Quiver Varieties and Demazure Modules

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Why Quiver Varieties?

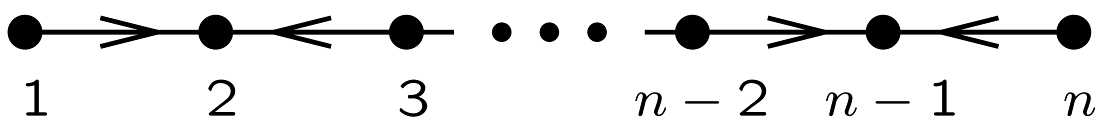
- give us a way to geometrically realize Kac-Moody algebras and their representations
- yield bases with very nice properties (integrality, positivity, etc.)
- yield crystal bases ($q \to 0$ limit of quantum groups)

quiver = oriented graph

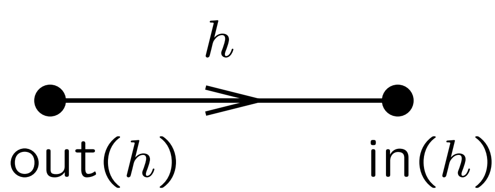
Let $\mathfrak{g}$ be a simply-laced Kac-Moody algebra.

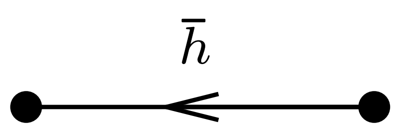
Consider the Dynkin graph of $\mathfrak{g}$ and fix an orientation.

Example: $\mathfrak{g} = \mathfrak{sl}_{n+1}$



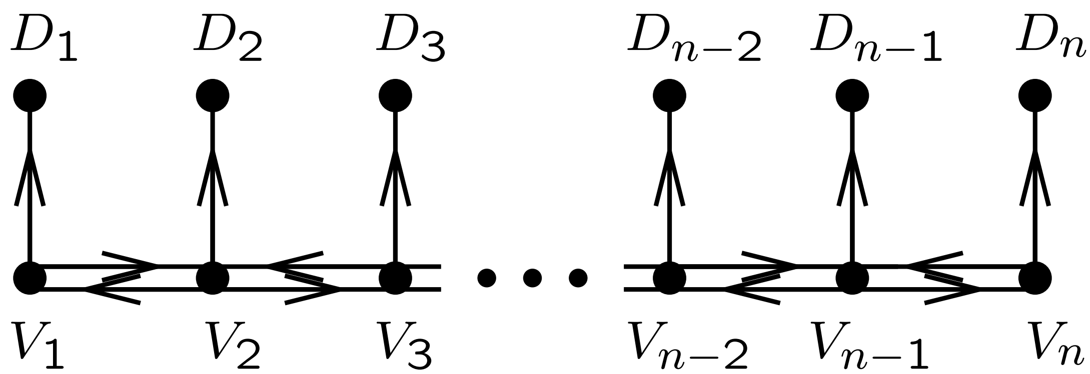
Notation:





I = set of vertices
Ω = (oriented) edges in our orientation
$H = \Omega \cup \bar{\Omega}$

For $\mathbf{v}, \mathbf{d} \in \mathbb{Z}^I_{\geq 0}$, let $\dim V_i = \mathbf{v}_i$, $\dim D_i = \mathbf{d}_i$.



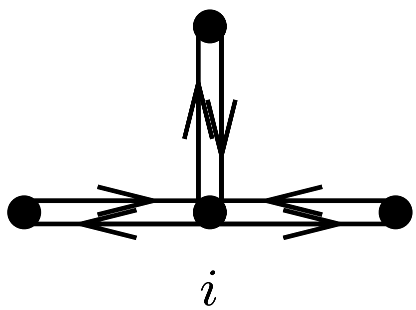
$$M(\mathbf{v}, \mathbf{d}) := \bigoplus_{h \in H} \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}) \oplus \bigoplus_{i \in I} \mathrm{Hom}(V_i, D_i)$$

Elements $((x_h), (t_i))$ of $M(\mathbf{v}, \mathbf{d})$ are called *representations* of the quiver.

We have notions of sums of reps, subreps, etc.

Definition. *We say* $((x_h),(t_i)) \in M(\mathbf{v},\mathbf{d})$ *is a* stable point *if*

$$\ker t_i \cap \bigcap_{\substack{h \in H \\ \text{out}(h)=i}} \ker x_h = 0 \ \forall\, i.$$



Definition. *For* $i \in I$, *define*

$$\psi_i((x_h),(t_i)) = \sum_{\substack{h \in H \\ \text{out}(h)=i}} \epsilon(h) x_{\overline{h}} x_h$$

where

$$\epsilon(h) = \begin{cases} +1 & h \in \Omega \\ -1 & h \in \bar{\Omega} \end{cases}$$

$G_V = \prod_{i \in I} GL(V_i)$ acts naturally on $M(\mathbf{v}, \mathbf{d})$

$$(g, ((x_h), (t_i))) \mapsto ((g_{\mathrm{in}(h)} x_h g_{\mathrm{out}(h)}^{-1}), (t_i g_i^{-1}))$$

Definition (Nakajima's Quiver Variety).

$$\mathcal{L}(\mathbf{v}, \mathbf{d}) := \left\{ \begin{matrix} (x,t) \\ \in M(\mathbf{v}, \mathbf{d}) \end{matrix} \middle| \begin{matrix} \psi_i((x,t)) = 0 \, \forall \, i \in I \\ (x,t) \text{ stable }, x \text{ nilpotent} \end{matrix} \right\} / G_V$$

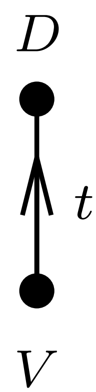
For $\mathbf{v}, \mathbf{d} \in \mathbb{Z}_{\geq 0}^I$, define

$$\Lambda_{\mathbf{d}} = \sum_{i \in I} \mathbf{d}_i \Lambda_i, \quad \alpha_{\mathbf{v}} = \sum_{i \in I} \mathbf{v}_i \alpha_i.$$

Theorem (Nakajima).

$$\# \operatorname{Irr} \mathcal{L}(\mathbf{v}, \mathbf{d}) = \dim L(\Lambda \mathbf{d})_{\Lambda_{\mathbf{d}} - \alpha_{\mathbf{v}}}$$

Example. $\mathfrak{g} = \mathfrak{sl}_2$



stability $\Rightarrow t$ *injective*

$$\begin{aligned}\mathcal{L}(\mathbf{v}, \mathbf{d}) &= \{t \in \mathrm{Hom}(V, D) \mid \ker t = 0\}/GL(V) \\ &\cong Gr(v, d) \qquad (t \mapsto \mathrm{Im}\, t)\end{aligned}$$

$Gr(v, d)$ *smooth, irreducible*
- *corresponds to* $(d - 2v)$ *weight space of* L_d.

Example. $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $d_1 = d$, $d_i = 0$ *for* $i \neq 1$. $v_1 \geq v_2 \geq \cdots \geq v_n$. *Then*

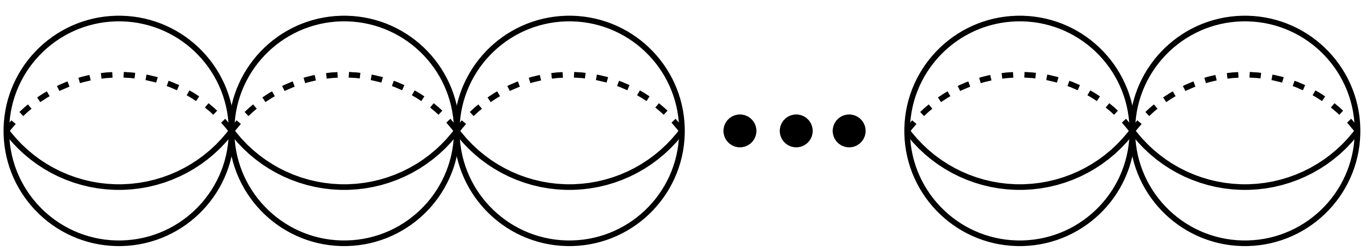
$$\mathcal{L}(\mathbf{v}, \mathbf{d}) \cong \{\mathbb{C}^n \supseteq V_1' \supseteq V_2' \supseteq \cdots \supseteq V_n' \mid \dim V_i' = v_i\}$$

is a partial flag variety.

Example. $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $d_1 = d_{n+1} = 1$, $d_i = 0$ *for* $i \neq 1, n+1$. *This is the* adjoint representation *of* $\mathfrak{g}$.

$v_1 = v_2 = \cdots = v_{n+1} = 1$ - *zero weight space*

Then $\mathcal{L}(\mathbf{v}, \mathbf{d})$ *is* $n+1$ *copies of* $\mathbb{C}P^1$.



Let $\mathbf{u}^i \in \mathbb{Z}^I_{\geq 0}$ have ith component 1 and other components zero. Define

$$Z(\mathbf{v}, \mathbf{v}+\mathbf{u}^i, \mathbf{d}) = \left\{ \begin{matrix} \left(\overline{(x',t')}, \overline{(x,t)}\right) \\ \in \mathcal{L}(\mathbf{v},\mathbf{d}) \times \mathcal{L}(\mathbf{v}+\mathbf{u}^i, \mathbf{d}) \end{matrix} \,\middle|\, (x',t') \text{ is a subrep of } (x,t) \right\}.$$

We have natural maps

$$\mathcal{L}(\mathbf{v},\mathbf{d}) \stackrel{\pi_1}{\leftarrow} Z(\mathbf{v}, \mathbf{v}+\mathbf{u}^i, \mathbf{d}) \stackrel{\pi_2}{\rightarrow} \mathcal{L}(\mathbf{v}+\mathbf{u}^i, \mathbf{d})$$

and pullback and pushfoward of constructible functions.

$$\pi_i^* f = f \circ \pi_i,$$
$$(\pi_i)_!(\mathbf{1}_A)(x) = \chi(A \cap \pi_i^{-1}(x)), \;\; A \text{ constructible}$$

Define

$$e_i = (\pi_1)_! \pi_2^*, \;\; f_i = (\pi_2)_! \pi_1^*$$

Let $\varphi = \mathbf{1}_{\mathcal{L}(0,\mathbf{d})}$ and let $L(\mathbf{d})$ be the space of functions spanned by the result of acting on φ by any combination of f_i's.

Theorem (Nakajima). *1.* $L(\mathbf{d}) \cong L(\Lambda_{\mathbf{d}})$ *with* e_i *and* f_i *as above giving action of the Chevalley generators.*

2. *For each irreducible component of* $\mathcal{L}(\mathbf{v}, \mathbf{d})$, $\exists!$ *function equal to 1 on an open dense subset of this irreducible component and equal to 0 on an open dense subset of all others. These functions span* $L(\mathbf{d})$ *and are called the* semi-canonical basis*.*

Nice Properties of Semicanonical Basis

1. Compatibiliy

    - have semicanonical basis $\mathcal{B}_{sc}$ of $U(\mathfrak{n}^-)$

    - let V be an irred h.w. module with h.w. vector v

    - $\{b \cdot v \mid v \in \mathcal{B}_{sc},\, b \cdot v \neq 0\}$ is the semicanonical basis of V.

2. Positivity and Integrality

Remarks

1. Can also do with homology or perverse sheaves (giving *canonical basis*).

2. Actual computation of these bases is difficult

3. Using equivariant K-theory, get f.d. reps of affine Lie algebras

4. Can give geometric realization of crystal graphs on set of irreducible components

Demazure Modules

Let W be the Weyl group of $\mathfrak{g}$
(e.g. $W = S_n$ for $g = \mathfrak{sl}_n$)

W acts on weight lattice of $\mathfrak{g}$.

$\dim L(\lambda)_{w\cdot\lambda} = 1$. Let $v_w \in L(\lambda_{w\cdot\lambda})$, $v_w \neq 0$.

Definition.

$$L_w(\lambda) = U(\mathfrak{n}^+)v_w \subseteq L(\lambda)$$

is called a Demazure Module.

Yields a filtration of reps:

$$L(\lambda) = \bigcup_{w\in W} L_w(\lambda)$$

$$w \leq w' \Rightarrow L_w(\lambda) \subseteq L_{w'}(\lambda)$$

Proposition ([S]).

$$\mathcal{L}(\mathbf{v},\mathbf{d}) \cong pt \Leftrightarrow \Lambda_{\mathbf{d}} - \alpha_{\mathbf{v}} = w\Lambda_{\mathbf{d}}$$

for some $w \in W$

Definition. *Let* $\mathcal{L}_w(\mathbf{v},\mathbf{d})$ *be the subvariety of* $\mathcal{L}(\mathbf{v},\mathbf{d})$ *consisting of* G_V*-orbits of quiver reps which are subreps of a representative of the point* $\Lambda(\mathbf{v}',\mathbf{d})$ *where* $\Lambda_{\mathbf{d}} - \alpha_{\mathbf{v}'} = w\Lambda_{\mathbf{d}}$*.*

We call $\mathcal{L}_w(\mathbf{v},\mathbf{d})$ *the* Demazure Quiver Variety*.*

Theorem ([S]). *The subspace of functions in* $L(\mathbf{d})$ *whose support lies in* $\sqcup_{\mathbf{v}}\mathcal{L}_w(\mathbf{v},\mathbf{d})$ *is isomorphic to* $L_w(\Lambda_{\mathbf{d}})$ *(with the action of* $U(\mathfrak{n}^+)$ *as above).*

Corollary. *The semicanonical basis is compatible with the filtration by Demazure modules. That is, each* $L_w(\lambda)$ *is spanned by a subset of the semicanonical basis.*

Note: The canonical basis is also compatible with the Demazure filtration.

Example. $\mathfrak{g} = \widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k$.

There are two elements of W of length m ($m \in \mathbb{Z}_{\geq 0}$).

$$w = \begin{cases} r_0 r_1 \dots r_\epsilon, & \text{or} \\ r_1 r_0 \dots r_\epsilon \end{cases}$$

Then

$$\mathcal{L}_w(\mathbf{v}, \mathbf{d}) \cong \{\overline{(x, t)} \in \mathcal{L}(\mathbf{v}, \mathbf{d}) \mid x^{m'} = 0\}$$

where $m' = m$ or $m - 1$.