal{C}) \cong A \qquad \text{(as a set/group/ring/algebra).}$$

Categorification is a much harder than decategorification since one must come up with an appropriate category.

Higher categorification

Recall: We can view a ring (with idempotents) as a category.

Question: What is the categorification of a ring, when that ring is itself viewed as a category?

Answer: A 2-category!

2-categories

A 2-category has

- objects,
- 1-morphisms between objects,
- 2-morphisms between 1-morphisms.

These are required to satisfy some compatibility axioms.

Why categorify?

Question: Why do we want to categorify algebraic objects (groups, rings, etc.)?

Answer:

- Uncovers hidden structure in the algebraic object (the category has much more structure than its decategorification).
- Provides tools for studying the categories involved.
- Applications to topology and physics.
- One obtains distinguished bases with integrality and positivity properties from categorification (classes of simple or indecomposable objects).

Integrality and positivity

Suppose

- A is a finite-dimensional algebra over a field,
- F is an exact functor $(A\text{-mod})^k \to A\text{-mod}$ for some $k \in \mathbb{N}$, $k > 0$.
  Example: tensor product, induction, restriction,... (when they are exact).

Then $K_0(A\text{-mod})$ has a basis given by classes of irreducible modules.

Since it is exact, F induces a map on Grothendieck groups.

If $V_1, \ldots, V_k$ are irreducible modules, then

$$F([V_1], \ldots, [V_k]) = [F(V_1, \ldots, V_k)]$$

is equal to a sum of the classes of the irreducible modules appearing in a Jordan-Hölder series for $F(V_1, \ldots, V_k)$.

So it is a positive integral sum of the (distinguished) basis elements of $K_0(A\text{-mod})$.

Lie algebras

Lie group

A Lie group is both a group and a manifold. The structures are compatible (e.g. multiplication is smooth).

Lie algebra

A Lie algebra is a “linearization” of a Lie group.

- As a vector space, it is the tangent space to the Lie group at the identity.
- Multiplication in the Lie group induces a Lie bracket operation on the tangent space.

Universal enveloping algebra

Each Lie algebra has a universal eveloping algebra which:

- is an associative algebra,
- has the same representation theory as the Lie algebra.

Lie algebras: Example

The group $\mathrm{SL}(n, \mathbb{C})$ of $n \times n$ complex matrices with determinant one is a Lie group. The group operation is matrix multiplication.

Its Lie algebra is $\mathfrak{sl}(n, \mathbb{C})$, the $n \times n$ matrices with trace zero. The Lie bracket is the commutator:

$$[A, B] = AB - BA$$

One can often study the properties (e.g. representations) of a Lie group via the Lie algebra, using the exponential map:

$$\exp \colon \mathfrak{sl}(n, \mathbb{C}) \to \mathrm{SL}(n, \mathbb{C}), \quad \exp(A) = \sum_{i=0}^{\infty} \frac{A^n}{n!}.$$

Quantum groups

Quantum group: A "deformation" of a universal enveloping algebra.

Suppose:

- $\mathfrak{g}$ is a Lie algebra,
- $U(\mathfrak{g})$ is its universal enveloping algebra.

The quantum group $U_q(\mathfrak{g})$ depends on a parameter q and reduces to $U(\mathfrak{g})$ when $q = 1$:

$$U_q(\mathfrak{g}) \overset{q \to 1}{\rightsquigarrow} U(\mathfrak{g}).$$

Applications:

- representation theory (Hopf algebras),
- physics,
- combinatorics (crystals: $q \to 0$),
- topology.

Categorification of quantum groups I

Lusztig slightly modified the quantum group $U_q(\mathfrak{g})$ to get an idempotented version $\dot{U}$.

As before, we can view $\dot{U}$ a category, with one object for each idempotent.

Lusztig then categorified $\dot{U}$ (more precisely, half of $\dot{U}$):

- defined an alegbraic variety (quiver variety) for each idempotent
- considered a certain category of perverse sheaves on these varieties
- considered a convolution of sheaves (multiplication)
- decategorification (Grothendieck group) recovers $\dot{U}$

This categorification yields canonical bases for quantum groups:

- positivity and integrality properties
- applications to combinatorics (crystals: $q \to 0$ limit)

Categorification of quantum groups II

One can extend the notion of Lie algebras to 2-Lie algebras.

2-Lie algebras are 2-categories that categorify quantum groups (themselves thought of as categories)

- algebraic definition (Rouquier)
- diagrammatic definition in terms of braid-like diagrams (Khovanov–Lauda)

Higher representation theory: 2-Functors from 2-Lie algebras into other 2-categories.

Category of cobordisms

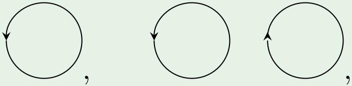
Category of k-cobordisms

Objects: oriented k-manifolds

Morphisms: $(k+1)$-manifolds with prescribed boundaries (cobordisms)

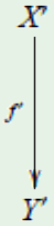
Example ($k = 1$)

Objects: collections of oriented circles



etc.

Morphisms:



etc.

Composition is given by gluing along boundaries.

TQFTS

Definition (TQFT)

A $(k+1)$-dimensional topological quantum field theory (TQFT) is a functor

category of k-cobordisms $\to$ category of R-modules (for some ring R).

So, a $(k+1)$-dim TQFT

- associates an R-module to each oriented k-manifold,
- associates an R-module homomorphism to each cobordism.

Example: Reshetikhin–Turaev invariant

The Reshetikhin–Turaev (RT) invariant is a $(0+1)$-dim TQFT.

Fix a simple Lie algebra $\mathfrak{g}$ and representation V.

Example: $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $V = \mathbb{C}^2$ (so $\mathfrak{g}$ acts on V by matrix multiplication).

The RT invariant is a functor

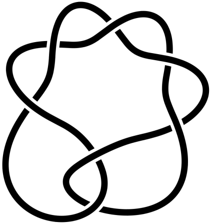
$$\text{category of 0-cobordisms} \to \text{category of } U_q(\mathfrak{g})\text{-modules.}$$

It sends

$$\text{empty 0-manifold} \mapsto \mathbb{Z}[q, q^{-1}] \quad \text{(the trivial } U_q(\mathfrak{g})\text{-module).}$$

Example: Reshetikhin–Turaev invariant

A knot is a cobordism from the empty 0-manifold to itself!



So, the RT invariant associates an endomorphism of $\mathbb{Z}[q, q^{-1}]$ to each knot.

Such an endomorphism must be given by multiplication by some polynomial in $\mathbb{Z}[q, q^{-1}]$.

So the RT invariant associates an element of $\mathbb{Z}[q, q^{-1}]$ to each knot!

If $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $V = \mathbb{C}^2$, this is the Jones polynomial.

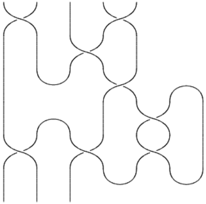
Khovanov homology

Khovanov homology is a functor

$$\text{category of 0-cobordisms} \to \text{category of categories.}$$

So it associates

- a category to each 0-manifold (i.e. collection of points),
- a functor to each 1-cobordism (i.e. tangle).



Decategorifying Khovanov homology recovers the RT invariant.

Conclusion: Khovanov homology categorifies the RT invariant.

The future

One can go beyond TQFTs to extended TQFTs.

Consider the 2-category with:

- objects: 0-dimensional manifolds
- 1-morphisms: tangles
- 2-morphisms: cobordisms between tangles

Goal: Construct functors from this 2-category to the 2-category of representations of a 2-quantum group.

Benefits:

- richer invariants of knots: to each knot, one assigns a homology theory instead of a polynomial
- polynomial invariants of surfaces: a surface is a cobordism from the empty tangle to itself