Alistair Savage

Fields Institute and University of Toronto

Quiver Varieties
and
Geometric Representation Theory
Geometric Representation Theory

- Integrable Models in Statistical Mechanics
- Moduli Spaces
- Remarkable Bases (canonical basis, semicanonical basis)
- Affine Grassmannian MV Cycles
- Hilbert Schemes
- Quantum Groups
- Instantons on ALE spaces
- "Flag Varieties" (Type A)
- Crystal Bases
- Crystal Graphs
- Quiver Varieties
- ...and other constructions
- Symplectic Geometry
- Braching Rules
- Tensor Product Multiplicities
- Combinatorics of Young Diagrams, Young Tableaux, etc.
A Sample Lie Algebra: $\mathfrak{sl}_n$

$\mathfrak{sl}_n = n \times n$ traceless matrices

$$[A, B] = AB - BA$$

Presentation in terms of *Chevalley generators*

$$e_i = E_{i,i+1}, \quad f_i = E_{i+1,i}, \quad i = 1, \ldots, n - 1$$

and certain relations.

A *representation* of $\mathfrak{sl}_n$ is

- A complex vector space $V$

- A map $\mathfrak{sl}_n \to \text{End}(V)$ such that

  $$[A, B](v) = A(B(v)) - B(A(v))$$
Dictionary

simply-laced Kac-Moody algebra $\leftrightarrow \mathfrak{sl}_n$

irreducible integrable highest weight rep $\leftrightarrow$ f.d. rep

weight space of a rep $\leftrightarrow$ eigenspace of diagonal matrices
Quiver Variety Approach

\[ g = \text{simply-laced Kac-Moody Lie algebra (e.g. } g = \mathfrak{sl}_n) \]

\[ L = \text{irreducible integrable highest weight rep of } g \]

\[ L \text{ decomposes into weight spaces:} \]

\[ L = \bigoplus_{\lambda} L_\lambda \]

\[ \lambda \leftrightarrow \text{quiver variety } QV(\lambda). \]
Quiver Variety Approach

Weight space $L_\lambda \leftrightarrow \text{“Homology” of } QV(\lambda)$

$\dim L_\lambda \leftrightarrow \# \text{ irr. comps. of } QV(\lambda)$

Action of $g \leftrightarrow \text{Correspondences}$

Correspondences:

$QV(\lambda_1) \leftrightarrow \text{“Intermediate Variety”} \rightarrow QV(\lambda_2)$
Quivers

quiver = oriented graph

Examples:

1. Quiver of type $A_n$

2. Quiver of type $A_n^{(1)}$
A representation of a quiver:

\[ \begin{align*} 
\text{vertex} & \longrightarrow \text{f.d. vector space} \\
\text{arrow} & \longrightarrow \text{linear map} 
\end{align*} \]

**Example:** A representation of the quiver

\[
\begin{array}{c}
\bullet & \Leftarrow & \bullet \\
1 & & 2
\end{array}
\]

consists of

\[ V_1, V_2 - \text{f.d. } \mathbb{C}\text{-vector spaces} \]

\[ x \in \text{Hom}(V_2, V_1) \]
\[ g = \text{simply-laced Kac-Moody Lie algebra (e.g. } g = sl_{n+1}) \]

Consider the Dynkin graph of \( g \)

Take \textit{both} orientations of each edge

Add in \textit{shadow vertices}

Call this quiver \( Q(g) \)
$I$ = set of vertices of Dynkin graph of $\mathfrak{g}$

Fix collections of v.s. $V = (V_k)_{k \in I}$, $W = (W_k)_{k \in I}$

Let $v = (\dim V_k)_{k \in I}$, $w = (\dim W_k)_{k \in I}$

Define (Nakajima)

$$\mathcal{L}(v, w) = \{\text{reps of } Q(\mathfrak{g}) \text{ with v.s. } V \text{ and } W \}
\setminus \prod_{k \in I} GL(V_k)$$

Quiver Varieties

\[ \begin{array}{cccccc}
W_1 & W_2 & W_3 & W_{n-2} & W_{n-1} & W_n \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
V_1 & V_2 & V_3 & V_{n-2} & V_{n-1} & V_n
\end{array} \]
Alternative Description

\( M(v, w) = \text{space of reps of quiver with fixed vector spaces} \)

\[
\begin{align*}
M(v, w) // \xi_1 \prod_k GL(V_k) & \hookrightarrow \text{hyper-Kähler quotient (smooth)} \\
\pi \downarrow & \\
M(v, w) // \xi_2 \prod_k GL(V_k) & \hookleftarrow \text{hyper-Kähler quotient (singular)}
\end{align*}
\]

\( L(v, w) = \pi^{-1}(0) \)

\( L(v, w) \) is a Lagrangian subvariety and a deformation retract of

\[
M(v, w) \overset{\text{def}}{=} \frac{M(v, w) // \xi_1 \prod_k GL(V_k)}{\xi_2 \prod_k GL(V_k)}
\]
Weights

Want to construct irred. integ. highest weight rep of $\mathfrak{g}$.

In $\mathcal{L}(v, w)$,

$$w \leftrightarrow \text{highest weight of rep}$$

$$w \leftrightarrow \sum_{k \in I} w_k \Lambda_k$$

$\Lambda_k$ – fundamental weights

$$v \leftrightarrow \text{weight space}$$

$$v \leftrightarrow \sum_{k \in I} w_k \Lambda_k - \sum_{k \in I} v_k \alpha_k$$

$\alpha_k$ – simple roots
Example: $\mathfrak{sl}_2$

Let $g = \mathfrak{sl}_2$.

\[ L(k, n) = \{ B \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^k) \mid B \text{ surjective} \} / GL(\mathbb{C}^k) \]
\[ \cong Gr(n - k, n) \]
Example: Adjoint Representation of $\mathfrak{sl}_3$

Adjoint rep of $\mathfrak{sl}_3$ has h.w. $\Lambda_1 + \Lambda_2 \Rightarrow w = (1,1)$.

$\mathcal{L}((1,0),(1,1))$  $\mathcal{L}((0,0),(1,1))$

$\mathcal{L}((2,1),(1,1))$  $\mathcal{L}((1,1),(1,1))$  $\mathcal{L}((0,1),(1,1))$

$\mathcal{L}((2,2),(1,1))$  $\mathcal{L}((1,2),(1,1))$
Other Examples of Quiver Varieties

- $\mathfrak{g} = \mathfrak{sl}_n$, highest weight $a\Lambda_1$
  $\rightarrow$ partial flag varieties

- $\mathfrak{g}$ affine, $w = 0$, $v \leftrightarrow$ imaginary root
  $\rightarrow$ ALE spaces (resolutions of simple singularities)

- $\mathfrak{g}$ affine
  $\rightarrow$ moduli spaces of instantons on ALE spaces

- Jordan quiver

  $\rightarrow$ Hilbert schemes of points in $\mathbb{C}^2$
Benefits of Geometric Approach

• Alternative (often simpler) geometric proofs of algebraic facts

• Rep theory organizes homological information

• Connection to crystal graphs ($q \to 0$ limit of quantum groups)

• Geometrically defined bases with remarkable properties
Geometrically Defined Bases

Recall,

weight space $\longleftrightarrow$ “homology” of $QV$

dimension of weight space $\longleftrightarrow$ # irr. comps. of $QV$

Classes of irr. comps. of $QV$ yield basis of representation

<table>
<thead>
<tr>
<th>Homology Theory</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constructible functions</td>
<td>Semicanonical basis</td>
</tr>
<tr>
<td>Top dim Borel-Moore homology</td>
<td>Semicanonical basis?</td>
</tr>
<tr>
<td>Perverse sheaves</td>
<td>Canonical basis</td>
</tr>
</tbody>
</table>

Nice positivity, integrality, and compatibility properties
Nice Properties of Geometric Bases

Positivity & Integrality:

\[ f_k \cdot b = \sum c_j b_j, \quad c_j \in \mathbb{Z}_{\geq 0} \]

Compatibility:

Have geometric basis \( B \) of \( U^-(g) \)

For any irred hw rep \( L \) of \( g \) with hw vec \( v \),

\[ \{ b \cdot v | b \in B, \; b \cdot v \neq 0 \} \]

is a basis of \( L \).
Geometric Boson-Fermion Correspondence

Bosonic Fock Space

Infinite-dim Heisenberg algebra:

\[ \mathfrak{s} = \bigoplus_{m \in \mathbb{Z} \setminus \{0\}} \mathbb{C} s_m \oplus \mathbb{C} K \]

\[ [\mathfrak{s}, K] = 0, \quad [s_m, s_n] = m \delta_{m,-n} K \]

Bosonic Fock Space:

\[ B = \mathbb{C}[p_1, p_2, \ldots] \]

Action of \( \mathfrak{s} \) on \( B \) by

\[ s_m \mapsto m \frac{\partial}{\partial p_m}, \quad s_{-m} \mapsto p_m, \quad m > 0 \]

\[ K \mapsto \text{Id} \]
Fermionic Fock Space

\[ F = \text{Span}_\mathbb{C}\{i_0 \wedge i_1 \wedge i_2 \wedge \ldots \mid i_k \in \mathbb{Z}, \ i_0 > i_1 > \ldots, \ i_k = -k \text{ for } k \gg 0\} = \text{“infinite wedge space”} \]

Part of a larger Fock space with action of an infinite Clifford algebra

\[ \mathfrak{gl}_\infty \text{ (and } \mathfrak{sl}_\infty \) act on } F \text{ by derivations:} \]

For \( a \in \mathfrak{gl}_\infty \),

\[ a(i_0 \wedge i_1 \wedge \ldots) = (a \cdot i_0) \wedge i_1 \wedge \ldots + i_0 \wedge (a \cdot i_1) \wedge \ldots + \ldots \]
Boson-Fermion Correspondence

Heisenberg Algebra $\mathfrak{h}$

Clifford Algebra $\mathfrak{s}l_\infty$

Bosonic Fock Space

Fermionic Fock Space

$\cong$
Boson-Fermion Correspondence

Action of $s$ on $F$: “Bosonization”

\[ s_m \mapsto \sum_{j \in \mathbb{Z}} E_{j,j+m}, \ m \in \mathbb{Z}\setminus\{0\}, \]

\[ K \mapsto \text{Id} \]

\[ F \cong B \text{ as } s\text{-modules} \]

Action of Clifford algebra on $B$ (generating functions, vertex operator algebras): “Fermionization”

\[ F \cong B \text{ as Clifford algebra modules} \]
Hilbert Schemes

\[ X_n = \{ (B_1, B_2, i) \mid B_j \in \text{End} \, V, \ i \in \text{Hom}(\mathbb{C}, V), \ + \text{ conditions} \}/\text{GL}(V) \]

Here, \( V \cong \mathbb{C}^n \).

**Conditions:**
- \( B_1 B_2 = B_2 B_1 \)
- \( \text{im} \, i \) generates \( V \) under \( B_1, B_2 \)
Hilbert Schemes

\( X_n \cong \) Hilbert scheme of \( n \) points in \( \mathbb{C}^2 \)

Hilbert scheme is a resolution of singularities:

\[
X_n \to (\mathbb{C}^2)^n / S_n
\]

Have \( T = \mathbb{C}^* \) action on \( X_n \):

\[
z \cdot (GL(V) \cdot (B_1, B_2, i)) = GL(V) \cdot (zB_1, z^{-1}B_2, i)
\]
Geometric Bosonic Fock Space

\[ \mathbb{H}_B = H_{2n}^T(X_n), \quad \mathbb{H}^B = \bigoplus_{n=0}^{\infty} \mathbb{H}_n^B \]

**Correspondences:** Natural projections

\[ X_{n+k} \xleftarrow{\pi_1} X_{n+k} \times X_n \xrightarrow{\pi_2} X_n \]

Define \( p_{-k} \in \text{End} \mathbb{H}^B, k \geq 0 \), by

\[ p_{-k}(\alpha) = (\pi_1)! (\pi_2^* \alpha \cup [\Sigma_{n,k}]), \quad \alpha \in \mathbb{H}_n^B \]

Here

\[ \Sigma_{n,k} \subset X_{n+k} \times X_n \]

\[ \leftrightarrow \text{“adding } k \text{ points” at } z \in \mathbb{C} \times \{0\} \subset \mathbb{C}^2 \]
Define adjoint operator $p_k$, $k > 0$

**Prop** (Nakajima, Vasserot):

$$[p_k, p_l] = k\delta_{k, -l}\text{Id}$$

So

$$s_k \mapsto p_k, \quad K \mapsto \text{Id}$$

defines action of $\mathfrak{s}$ on $\mathbb{H}^B$ and

$$\mathbb{H}^B \cong B$$ as $\mathfrak{s}$-modules
Quiver Varieties for $\mathfrak{sl}_\infty$

$V = \text{f.d. } \mathbb{Z}\text{-graded complex vector space}$

$v = \dim V = (\dim V_k)_{k \in I}$

$w = e_0$

$\mathcal{M}(v, e_0) = \{(B_1, B_2, i) \mid B_j \in \text{End } V, i \in \text{Hom}(\mathbb{C}, V_0)$

$+ \text{ conditions}\}/ \prod_k GL(V_k)$

Conditions:

- $B_1 \in \text{End } V$, $\deg B_1 = 1$ (i.e. $B(V_k) \subseteq V_{k+1}$)
- $B_2 \in \text{End } V$, $\deg B_2 = -1$ (i.e. $B(V_k) \subseteq V_{k-1}$)
- $B_1 B_2 = B_2 B_1$
- $\text{im } i \text{ generates } V \text{ under } B_1, B_2$
\[ \mathbb{H}^F = \bigoplus_v H^2_T(\mathcal{M}(v, e_0)) \quad \text{(trivial } T\text{-action)} \]

**Correspondences:** Have natural projections

\[ \mathcal{M}(v + e_k, e_0) \leftarrow \mathcal{M}(v + e_k, e_0) \times \mathcal{M}(v, e_0) \rightarrow \mathcal{M}(v, e_0) \]

Define geometric action of \( \mathfrak{sl}_\infty \) (similar to Hilbert scheme picture) and

\[ \mathbb{H}^F \cong F \text{ as } \mathfrak{sl}_\infty\text{-modules} \]
Torus Fixed Points of Hilbert Schemes

At a $T$-fixed point of $X_n$, weight decomposition of $V$ gives grading

$$V = \bigoplus_{m \in \mathbb{Z}} V_m$$

Can show that

- $B_1, B_2$ have degrees 1 and -1 resp.
- $\text{im } i \subseteq V_0$

Thus,

$T$-fixed points of H.S. = quiver varieties for $\mathfrak{sl}_\infty$
Localization and the Boson-Fermion Correspondence

\[ X \text{ smooth with a } T\text{-action} \]
\[ X^T = T\text{-fixed points of } X \]

Localization theorem states

\[ H^*_T(X) \otimes \mathbb{C}(t) \cong H^*_T(X^T) \otimes \mathbb{C}(t) \]

Thus (after a few technical steps),

\[ \mathbb{H}^B = \bigoplus_n H^2_T(X_n) \cong \bigoplus_n H^2_T(X^n_T) = \mathbb{H}^F \]

A geometric boson-fermion correspondence!
Fixed points of $\mathbb{Z}/n\mathbb{Z} \subset T$ yield quiver varieties for $\hat{\mathfrak{sl}}_n$

Vertex operator construction of basic rep of $\hat{\mathfrak{sl}}_n$ should fit into this geometric picture

Should help give algebraic description of nice geometric bases
Crystal Graphs

Quantum group $U_q(\mathfrak{g})$ is a $q$-deformation of $U(\mathfrak{g})$

Representations of $\mathfrak{g}$ (or $U(\mathfrak{g})$) have a $q$-deformation

usual rep theory $\xrightarrow{q \to 1}$ quantum groups $\xrightarrow{q \to 0}$ crystals

In $q \to 0$ limit, rep theory becomes combinatorics
$L = \text{rep of } U_q(g)$

Fix a basis of $L$ and depict elements of the basis by vertices:
Consider the action of a Chevalley generator, say \( f_2 \):

"Nice" basis:
- At most one coeff of 1 in each expression
- All other coeffs have positive power of \( q \)
Crystal graphs

Take $q \to 0$ limit

Label remaining edges by Chevalley generator index:
Crystal graphs

Repeat for remaining Chevalley generators:

Obtain the crystal graph of rep $L$
Crystal Graphs

- Connected graph $\implies$ irreducible rep

- Can compute characters by counting vertices of fixed weight

- Tensor product rule

“Nice” basis $=$ canonical basis !!

**Problem:** How do we “thaw” a crystal?
Let \( g = sl_2 \).

Consider \( V_3 \otimes V_4 \).

Crystal graphs are

\[
\begin{align*}
V_3 & \quad \begin{array}{c} 1 \\ 1 \\ \bullet \end{array} \quad \begin{array}{c} 1 \\ \bullet \end{array} \\
V_4 & \quad \begin{array}{c} 1 \\ 1 \\ 1 \\ \bullet \end{array} \quad \begin{array}{c} 1 \\ \bullet \end{array}
\end{align*}
\]
Tensor product rule yields

Thus

$$V_3 \otimes V_4 = V_2 \oplus V_4 \oplus V_6$$
Realizations of Crystal Graphs

Combinatorial Realizations:

- Young tableaux (classical Lie algebras)
- Young walls (affine Lie algebras)
- Kyoto path model (affine Lie algebras)
- Littelmann path model

Geometric Realization:

Vertices of crystal graph = irred. comps. of QVs

Crystal operators (edges) defined geometrically
Connections Between Realizations

Finite and affine types $A$ and $D$: Explicit isomorphism between Young tableaux/wall realizations and geometric realization (S)

Advantages:

- Explicit description of irr comps of QVs
- Gives geometric interpretation and suggests extensions of combinatorial constructions
- General QV theory gives universal method to “thaw” crystals
Connections Between Realizations
Other Constructions

Extension to non-simply laced case:

- Crystal structure – done (S)
- “Full” structure – open

Other constructions:

- Tensor products (Nakajima, Malkin)
- Fusion products (Schiffmann-S)
- Demazure modules (S)
- Spin representations, Clifford algebras (S)
- Virasoro algebra (Lehn)
- Others? Lie superalgebras? Jordan (super)algebras?
Connections to Affine Grassmannian Approach

Quiver Varieties (Type A) \[\xrightarrow{[\mathbf{S}]}\] Flag Varieties (Ginzburg, Beilinson–Lusztig–MacPherson)

Affine Grassmannian (Type A) \[\xleftarrow{[\mathbf{B–G–Vybornov}]}\] Dual Flag Variety Construction (Braverman–Gaitsgory)