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A Geometric Boson-Fermion Correspondence

Slides available at www.math.toronto.edu/alistair

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Overview

$$\begin{array}{ccc}
\text{Bosonic Fock Space} & \xrightarrow[\text{Grojnowski}]{\text{Nakajima}} & \text{Hilbert Schemes of Points in } \mathbb{C}^2 \\
\text{B-F corr}\downarrow & & \downarrow\text{Localization} \\
\text{Fermionic Fock Space} & \xrightarrow[\text{Nakajima}]{\text{Lusztig}} & \text{Quiver Varieties for } \mathfrak{sl}_\infty
\end{array}$$

- Geometric realizations give us nice bases with remarkable properties.

- Rep theory organizes cohomological information

Fermionic Fock Space

$$F = \text{Span}_{\mathbb{C}}\{\underline{i_0} \wedge \underline{i_1} \wedge \underline{i_2} \wedge \dots \mid i_k \in \mathbb{Z}, \\ i_0 > i_1 > \dots,\ i_k = -k \text{ for } k \gg 0\} \\ = \text{“infinite wedge space”}$$

Part of a larger Fock space with action of an infinite Clifford algebra

$\mathfrak{gl}_\infty$ (and $\mathfrak{sl}_\infty$) act on F by derivations:

For $a \in \mathfrak{gl}_\infty$,

$$a(\underline{i_0} \wedge \underline{i_1} \wedge \dots) = (a \cdot \underline{i_0}) \wedge \underline{i_1} \wedge \dots + \underline{i_0} \wedge (a \cdot \underline{i_1}) \wedge \dots + \dots$$

Bosonic Fock Space

Infinite-dim Heisenberg algebra:

$$\mathfrak{s} = \bigoplus_{m \in \mathbb{Z} \setminus \{0\}} \mathbb{C} s_m \oplus \mathbb{C} K$$

$$[\mathfrak{s}, K] = 0, \qquad [s_m, s_n] = m \delta_{m,-n} K$$

Bosonic Fock Space:

$$B = \mathbb{C}[p_1, p_2, \dots]$$

Action of $\mathfrak{s}$ on B by

$$s_m \mapsto m \frac{\partial}{\partial p_m}, \quad s_{-m} \mapsto p_m, \quad m > 0$$

$$K \mapsto \text{Id}$$

Boson-Fermion Correspondence

Action of $\mathfrak{s}$ on F: "Bosonization"

$$s_m \mapsto \sum_{j\in\mathbb{Z}} E_{j,j+m},\ \ m \in \mathbb{Z}\backslash\{0\},$$
$$K \mapsto \mathrm{Id}$$

$$F \cong B \text{ as } \mathfrak{s}\text{-modules}$$

Also have action of Clifford algebra on B (generating functions, vertex operator algebras): "Fermionization"

$$F \cong B \text{ as Clifford algebra modules}$$

Hilbert Schemes

$$X_n = \text{Hilbert scheme of } n \text{ points in } \mathbb{C}^2$$
$$= \{(B_1, B_2, i) \mid [B_1, B_2] = 0,\ (B_1, B_2, i) \text{ stable}\} / GL(V)$$

Here

$$B_j \in \operatorname{End} V,\ \ i \in \operatorname{Hom}(\mathbb{C}, V),\ \ V \cong \mathbb{C}^n$$

(B_1, B_2, i) *stable* iff $\operatorname{im} i$ generates V under B_1, B_2

Have $T = \mathbb{C}^*$ action on X_n:

$$z \cdot (GL(V) \cdot (B_1, B_2, i)) = GL(V) \cdot (zB_1, z^{-1}B_2, i)$$

Geometric Bosonic Fock Space

$$\mathbb{H}_n^B = H_T^{2n}(X_n), \quad \mathbb{H}^B = \bigoplus_{n=0}^{\infty} \mathbb{H}_n^B$$

Natural projections

$$X_{n+k} \xleftarrow{\pi_1} X_{n+k} \times X_n \xrightarrow{\pi_2} X_n$$

Define $\mathfrak{p}_{-k} \in \operatorname{End} \mathbb{H}^B$, $k > 0$, by

$$\mathfrak{p}_{-k}(\alpha) = (\pi_1)_!(\pi_2^* \alpha \cup [\Sigma_{n,k}]), \ \alpha \in \mathbb{H}_n^B$$

Here

$$\Sigma_{n,k} \subset X_{n+k} \times X_n$$
$$\leftrightarrow \text{"adding } k \text{ points" at } z \in \mathbb{C} \times \{0\} \subset \mathbb{C}^2$$

Geometric Bosonic Fock Space

Define adjoint operator $\mathfrak{p}_k$, $k > 0$

Prop (Nakajima, Vasserot):

$$[\mathfrak{p}_k, \mathfrak{p}_l] = k\delta_{k,-l}\text{Id}$$

So

$$s_k \mapsto \mathfrak{p}_k, \quad K \mapsto \text{Id}$$

defines action of $\mathfrak{s}$ on $\mathbb{H}^B$ and

$$\mathbb{H}^B \cong B \text{ as } \mathfrak{s}\text{-modules}$$

Quiver Varieties

V = f.d. $\mathbb{Z}$-graded complex vector space

$v = \dim V = (\dim V_k)_{k\in\mathbb{Z}}$, $|v| = \sum v_k$

$$\mathcal{M}(v) = \{(B_1, B_2, i) \mid [B_1, B_2] = 0, (B_1, B_2, i) \text{ stable}\} \Big/ \prod_k GL(V_k)$$

$B_1 \in \operatorname{End} V$, $\deg B = 1$ (i.e. $B(V_k) \subseteq V_{k+1}$)
$B_2 \in \operatorname{End} V$, $\deg B = -1$ (i.e. $B(V_k) \subseteq V_{k-1}$)
$i \in \operatorname{Hom}(\mathbb{C}, V_0)$

Geometric Fermionic Fock Space

$$\mathbb{H}^F = \bigoplus_v H_T^{2|v|}(\mathcal{M}(v)) \quad \text{(trivial } T\text{-action)}$$

Have natural projections

$$\mathcal{M}(v+e_k) \longleftarrow \mathcal{M}(v+e_k) \times \mathcal{M}(v) \longrightarrow \mathcal{M}(v)$$

Define geometric action of $\mathfrak{sl}_\infty$ (similar to Hilbert scheme picture) and

$$\mathbb{H}^F \cong F \text{ as } \mathfrak{sl}_\infty\text{-modules}$$

Torus Fixed Points

At a T-fixed point of X_n, weight decomposition of V gives grading

$$V = \bigoplus_{m \in \mathbb{Z}} V_m$$

Can show that

- B_1, B_2 have degrees 1 and -1 resp.
- $\operatorname{im} i \subseteq V_0$

Thus,

T-fixed points of H.S. = quiver varieties

Localization and the Boson-Fermion Correspondence

X smooth with a T-action
$X^T = T$-fixed points of X

Localization theorem states

$$H_T^*(X) \otimes \mathbb{C}(t) \cong H_T^*(X^T) \otimes \mathbb{C}(t)$$

Thus (after a few technical steps),

$$\mathbb{H}^B = \bigoplus_n H_T^*(X_n) \cong \bigoplus_n H_T^*(X_n^T) = \mathbb{H}^F$$

A geometric boson-fermion correspondence!

Further Directions

Fixed points of $\mathbb{Z}/n\mathbb{Z} \subset T$ yield quiver varieties for $\widehat{\mathfrak{sl}}_n$

Vertex operator construction of basic rep of $\widehat{\mathfrak{sl}}_n$ should fit into this geometric picture

Should help give algebraic description of nice geometric bases