A gentle introduction to categorification

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Slides available online: alistairsavage.ca/talks
1. Refresher on basic category theory
2. Common algebraic objects as categories
3. Categorification
4. Simple examples
5. Advanced example: quantum groups
6. Advanced example/application: TQFTs and knot invariants
7. The future
A (small) category $\mathcal{C}$ consists of

- a set of objects $\text{Ob} \mathcal{C}$,
- a set of morphisms $\text{Mor}_\mathcal{C}(X, Y)$ for all $X, Y \in \text{Ob} \mathcal{C}$,

一起 with a composition

$$\text{Mor}_\mathcal{C}(Y, Z) \times \text{Mor}_\mathcal{C}(X, Y) \rightarrow \text{Mor}_\mathcal{C}(X, Z), \quad (f, g) \mapsto f \circ g,$$

and an identity morphism $1_X \in \text{Mor}_\mathcal{C}(X, X)$ for all objects $X \in \text{Ob} \mathcal{C}$.

The composition must be associative:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

whenever $f \circ g$ and $g \circ h$ are defined.

The identity morphism has the property that

$$1_Y \circ f = f = f \circ 1_X$$

for all $f \in \text{Mor}_\mathcal{C}(X, Y)$. 
Categories (Examples)

Example (Sets)
- **Objects:** sets
- **Morphisms:** set maps

Example (Vector spaces)
- **Objects:** vector spaces over a fixed field
- **Morphisms:** linear maps

Example (Groups)
- **Objects:** groups
- **Morphisms:** group homomorphisms
Categories (Examples)

Example (Rings)
- **Objects:** rings
- **Morphisms:** ring homomorphisms

Example (Topological spaces)
- **Objects:** topological spaces
- **Morphisms:** continuous maps

Other examples
- modules over a fixed ring
- smooth manifolds
- algebraic varieties
- …
Categories with one object

Suppose $\mathcal{C}$ is a category with one object $X$.

Then we only have one set of morphisms:

$$\text{Mor}_\mathcal{C}(X, X)$$

Composition gives an associative operation on $\text{Mor}_\mathcal{C}(X, X)$.

The identity $1_X$ is an identity element for this operation.

So $\text{Mor}_\mathcal{C}(X, X)$ is a monoid!

Conclusion

Monoids are one-object categories.
Categories with one object

**Definition (Isomorphism)**

An **isomorphism** in a category \( \mathcal{C} \) is an element \( f \in \text{Mor}_\mathcal{C}(X, Y) \) such that there exists a \( g \in \text{Mor}_\mathcal{C}(Y, X) \) satisfying

\[
f \circ g = 1_Y, \quad g \circ f = 1_X.
\]

Suppose \( \mathcal{C} \) is a category with one object \( X \) and such that **all morphisms** are isomorphisms.

As before, \( \text{Mor}_\mathcal{C}(X, X) \) is a monoid. But now all elements are invertible! So it is a group!

**Conclusion**

Groups are one-object categories in which all morphisms are isomorphisms.
A category is preadditive if

- \( \text{Mor}_C(X, Y) \) is an abelian group for all \( X, Y \in \text{Ob} \, C \) (we write the group operation as addition), and

- composition is distributive:

\[
  f \circ (g + h) = f \circ g + f \circ h, \quad (g + h) \circ f = g \circ f + h \circ f,
\]

whenever the above compositions are defined.

**Example (Category of abelian groups)**

The category of abelian groups is preadditive.

We can add two group homomorphisms pointwise and composition is distributive over this addition.
Preadditive categories with one object

Suppose \( C \) is a preadditive category with one object \( X \).

Then

\[ \text{Mor}_C(X, X) \]

has

- an associative operation (composition),
- an abelian group structure (addition), and
- composition is distributive over addition.

Thus, \( \text{Mor}_C(X, X) \) is a ring, with multiplication given by composition!

Conclusion

Rings are one-object preadditive categories.
**C-linear categories**

A category is **C-linear** if

- $\text{Mor}_C(X, Y)$ is a complex vector space for all $X, Y \in \text{Ob} C$, and
- composition is bilinear: for all $f, g, h \in \text{Mor} C$, $\alpha, \beta \in \mathbb{C}$,

\[ f \circ (\alpha g + \beta h) = \alpha (f \circ g) + \beta (f \circ h), \]
\[ (\alpha g + \beta h) \circ f = \alpha (g \circ f) + \beta (h \circ f), \]

whenever the above compositions are defined.

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**Example (Category of vector spaces)**

The category of complex vector spaces is **C-linear**.

The space of linear maps between two vector spaces is itself a vector space, and composition of linear maps is bilinear.
**C-linear categories with one object**

Suppose $\mathcal{C}$ is a $\mathbb{C}$-linear category with one object $X$.

Then $\text{Mor}_\mathcal{C}(X, X)$ has an associative operation (composition), a $\mathbb{C}$-vector space structure, and composition is bilinear.

Thus, $\text{Mor}_\mathcal{C}(X, X)$ is a $\mathbb{C}$-algebra, with multiplication given by composition!

**Conclusion**

$\mathbb{C}$-algebras are one-object $\mathbb{C}$-linear categories.
Rings with idempotents

The above observations can be generalized to categories with several objects. Here’s one example:

Suppose $R$ is a ring.

An element $e \in R$ is called an idempotent if $e^2 = e$.

Let’s call a set $\{e_1, \ldots, e_n\}$ of idempotents of $R$ a system of idempotents if

- $e_i e_j = 0$ when $i \neq j$,
- $e_1 + e_2 + \cdots + e_n = 1$. 

Example: Ring of matrices

Let \( R = M_{n \times n}(\mathbb{C}) \).

Let \( E_{ij} \) be the matrix with a 1 in position \((i, j)\) and a 0 in every other position.

Then

- \( E_{ii}E_{jj} = \delta_{ij}E_{ii} \), and
- \( 1 = E_{11} + E_{22} + \cdots + E_{nn} \), where 1 \( \in R \) is the identity matrix.

So \( \{ E_{11}, E_{22}, \ldots, E_{nn} \} \) is a system of idempotents.

It is easy to check that

- \( E_{ii}RE_{jj} = \mathbb{C}E_{ij} \) is the set of matrices that are zero outside of position \((i, j)\),
- \( R = \bigoplus_{i,j} E_{ii}RE_{jj} \), and
- \( (E_{ii}RE_{jj})(E_{kk}RE_{\ell\ell}) \subseteq \delta_{jk} E_{ii}RE_{\ell\ell} \).
Example: Ring of matrices

Since

\[(E_{ii}RE_{jj})(E_{kk}RE_{\ell\ell}) \subseteq \delta_{jk} E_{ii}RE_{\ell\ell},\]

the only “interesting” multiplication is when \(j = k\).

So we can think of \(R = M_{n \times n}(\mathbb{C})\) as a category with

- objects \(\{E_{11}, \ldots, E_{nn}\}\),
- \(\text{Mor}(E_{ii}, E_{jj}) = E_{jj}RE_{ii} = \mathbb{C}E_{ji}\),
- composition given by matrix multiplication.

Generalization

A preadditive category with finitely many objects is equivalent to a ring together with a system of idempotents.
Groupoids

For preadditive categories, we had:

\[
\begin{array}{c}
\text{one object category} \rightsquigarrow \text{multiple object category} \\
\text{ring} \rightsquigarrow \text{ring with idempotents}
\end{array}
\]

For categories in which all morphisms are isomorphisms, we have:

\[
\begin{array}{c}
\text{one object category} \rightsquigarrow \text{multiple object category} \\
\text{group} \rightsquigarrow \text{groupoid}
\end{array}
\]
Functors

Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories.

A functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ consists of

- a map $F : \text{Ob} \, \mathcal{C} \to \text{Ob} \, \mathcal{D}$,
- for all $X, Y \in \text{Ob} \, \mathcal{C}$, a map $F : \text{Mor} \, \mathcal{C}(X, Y) \to \text{Mor} \, \mathcal{D}(F(X), F(Y))$.

We require that the map on morphisms respects composition:

$$F(f \circ g) = F(f) \circ F(g),$$

whenever the composition of $f, g \in \text{Mor} \, \mathcal{C}$ is defined.

We also require it to preserve identities:

$$F(1_X) = 1_{F(X)}.$$
Example (Forgetful functors)

We can define a functor

\[ F : \text{category of groups} \rightarrow \text{category of sets} \]

as follows:

- for a group \( G \), we define \( F(G) \) to be the underlying set of \( G \),
- for a group homomorphism \( f : G_1 \rightarrow G_2 \), we define \( F(f) \) to be the underlying set map.

So \( F \) just \textit{forgets} the group structure.

There are many other examples of forgetful functors.
Functors

Example (Double dual)

There is a functor from the category of complex vector spaces to itself that

- maps any vector space to its double dual (the dual of its dual space),
- maps any linear map to its double dual.

Example (Fundamental group)

Suppose

- \( \textbf{Top} \) is the category of pointed topological spaces (topological spaces together with a distinguished point),
- \( \textbf{Group} \) is the category of groups.

We have a functor \( \textbf{Top} \rightarrow \textbf{Group} \) that maps a pointed topological space to its fundamental group.
Group actions

Recall, from group theory, that one is often interested in actions of groups on sets.

Example: The group $S_n$ acts on a set with $n$ elements by permuting the elements.

Suppose $C$ is a group, thought of as a one-element category.

Question

What is a functor $C \to \text{Set}$? \hfill ($\text{Set} = \text{category of sets}$)

The single object of $C$ must be sent to an object of $\text{Set}$, i.e. to a set $A$.

Each morphism of $C$ (i.e. element of the group) must be sent to a map from $A$ to itself in a way that respects composition.

Answer

So a functor $C \to \text{Set}$ is just an action of the group on a set!
Other examples

As before, suppose $C$ is a group, thought of as a one-element category.

Example

A functor from $C$ to the category of topological spaces is an action of a group on a topological space.

Example

A functor from $C$ to the category of vector spaces is a representation of the group.

Using this idea, one can define representations of various algebraic objects (monoids, groups, rings, algebras) in any appropriate category.
Structure on sets of isomorphism classes

Given a category $\mathcal{C}$, define

$$\text{Iso} \mathcal{C} = \text{set of isomorphism classes of objects of } \mathcal{C}.$$  

Extra structure on the category (e.g. direct sums, tensor products) becomes extra structure on Iso $\mathcal{C}$.

Example (Finite sets)

Let $\mathcal{C}$ be the category of finite sets.

- Iso $\mathcal{C} \cong \mathbb{N}$ since every finite set is determined, up to isomorphism, by its cardinality,
- disjoint union on $\mathcal{C}$ becomes addition on $\mathbb{N}$ since $|A \sqcup B| = |A| + |B|$,
- cartesian product in $\mathcal{C}$ becomes multiplication on $\mathbb{N}$ since $|A \times B| = |A| \cdot |B|$,
Example (Vector spaces)

Let $\mathcal{C}$ be the category of finite dimensional vector spaces over $\mathbb{C}$.

- $\text{Iso} \mathcal{C} \cong \mathbb{N}$ since every f.d. vector space is determined, up to isomorphism by its dimension,

- direct sum on $\mathcal{C}$ becomes addition on $\mathbb{N}$ since
  \[ \dim(V \oplus W) = \dim V + \dim W, \]

- tensor product on $\mathcal{C}$ becomes multiplication on $\mathbb{N}$ since
  \[ \dim(V \otimes W) = \dim V \cdot \dim W. \]
Grothendieck groups

Typically, one is interested in a more sophisticated procedure.

**Split Grothendieck group**

If $\mathcal{C}$ is an additive category (have direct sums), then the split Grothendieck group $K^{\text{split}}(\mathcal{C})$ of $\mathcal{C}$ is

- the free abelian group generated by isomorphism classes of objects,
- modulo the relations

\[
[M \oplus N] = [M] + [N], \quad M, N \text{ objects of } \mathcal{C}.
\]

**Grothendieck group**

If $\mathcal{C}$ is an abelian category, the Grothendieck group $K_0(\mathcal{C})$ of $\mathcal{C}$ is

- the free abelian group generated by isomorphism classes of objects,
- modulo the relations $[M_2] = [M_1] + [M_3]$ for every short exact sequence

\[
0 \to M_1 \to M_2 \to M_3 \to 0.
\]
Example: Distinguished bases

Suppose $A$ is a finite-dimensional algebra over a field. Let

- $A\text{-mod}$ be the category of finite-dimensional $A$-modules, and
- $A\text{-pmod}$ be the category of finitely-generated projective $A$-modules.

$K_0(A\text{-mod})$ is a free abelian group generated by the classes of the (finitely many) f.d. irreducible $A$-modules.

In $K_0(A\text{-mod})$, the class of any f.d. module is equal to the sum of the classes (with multiplicity) appearing in a Jordan-Hölder series.

$K^{\text{split}}(A\text{-pmod})$ is a free abelian group generated by the classes of the indecomposable projective modules.
Decategorification

The process

\[ \mathcal{C} \rightsquigarrow \text{Iso} \mathcal{C} \text{ or } K^{\text{split}}(\mathcal{C}) \text{ or } K_0(\mathcal{C}) \]

is called **decategorification**.

In decategorification, one loses information:

category \( \rightsquigarrow \) “algebraic object” (set, group, ring, algebra)
Categorification is the process that is the reverse of decategorification:

\[
\text{Category} \quad \xrightarrow{\text{decategorification}} \quad \text{Algebraic object (set, group, ring, algebra)} \quad \xleftarrow{\text{categorification}}
\]

In other words, to categorify a set/group/ring/algebra \( A \) means to come up with a category \( C \) and some operations on that category such that

\[
\text{Iso} \ C \text{ or } K^{\text{split}}(C) \text{ or } K_0(C) \cong A \quad \text{(as a set/group/ring/algebra)}.
\]

Categorification is a much harder than decategorification since one must come up with an appropriate category.
Higher categorification

Recall: We can view a ring (with idempotents) as a category.

Question: What is the categorification of a ring, when that ring is itself viewed as a category?

Answer: A 2-category!

2-categories

A 2-category has

- objects,
- 1-morphisms between objects,
- 2-morphisms between 1-morphisms.

These are required to satisfy some compatibility axioms.
Why categorify?

**Question:** Why do we want to categorify algebraic objects (groups, rings, etc.)?

**Answer:**
- Uncovers hidden structure in the algebraic object (the category has much more structure than its decategorification).
- Provides tools for studying the categories involved.
- Applications to topology and physics.
- One obtains **distinguished bases** with integrality and positivity properties from categorification (classes of simple or indecomposable objects).
Integrality and positivity

Suppose

- $A$ is a finite-dimensional algebra over a field,
- $F$ is an exact functor $(A\text{-mod})^k \to A\text{-mod}$ for some $k \in \mathbb{N}$, $k > 0$.

**Example:** tensor product, induction, restriction,... (when they are exact).

Then $K_0(A\text{-mod})$ has a basis given by classes of irreducible modules.

Since it is exact, $F$ induces a map on Grothendieck groups.

If $V_1, \ldots, V_k$ are irreducible modules, then

$$F([V_1], \ldots, [V_k]) = [F(V_1, \ldots, V_k)]$$

is equal to a sum of the classes of the irreducible modules appearing in a Jordan-Hölder series for $F(V_1, \ldots, V_k)$.

So it is a **positive integral** sum of the (distinguished) basis elements of $K_0(A\text{-mod})$. 
Lie group

A Lie group is both a group and a manifold. The structures are compatible (e.g. multiplication is smooth).

Lie algebra

A Lie algebra is a “linearization” of a Lie group.

- As a vector space, it is the tangent space to the Lie group at the identity.
- Multiplication in the Lie group induces a Lie bracket operation on the tangent space.

Universal enveloping algebra

Each Lie algebra has a universal enveloping algebra which:

- is an associative algebra,
- has the same representation theory as the Lie algebra.
The group $\operatorname{SL}(n, \mathbb{C})$ of $n \times n$ complex matrices with determinant one is a Lie group. The group operation is matrix multiplication.

Its Lie algebra is $\mathfrak{sl}(n, \mathbb{C})$, the $n \times n$ matrices with trace zero. The Lie bracket is the commutator:

$$[A, B] = AB - BA$$

One can often study the properties (e.g. representations) of a Lie group via the Lie algebra, using the exponential map:

$$\exp: \mathfrak{sl}(n, \mathbb{C}) \rightarrow \operatorname{SL}(n, \mathbb{C}), \quad \exp(A) = \sum_{i=0}^{\infty} \frac{A^n}{n!}.$$
Quantum groups

Quantum group: A “deformation” of a universal enveloping algebra.

Suppose:

- \( \mathfrak{g} \) is a Lie algebra,
- \( U(\mathfrak{g}) \) is its universal enveloping algebra.

The quantum group \( U_q(\mathfrak{g}) \) depends on a parameter \( q \) and reduces to \( U(\mathfrak{g}) \) when \( q = 1 \):

\[
U_q(\mathfrak{g}) \xrightarrow{q \to 1} U(\mathfrak{g}).
\]

Applications:

- representation theory (Hopf algebras),
- physics,
- combinatorics (crystals: \( q \to 0 \)),
- topology.
Lusztig slightly modified the quantum group $U_q(g)$ to get an idempotent version $\dot{U}$.

As before, we can view $\dot{U}$ a category, with one object for each idempotent.

Lusztig then categorified $\dot{U}$ (more precisely, half of $\dot{U}$):
- defined an algebraic variety (quiver variety) for each idempotent
- considered a certain category of perverse sheaves on these varieties
- considered a convolution of sheaves (multiplication)
- decategorification (Grothendieck group) recovers $\dot{U}$

This categorification yields canonical bases for quantum groups:
- positivity and integrality properties
- applications to combinatorics (crystals: $q \to 0$ limit)
One can extend the notion of Lie algebras to 2-Lie algebras.

2-Lie algebras are 2-categories that categorify quantum groups (themselves thought of as categories)

- algebraic definition (Rouquier)
- diagrammatic definition in terms of braid-like diagrams (Khovanov–Lauda)

Higher representation theory: 2-Functors from 2-Lie algebras into other 2-categories.
Category of cobordisms

Category of $k$-cobordisms

**Objects:** oriented $k$-manifolds

**Morphisms:** $(k + 1)$-manifolds with prescribed boundaries (cobordisms)

**Example ($k = 1$)**

**Objects:** collections of oriented circles

Composition is given by gluing along boundaries.
Definition (TQFT)

A \((k+1)\)-dimensional topological quantum field theory (TQFT) is a functor
\[
\text{category of } k\text{-cobordisms} \rightarrow \text{category of } R\text{-modules (for some ring } R)\).
\]

So, a \((k + 1)\)-dim TQFT

- associates an \(R\)-module to each oriented \(k\)-manifold,
- associates an \(R\)-module homomorphism to each cobordism.
The Reshetikhin–Turaev (RT) invariant is a $(0 + 1)$-dim TQFT.

Fix a simple Lie algebra $\mathfrak{g}$ and representation $V$.

**Example:** $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $V = \mathbb{C}^2$ (so $\mathfrak{g}$ acts on $V$ by matrix multiplication).

The RT invariant is a functor

$$\text{category of 0-cobordisms} \rightarrow \text{category of } U_q(\mathfrak{g})\text{-modules}.$$ 

It sends

empty 0-manifold $\mapsto \mathbb{Z}[q, q^{-1}]$ (the trivial $U_q(\mathfrak{g})$-module).
Example: Reshetikhin–Turaev invariant

A knot is a cobordism from the empty 0-manifold to itself!

So, the RT invariant associates an endomorphism of $\mathbb{Z}[q, q^{-1}]$ to each knot.

Such an endomorphism must be given by multiplication by some polynomial in $\mathbb{Z}[q, q^{-1}]$.

So the RT invariant associates an element of $\mathbb{Z}[q, q^{-1}]$ to each knot!

If $g = \mathfrak{sl}(2, \mathbb{C})$ and $V = \mathbb{C}^2$, this is the Jones polynomial.
Khovanov homology

Khovanov homology is a functor
category of 0-cobordisms $\rightarrow$ category of categories.

So it associates
- a category to each 0-manifold (i.e. collection of points),
- a functor to each 1-cobordism (i.e. tangle).

Decategorifying Khovanov homology recovers the RT invariant.

Conclusion: Khovanov homology categorifies the RT invariant.
The future

One can go beyond TQFTs to **extended TQFTs**.

Consider the 2-category with:

- **objects**: 0-dimensional manifolds
- **1-morphisms**: tangles
- **2-morphisms**: cobordisms between tangles

**Goal**: Construct functors from this 2-category to the 2-category of representations of a 2-quantum group.

**Benefits**:

- **richer invariants of knots**: to each knot, one assigns a homology theory instead of a polynomial
- **polynomial invariants of surfaces**: a surface is a cobordism from the empty tangle to itself