Hecke algebras and a categorification of the Heisenberg algebra

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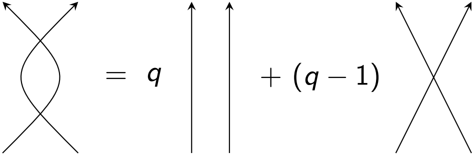


Joint work with Anthony Licata
(*Quantum Topology*, to appear. Preprint:arXiv:1101.0420).

Slides: www.mathstat.uottawa.ca/∼asavag2

Outline

Summary: We define a graphical category which acts naturally on modules of Hecke algebras and provides a categorification of the Heisenberg algebra in infinitely many variables.

Overview:

1. Overview of the theory of categorification
2. The Hecke algebra
3. The graphical category
4. Action of our graphical category on modules of Hecke algebras
5. A categorification of the Heisenberg algebra
6. A natural appearance of the affine Hecke algebra

A category theoretic point of view

Definition (Category)

A category $\mathcal{C}$ consists of:

- $\mathrm{Ob}\,\mathcal{C}$ – class of objects
- $\forall\ X, Y \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ – morphisms from X to Y
- associative composition:

$$\mathrm{Mor}_{\mathcal{C}}(Y, Z) \times \mathrm{Mor}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}_{\mathcal{C}}(X, Z)$$

- identity $1_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$ for each $X \in \mathrm{Ob}\,\mathcal{C}$

Some simple observations

1. Suppose $\mathcal{C}$ has only one object X. Then $\mathcal{C}$ (more precisely, $\mathrm{Mor}_{\mathcal{C}}(X, X)$) is a monoid. So monoids are just one object categories.
2. Similarly, groups are just one object categories where all morphisms are isomorphisms (i.e. invertible).

A category theoretic point of view

Definition (Preadditive category)

- for $X, Y \in \operatorname{Ob} \mathcal{C}$, $\operatorname{Mor}_{\mathcal{C}}(X, Y)$ is an abelian group
- composition is distributive

Simple observation

A preadditive category with one object is just a ring.

Fix a field k.

Definition (k-linear category)

- for $X, Y \in \operatorname{Ob} \mathcal{C}$, $\operatorname{Mor}_{\mathcal{C}}(X, Y)$ is a k-vector space
- composition is k-linear

Simple observation

A k-linear category with one object is just an associative k-algebra.

A category theoretic point of view

$\mathcal{C}$ a group (considered as a one object category)

Question: What is a functor $\mathcal{C} \to \underline{\mathrm{Set}}$?

- the single object of $\mathcal{C}$ is mapped to a set
- each morphism of this object is mapped to a set automorphism

Answer: A functor $\mathcal{C} \to \underline{\mathrm{Set}}$ is an action of a group on a set!

Similarly,

- a functor $\mathcal{C} \to \underline{\mathrm{Vect}}$ is a group representation
- $\mathcal{C}$ a monoid/ring/algebra, a functor $\mathcal{C} \to \underline{\mathrm{Vect}}$ is a rep
- can have reps in any appropriate category (e.g. category of topological spaces)

A category theoretic point of view

Question: If

- $\mathcal{C}$ is a monoid/group/ring/algebra (viewed as a category), and
- $F, G : \mathcal{C} \to \underline{\mathrm{Vect}}$ are functors,

what is a natural transformation $F \Rightarrow G$?

Answer: A homomorphism of representations.

Categorification and Decategorification

category $\mathcal{C}$

$\rightsquigarrow$

$A(\mathcal{C})$ = set of isom classes of objects of $\mathcal{C}$

decategorification $\downarrow$ $\uparrow$ categorification

Extra structure on $\mathcal{C}$ (e.g. tensor prod) becomes extra structure on $A(\mathcal{C})$.

Example (Categorification of $\mathbb{N}$)

- $\mathcal{C}$ = category of finite sets
- $A(\mathcal{C}) = \mathbb{N}$ (up to isom, a finite set determined by its cardinality)
- Cartesian product in $\mathcal{C} \rightsquigarrow$ product on $\mathbb{N}$
- Disjoint union (coproduct) in $\mathcal{C} \rightsquigarrow$ addition on $\mathbb{N}$
- Existence of an injection in $\mathcal{C} \rightsquigarrow$ order on $\mathbb{N}$

Categorification: Examples

Example (Another categorification of $\mathbb{N}$)

- $\mathcal{C}$ = category of f.d. vector spaces over some fixed field
- $A(\mathcal{C}) = \mathbb{N}$ (dimension)
- $\otimes \rightsquigarrow \times$
- $\oplus \rightsquigarrow +$

So categorifications are far from unique.

Example (Categorification of $\mathbb{N}[q, q^{-1}]$)

- $\mathcal{C}$ = category of f.d. $\mathbb{Z}$-graded vector spaces over some fixed field
- $A(\mathcal{C}) = \mathbb{N}[q, q^{-1}]$

Grothendieck group

$\mathcal{C}$ = abelian category

The Grothendieck group of $\mathcal{C}$ is

$$K_0 = (\text{free abelian group on objects})/R, \quad \text{where}$$
$$R = \langle B - A - C \mid 0 \to A \to B \to C \to 0 \text{ s.e.s.} \rangle$$

If $\mathcal{C}$ is a monoidal category (i.e. it has a tensor product), then $K_0(\mathcal{C})$ is a ring with multiplication given by

$$[A] \cdot [B] = [A \otimes B].$$

Important examples

Lusztig quiver varieties

- $U_q(\mathfrak{g})$ = quantized enveloping algebra associated to a Kac-Moody Lie algebra $\mathfrak{g}$
- $\dot{U}$ = certain "idempotented version" of $U_q(\mathfrak{g})$
- $\dot{U}_+$ = "upper half" of $\dot{U}$
- Lusztig defined varieties attached to quivers (directed graphs)
- $\mathcal{C}$ = certain category of perverse sheaves on these varieties
- convolution of sheaves induces multiplication on Grothendieck group
- passage to Grothendieck group recovers $\dot{U}_+$
- canonical basis: nice basis of $\dot{U}_+$ with integral structure coefficients hints at existence of categorification

Important examples

Categorification of quantum groups and their representations

- can define 2-Kac Moody algebras
  - algebraic (Rouquier)
  - diagrammatic (Khovanov-Lauda)
- these are 2-categories that categorify quantum groups
- have a notion of 2-representation: functors into other 2-categories (e.g. categories of bimodules)

Applications

- knot/surface invariants and TQFTs
- Khovanov homology: categorification of the Reshetikhin-Turaev invariant
- one gets richer invariants for knots: assign a homology theory to a knot instead of a polynomial
- should be able to get polynomial invariants of surfaces

The Heisenberg algebra

Definition

The rank one Heisenberg algebra has generators p, q and relation

$$pq - qp = 1.$$

It is the algebra of operators in the quantization of the harmonic oscillator.

Definition

The (infinite rank) Heisenberg algebra $\mathfrak{h}$ is the algebra with generators p_i, q_i, $i = 1, 2, 3, \ldots$, and relations

$$p_i q_j - q_j p_i = \delta_{i,j} 1, \quad p_i p_j - p_j p_i = 0, \quad q_i q_j - q_j q_i = 0.$$

Plays a fundamental role in QFT and the theory of affine Lie algebras. It is isomorphic to the Weyl algebra.

Our goal: Categorify $\mathfrak{h}$.

The Hecke algebra (of type A)

Fix $q \in \mathbb{C}^*$.

Definition (Hecke algebra H_n)

The $\mathbb{C}$-algebra generated by $t_1, \dots t_{n-1}$ with relations

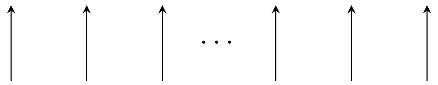
1. $t_i^2 = q + (q-1)t_i$,
2. $t_i t_j = t_j t_i \quad \text{if} \quad |j-i| > 1$,
3. $t_{i+1} t_i t_{i+1} = t_i t_{i+1} t_i$. (braid relation)

Remarks

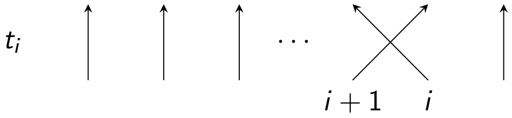
- Can work over more arbitrary fields (we choose $\mathbb{C}$ in this talk for simplicity).
- H_n specializes to $\mathbb{C}[S_n]$ when $q = 1$.
- t_i is invertible with inverse $t_i^{-1} = q^{-1} t_i + (q^{-1} - 1)$.

Hecke algebra H_n: graphical notation

Consider n (upward pointing) strands.



Generator t_i is a crossing of strands i and $i+1$ (numbered from the right).

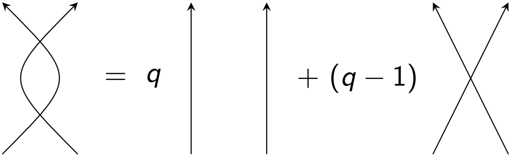


The Hecke algebra is then

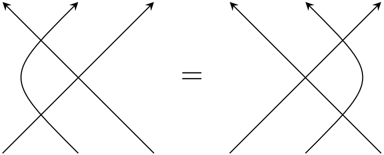
- the vector space spanned by all diagrams consisting of n upward pointing strands with crossings,
- with no maxima/minima (with respect to vertical coordinate) – i.e. strands keep "heading up",
- up to isotopy preserving the boundary (the endpoints of the strands),
- modulo the relations defining the Hecke algebra...

Hecke algebra H_n: graphical relations

1. $t_i^2 = q + (q-1)t_i$



2. $t_i t_j = t_j t_i$ if $|j-i| > 1$ becomes "distant crossings commute" (isotopy).
3. $t_{i+1}t_i t_{i+1} = t_i t_{i+1} t_i$ (braid relation)



Multiplication in the Hecke algebra is given by vertical composition (we read from bottom to top).

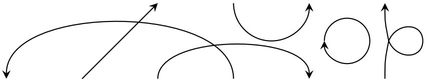
A graphical category $\mathcal{H}'$

Motivated by our graphical interpretation of the Hecke algebra, we define a category $\mathcal{H}'$ as follows.

- $\mathcal{H}'$ is a strict monodial category with two generating objects: $\uparrow$, $\downarrow$. In other words, objects of $\mathcal{H}'$ are sequences (tensor products) of $\uparrow$'s and $\downarrow$'s.

$$\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow, \quad \downarrow\uparrow\downarrow, \quad \varnothing, \quad \text{etc.}$$

- Morphisms of $\mathcal{H}'$ between two objects are $\mathbb{C}$-linear combinations of planar diagrams, agreeing with the two sequences at the boundary, modulo isotopy preserving the boundary



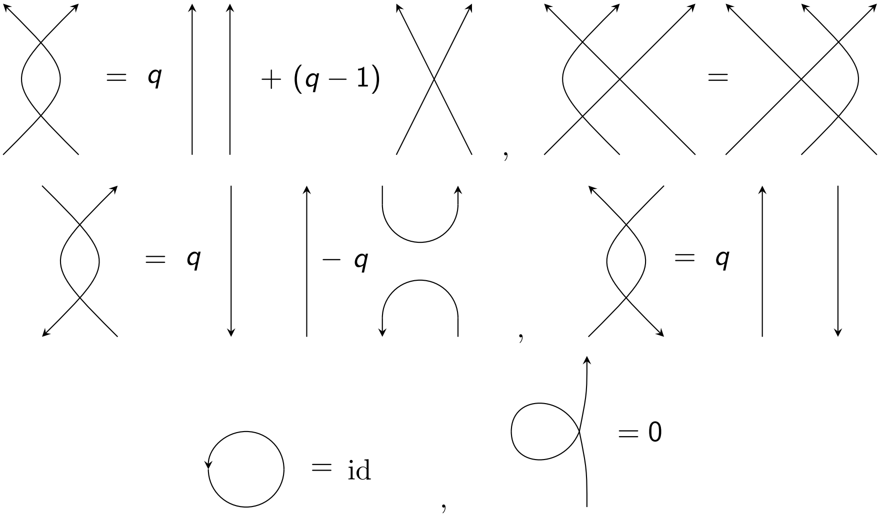
$$: \downarrow\uparrow\uparrow\uparrow\downarrow\uparrow \longrightarrow \uparrow\downarrow\uparrow\uparrow$$

modulo some local relations...

- Composition of morphisms is given by vertical composition of diagrams.

A graphical category $\mathcal{H}'$

The local relations on the space of morphisms are:



A graphical category $\mathcal{H}'$

Remarks

1. The relations for upward pointing strands are simply the Hecke algebra relations.
2. Other relations are motivated by certain functors on the category of modules over Hecke algebras (as we shall see).
3. At $q = 1$, the category $\mathcal{H}'$ reduces to a category studied by Khovanov. Our construction is motivated by his.

Properties of our category

1. Acts on the category of modules over Hecke algebras.
2. Acts on the category of modules over $GL_n(\mathbb{F}_q)$.
3. (Conjecturally) yields a categorification of the Heisenberg algebra.
4. (A subalgebra of) the affine Hecke algebra appears naturally.

Notation for bimodules over Hecke algebras

We view H_{n-1} as a subalgebra of H_n via the map $t_i \mapsto t_i$, $1 \leq i \leq n-2$.

- (n) denotes H_n as an (H_n, H_n)-bimodule.
- $(n)_{n-1}$ denotes H_n as an (H_n, H_{n-1})-bimodule.
- ${}_{n-1}(n)$ denotes H_n as an (H_{n-1}, H_n)-bimodule.
- ${}_{n-1}(n)_{n-1}$ denotes H_n as an (H_{n-1}, H_{n-1})-bimodule.

Tensor product of bimodules is denoted by juxtaposition. E.g.

$$(n)_{n-1}(n-1)_{n-2}(n-2)_{n-2}(n-1) := H_n \otimes_{H_{n-1}} H_{n-1} \otimes_{H_{n-2}} H_{n-2} \otimes_{H_{n-2}} H_{n-1}.$$

Bimodules over Hecke algebras

For $k, \ell \geq 0$, let ${}_\ell\mathrm{Bimod}_k$ be the category of (H_ℓ, H_k)-bimodules.

Let $\mathrm{Bimod}_k = \bigoplus_{\ell \geq 0} {}_\ell\mathrm{Bimod}_k$.

Remarks

- The functor of tensoring on the left with ${}_{n-1}(n)$ is restriction from the category of H_n-modules to the category of H_{n-1}-modules.
- The functor of tensoring on the left with $(n)_{n-1}$ is induction from the category of H_{n-1}-modules to the category of H_n-modules.
- Similarly, bimodules such as $(n)_{n-1}(n-1)_{n-2}(n-2)_{n-3}(n-2)$ correspond to compositions of induction and restriction functors.
- Under this correspondence, morphisms of bimodules are natural transformations of functors.

Action of $\mathcal{H}'$ on modules over Hecke algebras

Fix $k \geq 0$. We define a functor from our graphical category to Bimod_k.

Functor on objects

$\uparrow$ is sent to the induction bimodule.

$\downarrow$ is sent to the restriction bimodule.

Action of the functor is easiest to see if we label the regions between $\uparrow$'s and $\downarrow$'s by

- labeling the rightmost region by k,
- increasing labels by one as we move left across $\uparrow$, and
- decreasing labels by one as we move left across $\downarrow$.

Example

$$\uparrow\downarrow\downarrow\uparrow\uparrow \quad \rightsquigarrow \quad k+1 \uparrow k \downarrow k+1 \downarrow k+2 \uparrow k+1 \uparrow k$$

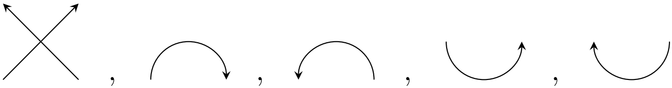
So our functor maps $\uparrow\downarrow\downarrow\uparrow\uparrow$ to the bimodule

$$(k+1)_k (k+1)_{k+1} (k+2)_{k+2} (k+2)_{k+1} (k+1)_k.$$

Action of $\mathcal{H}'$ on modules over Hecke algebras

It remains to define our functor on morphisms.

Our planar diagrams are built from compositions of crossings, cups, and caps:

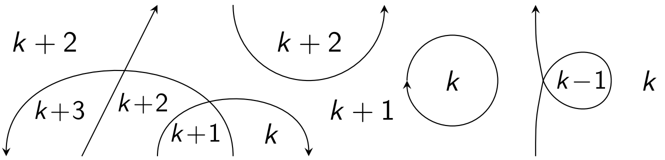


So we define our functor on these building blocks.

Action of $\mathcal{H}'$ on modules over Hecke algebras

Give a planar diagram (a morphism in $\mathcal{H}'$), label the regions by

- labeling the rightmost region by k,
- increasing labels by one as we move left across $\uparrow$, and
- decreasing labels by one as we move left across $\downarrow$.



We then assign a bimodule morphism to each of our building blocks.

Action of our functor on generating morphisms

First note that

$$ {}_n(n+1)_n \cong (n) \oplus (n)_{n-1}(n) $$

as (H_n, H_n)-bimodules.

We associate to our diagram building blocks, the following bimodule maps:

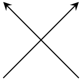

n : $(n)_{n-1}(n) \to (n), \quad x \otimes y \mapsto xy$ (multipication)


n : $(n) \hookrightarrow {}_n(n+1)_n,$


n : ${}_n(n+1)_n \twoheadrightarrow (n),$


n : $(n) \to (n)_{n-1}(n),$

$$ 1 \mapsto \sum_{i=1}^{n} q^{i-n} t_i \dots t_{n-2} t_{n-1} \otimes t_{n-1} t_{n-2} \dots t_i, $$


n : $(n+2)_n \to (n+2)_n, \quad z \mapsto z t_{n+1}.$

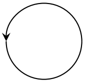
Action of $\mathcal{H}'$ on modules over Hecke algebras

Question: Have we really defined a functor from $\mathcal{H}'$ to the category of bimodules?

One needs to check:

1. the bimodule map associated to a diagram is invariant under isotopy preserving the boundary, and
2. the defining relations of the graphical category are satisfied.

Checking the defining relations is just a matter of computation. E.g.



is the composition

$$(n) \hookrightarrow {}_n(n+1)_n \cong (n) \oplus {}_{n-1}(n)_{n-1} \twoheadrightarrow (n)$$

which is the identity.

Isotopy: cups and caps

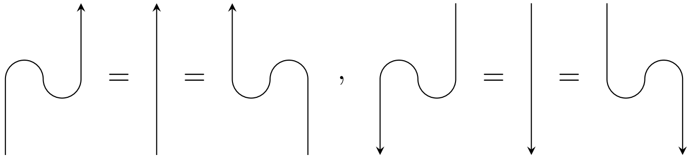
The cups and caps

correspond to natural transformations

$$\text{induction} \circ \text{restriction} \Rightarrow \mathrm{id}, \quad \text{restriction} \circ \text{induction} \Rightarrow \mathrm{id},$$
$$\mathrm{id} \Rightarrow \text{restriction} \circ \text{induction}, \quad \mathrm{id} \Rightarrow \text{induction} \circ \text{restriction}.$$

These natural transformations give adjunction data implying that induction and restriction are biadjoint.

In particular, this means that they satisfy the following relations:



Isotopy

Furthermore, one can check that the adjunction data is cyclic.

This implies that any two isotopic diagrams involving crossings, cups, and caps correspond to the same map of bimodules.

Therefore, our functor is well-defined.

So

- our category maps (via our functor) to the category of bimodules for the Hecke algebras, and
- the category of bimodules acts on the category of modules for the Hecke algebras (via tensoring on the left).

Thus, our category acts on the category of modules of the Hecke algebras.

Recall: The Heisenberg algebra

Definition

The rank one Heisenberg algebra has generators p, q and relation

$$pq - qp = 1$$

Definition

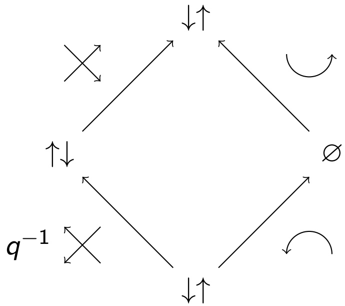
The (infinite rank) Heisenberg algebra $\mathfrak{h}$ is the algebra with

- generators p_i, q_i, $i = 1, 2, 3, \dots$, and
- relations

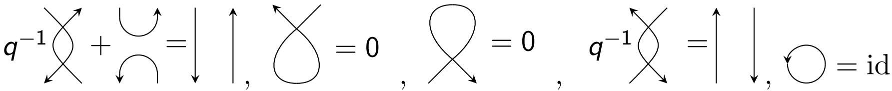
$$p_i q_j - q_j p_i = \delta_{i,j} 1, \quad p_i p_j - p_j p_i = 0, \quad q_i q_j - q_j q_i = 0.$$

Our graphical category bears a close relation to the Heisenberg algebra.

A categorification of the Heisenberg algebra relation



It follows from our defining relations that



Thus

$$\downarrow\uparrow \;\cong\; \uparrow\downarrow \;\oplus\; \varnothing.$$

A categorification of the key Heisenberg algebra relation

$$\downarrow\uparrow \quad \cong \quad \uparrow\downarrow \quad \oplus \quad \varnothing.$$

In our action on modules of the Hecke algebra, this corresponds to the fact that

$${}_n(n+1)_n \cong (n)_{n-1}(n) \oplus (n).$$

In the language of functors, it corresponds to

$$\text{restriction} \circ \text{induction} \cong \text{induction} \circ \text{restriction} \oplus \mathrm{id}.$$

Passing to the (split) Grothendieck group of our graphical category, we obtain the Heisenberg relation

$$[\downarrow]\ [\uparrow] = [\uparrow]\ [\downarrow] + 1.$$

or

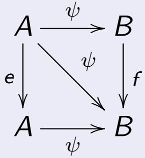
$$pq - qp = 1, \qquad \text{where} \quad p = [\downarrow], \quad q = [\uparrow].$$

A categorification of the Heisenberg algebra

Definition: Karoubi envelope

The Karoubi envelope of a category $\mathcal{C}$ is the category whose

- objects are pairs (A, e) where $A \in \operatorname{Ob}\mathcal{C}$ and $e \in \operatorname{Mor}\mathcal{C}$ is an idempotent ($e^2 = e$), and
- morphisms from (A, e) to (B, f) are $\psi \in \operatorname{Mor}\mathcal{C}$ such that the following diagram commutes:



Intuition: One thinks of passing to the Karoubi envelope as adding in objects such that the idempotents correspond to projections onto direct summands.

A categorification of the Heisenberg algebra

Let $\mathcal{H}$ be the Karoubi envelope of our graphical category $\mathcal{H}'$.

Theorem (Licata-S.)

There is a natural homomorphism

$$\mathfrak{h} \to \text{(split) Grothendieck group of } \mathcal{H}.$$

When q is generic (i.e. not equal to a root of unity), this map is injective.

Remarks

- We expect that the map is also surjective and hence that (the Karoubi envelope of) our graphical category is a categorification of the Heisenberg algebra.
- This is a q-deformed version of a result of Khovanov (which inspired the current work).

A categorification of bosonic Fock space

Bosonic Fock space is the representation of $\mathfrak{h}$ on

$$\mathbb{C}[x_1, x_2, \dots]$$

given by

$$p_i \mapsto \frac{\partial}{\partial x_i}, \qquad q_i \mapsto x_i.$$

The action of $\mathcal{H}$ on the category of modules for Hecke algebras is a categorification of this important representation.

The affine Hecke algebra

Definition (The affine Hecke algebra)

The affine Hecke algebra is the algebra

$$H_n^{\mathrm{aff}} = H_n \otimes_{\mathbb{C}} \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

Here H_n and $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ are subalgebras and

$$t_i x_k = x_k t_i, \quad i \neq k, k+1,$$
$$t_i x_i t_i = q x_{i+1}.$$

The affine Hecke algebra

Define elements

$$y_i = (q-1)x_i - \frac{q}{q-1}, \quad i = 1, \dots, n.$$

Then the y_i, t_i satisfy

$$\begin{aligned} y_i t_j &= t_j y_i, \quad i \neq j, j+1, \\ t_i y_{i+1} &= y_i t_i + (q-1)y_{i+1} + q, \\ y_{i+1} t_i &= t_i y_i + (q-1)y_{i+1} + q. \end{aligned}$$

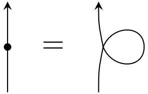
When $q = 1$, these are the relations of the degenerate affine Hecke algebra.

Definition (A_n)

Let $A_n \subseteq H_n^{\mathrm{aff}}$ denote the subalgebra of H_n^{aff} generated by y_i, t_i, $i = 1, \dots, n$.

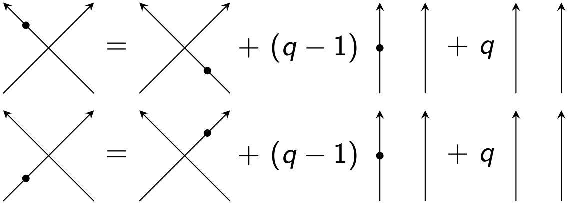
The affine Hecke algebra

Let a dot denote a right curl:



In our action on modules of Hecke algebras, right curls (or dots) correspond to multiplication by q-deformed Jucys-Murphy elements (i.e. Hecke algebra analogues of Jucys-Murphy elements in the group algebra of the symmetric group).

One can check from our defining relations that we have the following equalities:



The affine Hecke algebra

Algebraically, if we let y_i denote a dot on the ith strand, we have

$$y_i t_j = t_j y_i, \quad i \neq j, j+1,$$
$$t_i y_{i+1} = y_i t_i + (q-1) y_{i+1} + q,$$
$$y_{i+1} t_i = t_i y_i + (q-1) y_{i+1} + q.$$

Thus, we have a homomorphism from A_n to the algebra of endomorphisms of

$$\uparrow\uparrow \cdots \uparrow.$$

So the affine Hecke algebra appears naturally in our category.

Remarks

We can give a precise description of the entire algebra of endomorphisms of the above object. It is $A_n \otimes_{\mathbb{C}} \mathbb{C}[c_0, c_1, \dots]$, where c_k denotes a clockwise circle with k dots.

Additional remarks

1. On the level of Grothendieck groups, the fact that induction and restriction on modules of the symmetric group satisfy the Heisenberg relation is well known. The current work is a categorification of this structure.

2. Our categorification is actually more natural in the language of 2-categories (which we chose to avoid for the purposes of this talk).

3. Our category also acts on the category of modules over $GL_n(\mathbb{F}_q)$ via parabolic induction and restriction functors. This gives another categorification of bosonic Fock space. In fact, the two are equivalent.

Summary

- Inspired by the Hecke algebra, we defined a graphical category $\mathcal{H}'$.
- The Karoubi envelope $\mathcal{H}$ is (conjecturally) a categorification of the Heisenberg algebra in infinitely many variables.
- This category acts on modules for Hecke algebras and finite general linear groups, yielding two categorifications of bosonic Fock space.
- Jucys-Murphy elements and a subalgebra of the affine Hecke algebra (a q-deformed degenerate affine Hecke algebra) appear naturally from our graphical calculus.