Irreducible finite-dimensional representations of equivariant map algebras

Alistair Savage

University of Ottawa

May 12, 2010

Joint work with Erhard Neher and Prasad Senesi

Slides: www.mathstat.uottawa.ca/~asavag2

Full details: arXiv:0906.5189
Outline

Goal: Classify the irreducible finite-dimensional representations of a certain class of Lie algebras.

Overview:

1. Equivariant map algebras
2. Examples
3. Evaluation representations
4. Classification theorem
5. Applications
   - recover some known classifications (often in a simplified manner)
   - produce some new classifications

Terminology:

small = irreducible finite-dimensional
(Untwisted) Map algebras

Notation

- \(k\) - algebraically closed field of characteristic zero
- \(X\) - scheme (or algebraic variety) over \(k\)
- \(A = A_X = \mathcal{O}_X(X)\) - coordinate ring of \(X\)
- \(\mathfrak{g}\) - finite-dimensional Lie algebra over \(k\)

Definition (Untwisted map algebra)

\(M(X, \mathfrak{g}) = \text{Lie algebra of regular maps from } X \text{ to } \mathfrak{g}\)

Pointwise multiplication:

\[
[\alpha, \beta]_{M(X, \mathfrak{g})}(x) = [\alpha(x), \beta(x)]_{\mathfrak{g}} \text{ for } \alpha, \beta \in M(X, \mathfrak{g})
\]

Note: \(M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes A_X\)
Examples

Discrete spaces

If $X$ is a discrete variety, then

$$M(X, g) \cong \prod_{x \in X} g, \quad \alpha \mapsto (\alpha(x))_{x \in X}, \quad \alpha \in M(X, g).$$

In particular, if $X = \{x\}$ is a point, then

$$M(X, g) \cong g, \quad \alpha \mapsto (\alpha(x)), \quad \alpha \in M(X, g).$$

The isomorphisms are given by evaluation.

Current algebras

$$X = k^n \implies A_X = k[t_1, \ldots, t_n]$$

Thus, $M(X, g) \cong g \otimes k[t_1, \ldots, t_n]$ is a current algebra.
Untwisted multiloop algebras

\[ X = (k^\times)^n \implies A_X = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \]

Thus,

\[ M(X, g) \cong g \otimes k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \]

is the untwisted multiloop algebra.

If \( n = 1 \), this is called the untwisted loop algebra and plays an important role in the theory of (untwisted) affine Lie algebras.
Examples

Three point algebras

\[ X = k \setminus \{0, 1\} = \mathbb{P}^1 \setminus \{0, 1, \infty\} \]
\[ \implies A_X \cong k[t, t^{-1}, (t - 1)^{-1}] \]

Thus,
\[ M(X, \mathfrak{sl}_2) \cong \mathfrak{sl}_2 \otimes k[t, t^{-1}, (t - 1)^{-1}] \]
is the three point \(\mathfrak{sl}_2\) loop algebra.

Remarks

- Removing any 2 points from \(k\) results in an isomorphic map algebra.
- \(M(X, \mathfrak{sl}_2)\) is isomorphic to the tetrahedron Lie algebra and to a direct sum of 3 copies of the Onsager algebra (Hartwig-Terwilliger 2007).
Equivariant map algebras

- $\Gamma$ - finite group
- Suppose $\Gamma$ acts on $X$ and $g$ by automorphisms

**Definition (equivariant map algebra)**

The **equivariant map algebra** is the Lie algebra of $\Gamma$-equivariant maps from $X$ to $g$:

$$M(X, g)^\Gamma = \{ \alpha \in M(X, g) : \alpha(g \cdot x) = g \cdot \alpha(x) \ \forall \ x \in X, \ g \in \Gamma \}$$

**Note:** If $X$ is any scheme, then $M(X, g)^\Gamma \cong M(X_{\text{aff}}, g)^\Gamma$ where $X_{\text{aff}} = \text{Spec} \ A_X$ is the affine scheme with the same coordinate ring as $X$. So we often assume $X$ is affine.
Equivariant map algebras – algebraic description

- Induced action on $A_X$ given by
  \[(g \cdot f)(x) = f(g^{-1} \cdot x), \quad f \in A_X, \quad x \in X, \quad g \in \Gamma\]

- $\Gamma$ acts diagonally on $g \otimes A_X$:
  \[g \cdot (u \otimes f) = (g \cdot u) \otimes (g \cdot f)\]

- Then
  \[M(X, g)^\Gamma \cong (g \otimes A_X)^\Gamma\]
Example: Trivial $\Gamma$-action on $g$

If $\Gamma$ acts trivially on $g$, then

$$M(X, g)_\Gamma \cong M(X/\!/\Gamma, g) \cong g \otimes A^\Gamma_X$$

where $X/\!/\Gamma = \text{Spec } A^\Gamma_X$ is the quotient of $X$ by $\Gamma$.

Thus $M(X, g)_\Gamma$ is isomorphic to an untwisted map algebra.
Example: multiloop algebras

\[ \Gamma = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}, \quad X = (k^\times)^n \]

- For \( i = 1, \ldots, n \), let \( \xi_i \) be a primitive \( m_i \)-th root of unity.
- Define action of \( \Gamma \) on \( X \) by
  \[
  (a_1, \ldots, a_n) \cdot (z_1, \ldots, z_n) = (\xi_1^{a_1} z_1, \ldots, \xi_n^{a_n} z_n)
  \]
- Define action of \( \Gamma \) on \( \mathfrak{g} \) by specifying commuting automorphisms \( \sigma_i \), \( i = 1, \ldots, n \), such that \( \sigma_i^{m_i} = 1 \).

Then \( M(X, \mathfrak{g})^\Gamma \) is the (twisted) multiloop algebra.

If \( n = 1 \), this is the (twisted) loop algebra.

Affine Lie algebras

The affine Lie algebras can be constructed as central extensions of loop algebras plus a differential:

\[ \hat{\mathfrak{g}} = M(X, \mathfrak{g})^\Gamma \oplus kc \oplus kd \quad (n = 1) \]
Example: generalized Onsager algebra

\[ \Gamma = \mathbb{Z}_2 = \{1, \sigma\}, \quad X = k^\times, \quad g = \text{simple Lie algebra} \]

- \( \Gamma \) acts on \( X \) by \( \sigma \cdot x = x^{-1} \)
- \( \Gamma \) acts on \( g \) by any involution

When \( \Gamma \) acts on \( g \) by the Chevalley involution, we write

\[ \Theta(g) = M(X, g)^\Gamma \]

Remarks

- If \( k = \mathbb{C} \), \( \Theta(\mathfrak{sl}_2) \) is isomorphic to the Onsager algebra (Roan 1991)
  - Key ingredient in Onsager’s original solution of the 2D Ising model
- For \( k = \mathbb{C} \), \( \Theta(\mathfrak{sl}_n) \) was studied by Uglov and Ivanov (1996)
Evaluation

If \( x = \{x_1, \ldots, x_n\} \subseteq X \), we have the evaluation map

\[
ev_x : M(X, g)^\Gamma \to g^{\oplus n}, \quad \alpha \mapsto (\alpha(x_i))_i
\]

**Important:** This map is not surjective in general!

For \( x \in X \), define

\[
\Gamma_x = \{ g \in \Gamma : g \cdot x = x \}
\]
\[
g^x = \{ u \in g : \Gamma_x \cdot u = u \}
\]

**Lemma**

*For \( X \) affine, \( x = \{x_1, \ldots, x_n\} \subseteq X \), \( x_i \notin \Gamma \cdot x_j \) for \( i \neq j \),

\[
\text{im} \ ev_x = \bigoplus_i g^{x_i}.
\]
Evaluation representations

Given

- \( x = \{x_1, \ldots, x_n\} \subseteq X \), and
- representations \( \rho_i : g^{x_i} \to \text{End}_k V_i, \ i = 1, \ldots, n \),

we define the (twisted) evaluation representation as the composition

\[
M(X, g) \Gamma \xrightarrow{\text{ev}_x} \bigoplus_i g^{x_i} \xrightarrow{\otimes_i \rho_i} \text{End}_k (\otimes_i V_i).
\]
Important remarks

This notion of evaluation representation differs from the classical definition.

- Some authors use the term *evaluation representation* only for the case when evaluation is at a single point and call the general case a tensor product of evaluation representations.
- To a point $x \in X$, we associate a representation of $g^x$ instead of $g$. If $\Gamma$ acts freely, this coincides with the usual definition.
- Recall that (when $g^x \not\subseteq g$) not all reps of $g^x$ extend to reps of $g$ – so the new definition is more general.
- We do not require the representations $\rho_i$ to be faithful.

We will see that the more general definition allows for a more uniform classification of representations.
Evaluation representations

\[ \mathcal{R}_x = \{ \text{isomorphism classes of small reps of } g^x \} \]

\[ \mathcal{R}_X = \bigsqcup_{x \in X} \mathcal{R}_x \]

Since \( \Gamma_{g \cdot x} = g \Gamma_x g^{-1} \), we have

\[ g \cdot g^x = g^{g \cdot x}. \]

We have an action of \( \Gamma \) on \( \mathcal{R}_X \): if \([\rho] \in \mathcal{R}_x\), then

\[ g \cdot [\rho] = [\rho \circ g^{-1}] \in \mathcal{R}_{g \cdot x}. \]

---

**Definition (\( \mathcal{F} \))**

\( \mathcal{F} \) is set of all \( \Psi : X \to \mathcal{R}_X \) such that

1. \( \Psi \) is \( \Gamma \)-equivariant,
2. \( \Psi(x) \in \mathcal{R}_x \) for all \( x \in X \), and
3. \( \text{supp } \Psi = \{ x \in X : \Psi(x) \neq 0 \} \) is finite.
Evaluation representations

We think of $\Psi \in \mathcal{F}$ as assigning a finite number of (isom classes of) reps of $g^x$ to points $x \in X$ in a $\Gamma$-equivariant way.
Evaluation representations

For each \( \Psi \in \mathcal{F} \), define

\[
ev \Psi = ev_x(\Psi(x_i))_{i=1}^n = ev_{x_1} \Psi(x_1) \otimes \cdots \otimes ev_{x_n} \Psi(x_n)
\]

where \( x = (x_1, \ldots, x_n) \) is an \( n \)-tuple of points of \( X \) containing one point from each \( \Gamma \)-orbit in \( \text{supp} \Psi \) (the isom class is independent of this choice).

For \( \Psi \in \mathcal{F} \), \( \ev \Psi \) is the isomorphism class of a small representation of \( M(X, g)^\Gamma \).

**Proposition**

The map

\[
\mathcal{F} \longrightarrow \{ \text{isom classes of small reps of } M(X, g)^\Gamma \}, \quad \Psi \mapsto \ev \Psi
\]

is injective.
One-dimensional representations

Recall: Any 1-dimensional rep of a Lie algebra $L$ corresponds to a linear map $\lambda : L \to k$ such that $\lambda([L, L]) = 0$.

We identify such 1-dimensional reps with elements

$$\lambda \in (L/[L, L])^*.$$ 

Two 1-dimensional reps are isomorphic if and only if they are equal as elements of $(L/[L, L])^*$. 
Classification Theorem

**Theorem (Neher-S.-Senesi 2009)**

Suppose $\Gamma$ is a finite group acting on an affine scheme (or variety) $X$ and a finite-dimensional Lie algebra $\mathfrak{g}$. Let $M = M(X, \mathfrak{g})^\Gamma$.

Then the map

$$(\lambda, \Psi) \mapsto \lambda \otimes \text{ev}_\Psi, \quad \lambda \in (M/[M,M])^*, \quad \Psi \in \mathcal{F}$$

gives a surjection

$$(M/[M,M])^* \times \mathcal{F} \rightarrow \{\text{isom classes of small representations of } M\}.$$

In particular, all small representations are of the form

$$(1\text{-dim rep}) \otimes (\text{evaluation rep}).$$
\[(\lambda, \Psi) \mapsto \lambda \otimes \text{ev}_\Psi, \quad \lambda \in (\mathcal{M}/[\mathcal{M}, \mathcal{M}])^*, \quad \Psi \in \mathcal{F}\]

1. This map is not injective in general since we can have nontrivial evaluation reps which are 1-dimensional. This happens when $g^\times$ is not perfect (e.g. reductive but not semisimple).

Example: $\mathfrak{g} = \mathfrak{sl}_2$, $\Gamma = \mathbb{Z}_2$, $X = k = \mathbb{C}$

- $\Gamma$ acts on $\mathfrak{g}$ by the Chevalley involution.
- $\Gamma$ acts on $X$ by multiplication by $-1$.
- Then $\mathfrak{g}^0 = \mathfrak{g}^\Gamma$ is one-dimensional and so has nontrivial 1-dim reps.

2. However, we can specify precisely when $\lambda \otimes \text{ev}_\Psi \cong \lambda' \otimes \text{ev}_{\Psi'}$.

3. The restriction of the map to either factor is injective.
Classification

\[(\lambda, \Psi) \mapsto \lambda \otimes \text{ev}_\Psi, \quad \lambda \in (\mathcal{M}/[\mathcal{M}, \mathcal{M}])^*, \quad \Psi \in \mathcal{F}\]

Corollary

1. If \(\mathcal{M}\) is perfect (i.e. \(\mathcal{M} = [\mathcal{M}, \mathcal{M}]\)), then we have a bijection

\[\mathcal{F} \leftrightarrow \{\text{isom classes of small reps}\}, \quad \Psi \mapsto \text{ev}_\Psi.\]

In particular, all small reps are evaluation reps.

2. If \([g^\Gamma, g] = g\), then \(\mathcal{M}\) is perfect and the above bijection holds.

3. If \(\Gamma\) acts on \(g\) by diagram automorphisms, then \([g^\Gamma, g] = g\) and the above bijection holds.

Note: Being perfect is not a necessary condition for the all small reps to be evaluation reps (as we will see).
If $\Gamma$ is trivial, then

$$M(X, g)^\Gamma = M(X, g), \quad g^\Gamma = g.$$  

Thus, if $g$ is perfect,

$$[g^\Gamma, g] = [g, g] = g$$

and so all small reps are evaluation reps.
Application: multiloop algebras

**Corollary**

*If $\mathcal{M}$ is a (twisted) multiloop algebra, then $\mathcal{M}$ is perfect and so we have a bijection

$$\mathcal{F} \leftrightarrow \{\text{isom classes of small reps}\}, \quad \Psi \mapsto \text{ev}_\Psi.$$*

*In particular, all small reps are evaluation reps.*

**Remarks**

1. This recovers results of Chari-Pressley (for loop algebras) and Batra, Lau (multiloop algebras), but with a different description.

2. The description given above (in terms of $\mathcal{F}$) gives a simple and uniform description of the somewhat technical conditions appearing in previous classifications.

3. Action of $\Gamma$ on $X$ is free and so $g^x = g$ for all $x \in X$. So the more general notion of evaluation rep does not play a role.
Application: generalized Onsager algebra

\[ \Gamma = \mathbb{Z}_2 = \{1, \sigma\}, \quad X = k^\times, \quad \mathfrak{g} = \text{simple Lie algebra} \]

- \( \Gamma \) acts on \( X \) by \( \sigma \cdot x = x^{-1} \)
- \( \Gamma \) acts on \( \mathfrak{g} \) by any involution

**Corollary**

*With \( \Gamma, X, \mathfrak{g} \) as above, we have a bijection*

\[ \mathcal{F} \leftrightarrow \{\text{isom classes of small reps}\}, \quad \Psi \mapsto \text{ev}_\Psi. \]

*In particular, all small reps are evaluation reps.*
Remarks – generalized Onsager algebra

- There are two types of points of $X$:
  - $x \in \{\pm 1\} \implies \Gamma_x = \Gamma = \mathbb{Z}_2$, $g^x = g^\Gamma$
  - $x \not\in \{\pm 1\} \implies \Gamma_x = \{1\}$, $g^x = g$

- $g^\Gamma$ can be semisimple or have a one-dimensional center

When $g^\Gamma$ has a one-dimensional center:
- the generalized Onsager algebra is not perfect
- we can place (nontrivial) one-dim reps of $g^\Gamma$ at the points $\pm 1$
- under our more general definition of evaluation rep, all small reps are evaluation reps
- under classical notion of evaluation rep, there are small reps which are not evaluation reps

**Moral:** The more general definition of evaluation rep allows for a more uniform classification.
Special case: Onsager algebra

- When $k = \mathbb{C}$ and $\Gamma$ acts on $g = \mathfrak{sl}_2$ by the Chevalley involution, then

$$\mathcal{O}(\mathfrak{sl}_2) \overset{\text{def}}{=} \mathcal{M}(X, \mathfrak{sl}_2)^\Gamma$$

is the Onsager algebra.

- $g^{\{\pm 1\}}$ is one-dimensional abelian and $\mathcal{O}(\mathfrak{sl}_2)$ is not perfect.

- Small reps of $\mathcal{O}(\mathfrak{sl}_2)$ were classified previously (Date-Roan 2000)
  - classical definition of evaluation rep was used
  - not all small reps were evaluation reps
  - this necessitated the introduction of the type of a representation

**Note:** For the other cases, the classification seems to be new.
Application: a nonabelian example

\[ \Gamma = S_3, \quad X = \mathbb{P}^1 \setminus \{0, 1, \infty\}, \quad g = \mathfrak{so}_8 \text{ (type } D_4) \]

- symmetry group of Dynkin diagram of \( g \) is \( S_3 \)
- so \( \Gamma \) acts on \( g \) by diagram automorphisms
- for any permutation of the set \( \{0, 1, \infty\} \), \( \exists! \) Möbius transformation of \( \mathbb{P}^1 \) inducing that permutation
- so \( \Gamma \) acts naturally on \( X \)

Thus we can form the equivariant map algebra \( M(X, g)_{\Gamma} \) and show that it is perfect.

Our classification tells us all small reps are eval reps and gives a bijection between these and the set \( \mathcal{F} \).
Application: a nonabelian example

It is straightforward to find the points with nontrivial stabilizer:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\Gamma_x$</th>
<th>Type of $g^x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>${\text{Id, } (0 \infty)} \cong \mathbb{Z}_2$</td>
<td>$B_3$</td>
</tr>
<tr>
<td>$2$</td>
<td>${\text{Id, } (1 \infty)} \cong \mathbb{Z}_2$</td>
<td>$B_3$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>${\text{Id, } (0 1)} \cong \mathbb{Z}_2$</td>
<td>$B_3$</td>
</tr>
<tr>
<td>$e^{\pm \pi i/3}$</td>
<td>${\text{Id, } (0 1 \infty), (0 \infty 1)} \cong \mathbb{Z}_3$</td>
<td>$G_2$</td>
</tr>
</tbody>
</table>

The sets

$$\left\{-1, 2, \frac{1}{2}\right\} \quad \text{and} \quad \left\{e^{\pi i/3}, e^{-\pi i/3}\right\}$$

are $\Gamma$-orbits.

So elements of $\mathcal{F}$ can assign

- irreps of type $B_3$ to the 3-element orbit,
- irreps of type $G_2$ to the 2-element orbit,
- irreps of type $D_4$ to the other points (6-element orbits).
Application: a nonabelian example

\[ B_3 \quad \infty \quad 0 \quad 1 \quad \frac{1}{2} \quad 2 \quad G_2 \quad e^{-\pi i/3} \quad e^{\pi i/3} \quad \frac{1}{2} \quad B_3 \quad D_4 \]
When are all small representations evaluation representations?

We have seen

- perfect (i.e. $\mathcal{M} = [\mathcal{M}, \mathcal{M}]$) $\Rightarrow$ all small reps are eval reps
- the converse is not true (e.g. the Onsager algebra)

Reduction

Since all reps are of the form

$$(1\text{-dim rep}) \otimes (\text{eval rep})$$

it suffices to know when there are one-dimensional reps that are not evaluation reps.
When are all small reps eval reps?

**Definition**

\[ \tilde{X} = \{ x \in X \mid g_x \neq [g_x, g_x] \} \]

**Recall**

\[ g_x = [g_x, g_x] \iff \text{all one-dimensional reps of } g_x \text{ are trivial} \]

Thus, \( \tilde{X} \) is precisely the set of points where we can place nontrivial one-dimensional evaluation representations.

**Proposition (Neher-S-Senesi 2009)**

If \( X \) is a Noetherian scheme (i.e. \( A \) is finitely generated) and \( |\tilde{X}| = \infty \), then \( M(X, g)^\Gamma \) has one-dimensional representations that are not evaluation representations.
When are all small reps eval reps?

Example

\[ \Gamma = \mathbb{Z}_2 = \{1, \sigma\}, \quad X = k^2, \quad g = \mathfrak{sl}_2(k) \]

- \( \sigma \cdot (x_1, x_2) = (x_1, -x_2), \quad (x_1, x_2) \in k^2 \)
- \( \sigma \) acts as Chevalley involution on \( g \)

Then

- \( x = (x_1, x_2) \in k^2, \quad x_2 \neq 0 \implies \Gamma_x = \{1\} \implies g^x = g \)
- \( x = (x_1, 0) \in k^2 \implies \Gamma_x = \Gamma \implies g^x = \mathbb{C} \) (abelian)

Thus

\[ \tilde{X} = \{(x_1, 0) \mid x_1 \in k\} \quad \text{and so} \quad |\tilde{X}| = \infty. \]

Therefore \( M(X, g)^\Gamma \) has one-dimensional reps that are not eval reps.
When are all small reps eval reps?

Question

$|\tilde{X}| < \infty$ is a necessary condition for all small reps to be eval reps.

Is it sufficient?

Answer: NO
When are all small reps eval reps?

Let \( x \) be a finite subset of \( X \) and consider the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\text{ev}_x} & \bigoplus_{x \in x} \mathcal{G}^x \\
& & \downarrow \\
& & \bigoplus_{x \in x} \mathcal{G}^x / [\mathcal{G}^x, \mathcal{G}^x] \\
& & \downarrow \gamma \\
\mathcal{M} / [\mathcal{M}, \mathcal{M}] & \leftarrow & \\
\end{array}
\]

**Theorem**

*If \(|\tilde{X}| < \infty\), then*

\[
(\lambda, \Psi) \mapsto \lambda \otimes \text{ev}_\Psi, \quad \lambda \in (\ker \gamma)^*, \quad \Psi \in \mathcal{F},
\]

*is a bijection*

\[
(\ker \gamma)^* \times \mathcal{F} \longleftrightarrow \{\text{isom classes of small reps}\}
\]
Corollary

If $|\tilde{X}| < \infty$, all small reps are eval reps if and only if $\ker \gamma = 0$. This is true if and only if

$$[M, M] = M^d := \{ \alpha \in M \mid \alpha(x) = [g^x, g^x] \ \forall \ x \in X \}.$$ 

Note: $[M, M] \subseteq M^d$ is always true.
Example: $|\tilde{X}| < \infty$ with small reps that are not eval reps

- $g = \mathfrak{sl}_2(k)$
- $X = Z(y^2 - x^3) = \{(x, y) \mid y^2 = x^3\} \subseteq k^2$
- So $A = k[y, x]/(y^2 - x^3)$
- $\Gamma = \mathbb{Z}_2 = \{1, \sigma\}$
- $\sigma \cdot (y, x) = (-y, x)$
- This action fixes $y^2 - x^3$ and so induces an action of $\Gamma$ on $X$.
- Only fixed point is the origin.
- Thus $\tilde{X} = \{0\}$ and so $|\tilde{X}| < \infty$

Then one can easily show that

$$M^d/[M, M] \cong xk[x]/(x^3) \neq 0$$

Thus $[M, M] \subsetneq M^d$ and so $M$ has small reps that are not eval reps.

Note: $X$ has a singularity at 0.
Further directions (work in progress)

- The category of finite-dimensional representations of an equivariant map algebra is not semisimple in general.

- Can one describe the finite-dimensional representations (not necessarily irreducible)?
  - (twisted) Weyl modules (untwisted case considered by Chari-Fourier-Khandai 2009)
  - block decompositions
    - untwisted loop (Chari-Moura)
    - twisted loop (Senesi)
    - untwisted map algebras (Kodera)

- Current work (with E. Neher): Describe extensions between small reps (and then block decompositions). Untwisted case done by Kodera.