Quivers and the Euclidean Group

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Euclidean group

**Definition (Euclidean group)**

Group of orientation-preserving isometries of $n$-dim Euclidean space:

$$E(n) = \mathbb{R}^n \rtimes SO(n)$$

Study (at least for $n = 2, 3$) predates even concept of group.

We will focus on $E(2)$ – much still unknown about rep theory.
Representations of the Euclidean group

- $E(2)$ solvable $\Rightarrow$ all finite-dim irreps are 1-dim

- finite-dim unitary reps (of interest in quantum mechanics) are completely reducible $\Rightarrow$ isom to direct sum of one-dim reps

- infinite-dim unitary reps have received considerable attention

- $\exists$ finite-dim nonunitary indecomp reps (not irreducible)
  - much less known about these
  - play important role in math physics and rep theory of Poincaré group
Poincaré Group

Group of isometries of Minkowski spacetime

\[ \text{Poincaré group} = \{\text{translations}\} \rtimes \text{Lorentz group} \]

The Little Group (Wigner 1939)

**Def:** maximal subgroup of Lorentz group leaving invariant the four-momentum of a particle

- governs internal space-time symmetries of particle
- **massive particles:** little group locally isom to \( O(3) \)
- **massless particles:** little group locally isom to \( E(2) \)
**Gravity**

Consider

- Chern-Simons formulation of Einstein gravity
- $2 + 1$ dimensions
- space-time with Euclidean signature
- vanishing cosmological constant

Then phase space of gravity is moduli space of flat $E(2)$-connections
Recall $E(2) = \mathbb{R}^2 \rtimes SO(2)$

The Euclidean Algebra

$\mathfrak{e}(2) = \text{complexification of Lie alg of } E(2)$

Has basis $\{p_+, p_-, l\}$ and relations

$$[p_+, p_-] = 0, \quad [l, p_{\pm}] = \pm p_{\pm}$$

Representation Theory

$SO(2)$ compact $\Rightarrow$ finite-dim $E(2)$-modules equiv to finite-dim $\mathfrak{e}(2)$-modules where $l$ acts semisimply with integer eigenvalues

We use term $\mathfrak{e}(2)$-module to mean such a module
Weight decompositions

$V$ an $\varepsilon(2)$-module

We have weight decomposition into $l$-eigenspaces

$$V = \bigoplus_{k \in \mathbb{Z}} V_k, \quad V_k = \{ v \in V \mid l \cdot v = kv \}$$

and

$$p_+ V_k \subseteq V_{k+1}, \quad p_- V_k \subseteq V_{k-1}$$

We define

$$\dim V = (\dim V_k)_{k \in \mathbb{Z}} \in (\mathbb{Z}_{\geq 0})^\mathbb{Z}$$
Modified enveloping algebra

\[ U = \text{universal enveloping algebra of } \mathfrak{e}(2) \]

\[ U^+, U^-, U^0 \text{ subalgebras generated by } p_+, p_-, l \]

Have triangular decomp \( U \cong U^+ \otimes U^0 \otimes U^- \)

Following idea of Lusztig, define \textit{modified enveloping algebra}

\[ \tilde{U} = U^+ \otimes \left( \bigoplus_{k \in \mathbb{Z}} \mathbb{C} a_k \right) \otimes U^- \]

with multiplication

\[ a_k a_l = \delta_{kl} a_k \]

\[ p_+ a_k = a_{k+1} p_+, \quad p_- a_k = a_{k-1} p_-, \]

\[ p_+ p_- a_k = p_- p_+ a_k \]
The Euclidean group

Euclidean algebra

Preprojective algebras

Quiver varieties

Conclusion

Representation theory

\[ \tilde{U} = U^+ \otimes \left( \bigoplus_{k \in \mathbb{Z}} \mathbb{C} a_k \right) \otimes U^- \]

\( a_k \sim \) projection to \( k \)th weight space

Definition

A \( \tilde{U} \)-module is unital if

1. \( \forall \, v \in V, \ a_k \, v = 0 \) for almost all \( k \in \mathbb{Z} \)
2. \( \forall \, v \in V, \ \sum_{k \in \mathbb{Z}} a_k \, v = v \)

\( \tilde{U} \)-module \( \sim \) \( U \)-module with weight decomp

Proposition

\[ \text{Mod} \ \tilde{U} \cong \text{Mod} \ U \cong \text{Mod} \ \varepsilon(2) \]
**Quivers**

A quiver is a directed graph.

\[ Q = (I, H) \]
- \( I \) is the vertex set
- \( H \) is the (directed) edge set

**Representations of quivers**

- \( I \)-graded vector space \( V = (V_i)_{i \in I} \)
- Linear map \( x_h : V_{\text{out}(h)} \to V_{\text{in}(h)} \) for each \( h \in H \)

\[ \text{rep}(Q, V) = \bigoplus_{h \in H} \text{Hom}_\mathbb{C}(V_{\text{out}(h)}, V_{\text{in}(h)}) \]
Quivers

The quiver $Q_{a,b}$

$l = \{ k \in \mathbb{Z} \mid a \leq k \leq b \}$

$H = \{ h_i \mid a \leq i \leq b - 1 \}$,  \( \text{out}(h_i) = i, \text{in}(h_i) = i + 1 \)

The quiver $Q_{\infty}$

$l = \mathbb{Z}$

$H = \{ h_i \mid i \in \mathbb{Z} \}$,  \( \text{out}(h_i) = i, \text{in}(h_i) = i + 1 \)
Path algebra and double quiver

**Path algebra**

\( \mathbb{C}Q = \text{algebra spanned by paths with multiplication given by concatenation} \)

\[
\text{cat of reps of } Q \cong \textbf{Mod} \mathbb{C}Q
\]

**Double quiver**

\( Q^* = \text{double quiver of } Q \)

\[
I_{Q^*} = I_Q, \quad H_{Q^*} = H_Q \cup \bar{H}_Q, \quad \bar{H}_Q = \{ \bar{h} \mid h \in H_Q \}
\]
For \( i \in I \), define

\[
 r_i = \sum_{h \in H, \text{out}(h) = i} \bar{h} h - \sum_{h \in H, \text{in}(h) = i} h \bar{h}
\]

**Preprojective algebra**

\[ P(Q) = \mathbb{C} Q^* / J \]

\( J = \) two-sided ideal generated by \( r_i, \ i \in I \)

**Representations of the preprojective algebra**

\[ \text{mod}(P(Q), V) = \{ P(Q)-\text{modules with underlying v.s. } V \} \]

Equivalent to set of elements of \( \text{rep}(Q^*, V) \) such that

\[
\sum_{h \in H, \text{out}(h) = i} \bar{x}_h x_h - \sum_{h \in H, \text{in}(h) = i} x_h \bar{x}_h = 0 \quad \forall i \in I
\]
Proposition (Crawley-Boevey, Lusztig and others)

The following are equivalent for a quiver $Q$:

1. $P(Q)$ is finite-dimensional
2. All elements of $\text{rep}(P(Q), V)$ are nilpotent
3. $Q$ is a Dynkin quiver (underlying graph of $ADE$ type)

Proposition

If $Q$ is a finite quiver then

1. $P(Q)$ is of **finite rep type** iff $Q$ is of Dynkin type $A_n$, $n \leq 4$
2. $P(Q)$ is of **tame rep type** iff $Q$ is of Dynkin type $A_5$ or $D_4$
3. $P(Q)$ is of **wild rep type** for other types
Corollary

- $Q_{a,b}$ has finite rep type iff $b - a \leq 3$, and all reps are nilpotent
- $Q_\infty$ is of wild rep type and all reps are nilpotent
The Euclidean group
Euclidean algebra
Preprojective algebras
Quiver varieties
Conclusion

Preprojective algebras and the Euclidean algebra

Theorem

The map \( \psi : \mathbb{C}Q^*_\infty \rightarrow \tilde{U} \) given by (\( \epsilon_i = \) trivial path at \( i \))

\[
\begin{align*}
\psi(\epsilon_i) &= a_i, \\
\psi(h_i) &= p_+ a_i = a_{i+1} p_+, \\
\psi(\bar{h}_i) &= a_i p_- = p_- a_{i+1}
\end{align*}
\]

extends to a surjective map of algebras with kernel \( J \). Thus

\[ P(Q_\infty) \cong \tilde{U} \]

Corollary

\( \text{Mod} \, \epsilon(2) \cong \text{Mod} \, P(Q_\infty) \)

and

\( \text{Mod}_{a,b} \, \epsilon(2) \cong \text{Mod} \, P(Q_{a,b}) \)

where \( \text{Mod}_{a,b} \, \epsilon(2) \) is category of \( \epsilon(2) \)-modules with weights lying between \( a \) and \( b \).
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Representation theory of the Euclidean algebra

**Theorem**

- $e(2)$ (and hence $E(2)$) has **wild representation type**
- for $0 \leq b - a \leq 3$, $\exists$ a finite number of isom classes of indecomposable $e(2)$-modules whose weights lie between $a$ and $b$
**Lusztig quiver variety**

**Definition (Lusztig quiver variety)**

\( \Lambda_{V,Q} \) is set of all nilpotent \((x_h) \in \text{mod}(P(Q), V)\)

Recall, for \(Q\) of Dynkin type

\[
\text{mod}(P(Q), V) = \Lambda_{V,Q}
\]

**Relation to Kac-Moody algebras**

Let \(g_Q = \text{Kac-Moody algebra whose Dynkin graph is underlying graph of } Q\).

\[
\# \text{ irred comps of } \Lambda_{V,Q} = \dim \text{ of } (\ - \sum (\dim V_i)\alpha_i)\text{-weight space of } U(g_Q)^-\]
Nakajima quiver variety

Let $Q$ be $Q_\infty$ or $Q_{a,b}$.

For $I$-graded vec spaces $V$ and $W$, define

$$L_Q(V, W) = \Lambda_{V,Q} \oplus \bigoplus_{i \in I} \operatorname{Hom}_\mathbb{C}(W_i, V_i)$$

For $(x, s) = ((x_h)_{h \in H}, (s_i)_{i \in I})$ we say

- $I$-graded $S \subseteq V$ is $x$-invariant if

$$x_h(S_{\text{out}(h)}) \subseteq S_{\text{in}(h)} \quad \forall h \in H$$

- $(x, s)$ is stable if $\not\exists$ proper $x$-invariant subspace of $V$ containing $\text{im } s$

Let $L_Q(V, W)^{\text{st}} = \text{set of stable points}$
Nakajima quiver variety

\[ G_V = \prod_{i \in I} GL(V_i) \] acts on \( L_Q(V, W) \) by

\[ g \cdot (x, s) = \left( (g_{\text{in}}(h)x_hg_{\text{out}}^{-1}(h)), (g_is_i) \right) \]

Stabilizer in \( G_V \) of a stable point is trivial

**Definition (Nakajima quiver variety)**

\[ \mathcal{L}_Q(V, W) = L_Q(V, W)^{\text{st}} / G_V \]
Nakajima quiver varieties and Kac-Moody algebras

\[
\bigoplus_{V} H_{\text{top}}(\mathcal{L}_Q(V, W)) \cong \text{irrep of } g_Q \text{ of hw } \sum_{i \in I} (\dim W_i)\omega_i
\]

where \(\omega_i\) are fundamental weights

\[
H_{\text{top}}(\mathcal{L}_Q(V, W)) = \sum_{i \in I} (\dim W_i)\omega_i - \sum_{i \in I} (\dim V_i)\alpha_i \text{ weight space}
\]

\# \text{irred comps of } \mathcal{L}_Q(V, W) = \dim \text{of weight space}
Representation theory of $\varepsilon(2)$

- recall $\varepsilon(2)$ has \textcolor{red}{wild} rep type

- restrict attention to subclasses of modules and attempt a classification

- impose restriction on number of generators of a module

- moduli spaces of such modules related to Nakajima quiver varieties
Let $V$ be a rep of $\mathfrak{e}(2)$

We say $\{u_1, \ldots, u_n\} \subseteq V$ is a set of generators of $V$ if

1. each $u_i$ is a weight vector
2. $\not\exists$ proper submodule of $V$ containing all $u_i$

Definition

For $v, w \in (\mathbb{Z}_{\geq 0})^\mathbb{Z}$, let $E(v, w)$ be set of all

$$(V, (u^i_k)_{k \in \mathbb{Z}, 1 \leq j \leq w_k})$$

where

- $V$ is an $\mathfrak{e}(2)$-module with $\text{dim } V = v$
- $(u^i_k)_{k \in \mathbb{Z}, 1 \leq j \leq w_k}$ is a set of generators of $V$ with $\text{wt } u^i_k = k$
Moduli spaces of representations of \( \varepsilon(2) \)

**Definition**

We say

\[
(V, (u^i_j)) \sim (\tilde{V}, (\tilde{u}^i_j))
\]

if \( \exists \varepsilon(2) \)-module isom \( \phi : V \xrightarrow{\sim} \tilde{V} \), \( \phi(u^i_j) = \tilde{u}^i_j \) \( \forall j, k \)

Let

\[
\mathcal{E}(v, w) = E(v, w) / \sim
\]
Moduli spaces of representations of $\varepsilon(2)$

**Theorem**

There is a natural one-to-one correspondence

$$\mathcal{E}(v, w) \leftrightarrow \mathcal{L}_{Q_{\infty}}(V, W)$$

if $\dim V = v$, $\dim W = w$

**Idea of proof**

Given $(V, (u^i_k)) \in E(v, w)$, define a point $(x, s) \in L_{Q_{\infty}}(V, W)$ by

$$x_{h_i} = p_+ |v_i|, \quad x_{\bar{h}_i} = p_- |v_{i+1}|, \quad k \in \mathbb{Z}$$

$$s(w^j_k) = u^j_k, \quad k \in \mathbb{Z}, \quad 1 \leq j \leq w_k$$

where $\{w^j_k\}_{1 \leq j \leq w_k}$ is a basis of $W_k$. Then

generating set $\leftrightarrow$ stability $\sim \leftrightarrow G_V - \text{orbits}$
Remarks

- relationship between rep theory of Euclidean group and rep theory of $\mathfrak{sl}_\infty$ (or groups $SL(n)$)

- moduli space of reps of Euclidean group along with a set of generators closely related to rep theory of $\mathfrak{sl}_\infty$ and $SL(n)$

- although Euclidean group has wild rep type, we have method of approaching classification:
  - fix cardinality and weights of a generating set
  - resulting moduli space enumerated by countable number of varieties – one variety for reps of each graded dimension
Further directions

Positive characteristic

- consider Euclidean group over field of characteristic $p$
- weights now lie in $\mathbb{Z}/p\mathbb{Z}$ instead of $\mathbb{Z}$
- category of reps equivalent to category of reps of preprojective algebra of quiver of affine type $\hat{A}_{p-1}$
- quiver varieties related to moduli spaces of solutions to anti-self-dual Yang-Mills equations and Hilbert schemes of points in $\mathbb{C}^2$
Further directions

Crystals and Jordan-Hölder decompositions

- can define crystal structure on set of irred comps of Lusztig and Nakajima quiver varieties
- each irred comp can be identified with a sequence of crystal operators
- sequence corresponds to Jordan-Hölder decomposition of $\varepsilon(2)$-modules