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Branching Rules and Quiver Varieties

(Joint with I. Dimitrov)

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The Lie Algebra $\mathfrak{gl}_n$

\[ \mathfrak{gl}_n = n \times n \text{ matrices} \]

\( \{ E_{ij} \}_{i,j=1}^n \) is the standard basis of $\mathfrak{gl}_n$.

\[ E_i = E_{i,i+1}, \quad F_i = E_{i+1,i} \]

\( \{ E_i, F_i \}_{i=1}^{n-1} \) are Chevalley generators (of $\mathfrak{sl}_n \subset \mathfrak{gl}_n$).

We have

\[ \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n \]

\[ \mathfrak{gl}_{n-1} = \text{Span}\{ E_{ij} \mid 1 \leq i, j \leq n - 1 \}. \]
Representations of $\mathfrak{gl}_n$

F.d. irred. (h.w.) reps of $\mathfrak{gl}_n \overset{1 \rightarrow 1}{\leftrightarrow}$ partitions

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$$

Let $L_n(\lambda)$ be the rep. corr. to $\lambda$.

A vector $v \in L_n(\lambda)$ has weight $\mu = (\mu_i)_{i=1}^n$ if

$$E_{ii}v = \mu_i v.$$ 

We have the weight space decomposition

$$L_n(\lambda) = \bigoplus_{\mu} L_n(\lambda)_\mu$$

where $L_n(\lambda)_\mu$ is the space of vectors of weight $\mu$. 

2
Branching Rules for $\mathfrak{gl}_n$

We can restrict a rep of $\mathfrak{gl}_n$ to a rep of $\mathfrak{gl}_{n-1}$.

\[ L_n(\lambda)|_{\mathfrak{gl}_{n-1}} \simeq \bigoplus_{\mu} L_{n-1}(\mu) \]

where the sum is over all partitions $\mu$ such that

\[ \lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n. \]

This is called the branching rule.

**NOTE:** The restriction is multiplicity free.

Continuing restrictions gives natural Gelfand-Tsetlin basis.
Assume $\lambda_n = 0$. Let $d = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Let $x \in \text{End}(\mathbb{C}^d)$ be nilpotent with $\lambda_i - \lambda_{i+1}$ Jordan blocks of size $i \times i$.

Equivalently,

$$\lambda_i = \sum_{j \geq i} \# j \times j \text{ Jordan blocks in } x.$$ 

Let $\mathcal{F}^n_x$ be the Spaltenstein variety

$$\{(0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = \mathbb{C}^d) \mid x(F_i) \subset F_{i-1}\}$$

$\mathcal{F}^n_x$ has connected components

$$\mathcal{F}^n_{d,x} = \{(F_i) \in \mathcal{F}^n_x \mid \dim F_i/F_{i-1} = d_i\}.$$
We are going to use $\mathcal{F}_x^n$ to construct $L_n(\lambda)$.

$$\mathcal{F}_{d,x}^n \leftrightarrow L_n(\lambda)_{\mu}$$

$$\mu_i = d_i - d_i + 1$$

To construct $E_i, F_i$ define

$$i \mathcal{F}_{d,x}^n = \{ (F, F') \in F_{d,x}^n \times F_{d+e^i,x}^n \mid F_j = F'_j, j \neq i, \quad F_i \subset F'_i, \quad \dim F'_i / F_i = 1 \}.$$
Geometric Construction of $L_n(\lambda)$

$M(F^n_x) = \text{space of "constructible" functions on } F^n_x$.

Define action of Chevalley generators $E_i$ and $F_i$ on $M(F^n_x)$ by

$$E_i f = (\pi_2)_! \pi_1^* f, \quad F_i f = (\pi_1)_! \pi_2^* f,$$

$$\pi_k^* = \text{pullback},$$

$$(\pi_k)_! = \text{"push-forward"}.$$
Geometric Construction of $L_n(\lambda)$

Highest weight space of $L_n(\lambda)$ corresponds to constant functions on the point $\mathcal{F}_{d_{\text{max}},x}^n = \{ (F_i^{\text{max}}) \}$, where

$$F_i^{\text{max}} = \ker x^i.$$ 

Let $\tilde{M}(\mathcal{F}_x^n)$ be the functions in $M(\mathcal{F}_x^n)$ generated by action of $F_i$ on constant functions on $\mathcal{F}_{d_{\text{max}},x}^n$.

**Theorem:**

$$\tilde{M}(\mathcal{F}_x^n) \simeq L_n(\lambda)$$

as $\mathfrak{sl}_n$-modules.
Want to create a natural map

\[ \tilde{M}(\mathcal{F}_x^n) \to \bigoplus_{x'} \tilde{M}(\mathcal{F}_{x'}^{n-1}) \]

which is an isomorphism of $\mathfrak{gl}_{n-1}$-modules.

Basic idea is

- restrict flags to the subspace $F_{n-1}

- set $x' = x|_{F_{n-1}}$

What are the possible Jordan normal forms of $x|_{F_{n-1}}$?
Geometric Construction of Branching

Consider an $i \times i$ Jordan block of $x$

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

$x(\mathbb{C}^d) = x(F_n) \subset F_{n-1} \Rightarrow F_{n-1}$ contains at least first $i - 1$ of these basis vectors.

Each Jordan block of $x$ can be reduced in size by 0 or 1 after restriction to $F_{n-1}$.
Recall

\[ \lambda_i = \sum_{j \geq i} \# j \times j \text{ Jordan blocks in } x. \]

Let

\[ \mu_i = \sum_{j \geq i} \# j \times j \text{ Jordan blocks in } x|_{F_{n-1}}. \]

Thus

\[ \lambda_i \geq \mu_i \geq \lambda_{i+1} \]

This is precisely the branching rule.
Further Directions

Geometric construction of representations can be extended to arbitrary symmetric Kac-Moody algebras using *quiver varieties* (Lusztig, Nakajima).

In type $D$ case, the natural restriction of reps is *not* multiplicity free.

However, there are different ways to construct same rep using geometry (in above, for $\mathfrak{sl}_n$, can add $n \times n$ Jordan blocks to $x$).

Above construction, suitably generalized, may yield only one copy of each realization, thus giving a natural way to deal with multiplicities and define Gelfand-Tsetlin type bases in other types.