COHOMOLOGY OF FLAG VARIETIES
FIELDS INSTITUTE WORKSHOP ON SCHUBERT VARIETIES AND SCHUBERT CALCULUS

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Abstract. In this introductory lecture, we discuss the cohomology ring of the full flag variety and note the relation to Schubert polynomials. We closely follow the presentation in [1]. The interested reader may find more details there.

1. Some facts on cohomology

We first state some properties of cohomology which will be used in what follows. Let Y be a nonsingular projective variety. We then have the following:

(1) An irreducible subvariety Z of codimension d in Y determines a cohomology class $[Z] \in H^{2d}(Y)$.
(2) If Y has dimension N, then $H^{2N}(Y) = \mathbb{Z}$, with the class of a point being a generator.
(3) If two subvarieties Z_1 and Z_2 of Y have complimentary dimension and meet transversally in t points, then the product of their classes is $t \in H^{2N}(Y) = \mathbb{Z}$. We write $\langle [Z_1], [Z_2] \rangle = t$.
(4) If Y has a filtration

$$Y = Y_0 \subset Y_1 \subset \cdots \subset Y_s = \emptyset$$

by closed algebraic subsets, and $Y_i \backslash Y_{i+1}$ is a disjoint union of varieties $U_{i,j}$, each isomorphic to an affine space $\mathbb{C}^{n(i,j)}$, then the classes $[\overline{U}_{i,j}]$ of the closures of these varieties yield an additive basis for $H^*(Y)$ over $\mathbb{Z}$.

We will also use the fact that any continuous map $f : X \to Y$ between two topological spaces defines a *pullback* homomorphism $f^* : H^iY \to H^iX$.

2. Schubert varieties

Fix once and for all a complex vector space E of dimension m. We are interested in the *(full) flag variety*

$$\mathcal{F}_m = \{E_\bullet = (E_1 \subset E_2 \subset \cdots \subset E_m = E) \mid \dim E_i = i\}.$$

Recall that this variety is split into cells called *Schubert cells* which are cut out of the flag variety by specifying the dimensions of the intersections of the steps in the flag E with the steps of some fixed flag. More precisely, fix a flag

$$F_1 \subset F_2 \subset \cdots \subset F_m = E, \ \dim F_q = q.$$

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When we identify E with $\mathbb{C}^m$ by picking a basis, we will take F_j to be the subspace spanned by the first j elements of this basis. Then, for any permutation w in the symmetric group S_m, we define the Schubert cell

$$X_w^\circ = \{E_\bullet \in \mathcal{F}_m \mid \dim(E_p \cap F_q) = \#\{i \le p : w(i) \le q\} \text{ for } 1 \le p, q \le m\}.$$

We can also describe the Schubert cells as follows. Every flag $E_\bullet \in X_w^\circ$ has E_p spanned by the first p rows of a unique row echelon matrix, where the p^{th} row has a 1 in the $w(p)^{\text{th}}$ column, with all 0's to the right of these 1's, and all 0's below these 1's. For example, if $w = 3 \quad 5 \quad 1 \quad 4 \quad 2$ in S_5, these matrices have the form

$$\begin{pmatrix} * & * & 1 & 0 & 0 \\ * & * & 0 & * & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where the stars denote arbitrary complex numbers. So we see that each Schubert cell is isomorphic to the affine space $\mathbb{C}^a$ where a is the number of stars appearing in the above description. It is an easy exercise to show that the number of stars is precisely the *length* of the permutation w defined by

$$l(w) = \#\{i < j \mid w(i) > w(j)\}.$$

Thus

$$X_w^\circ \cong \mathbb{C}^{l(w)},\ w \in S_m, \qquad \bigsqcup_{w \in S_m} X_w^\circ = \mathcal{F}_m$$

We also define the *dual Schubert cells* Ω_w° as follows. Let $\tilde{F}_q$ be the subspace of $E = \mathbb{C}^m$ spanned by the last q vectors of the basis. Then define

$$\Omega_w^\circ = \{E_\bullet \in \mathcal{F}_m \mid \dim(E_p \cap \tilde{F}_q) = \#\{i \le p : w(i) \ge m+1-q\} \text{ for } 1 \le p, q \le m\}.$$

In terms of the description above, Ω_w° consists of flags spanned by rows of a row echelon matrix with 1's in the $(p, w(p))^{\text{th}}$ position and 0's under these 1's, as before, but this time with 0's to the *left* of these 1's. Thus for the element $w = 3 \quad 5 \quad 1 \quad 4 \quad 2$ in S_5 considered above, these matrices are of the form

$$\begin{pmatrix} 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 \\ 1 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We see that $\Omega_w^\circ \cong \mathbb{C}^{n-l(w)}$, where $n = m(m-1)/2 = \dim \mathcal{F}_m$.

The *Schubert variety* X_w is defined to be the closure of the cell X_w°. Similarly, Ω_w is defined to be the closure of Ω_w°. These are irreducible subvarieties of $\mathcal{F}_m$ of dimensions $l(w)$ and $n - l(w)$, respectively. In particular,

$$X_w = \bigsqcup_{v \le w} X_v^\circ, \quad \Omega_w = \bigsqcup_{v \ge w} \Omega_v^\circ,$$

where $v \le w$ and $v \ge w$ are referring to the *Bruhat order* on S_m. Note that $v \le w$ implies $l(v) \le l(w)$.

For $1 \leq d \leq n$, let

$$Z_d = \bigcup_{w\,:\,l(w)\leq d} X_w^\circ = \bigcup_{w\,:\,l(w)\leq d} X_w.$$

So Z_d is a closed algebraic subspace of $\mathcal{F}_m$. Furthermore, $Z_d \backslash Z_{d-1}$ is a disjoint union of the cells $X_w^\circ \cong \mathbb{C}^d$ (i.e. those X_w° with $l(w) = d$). Thus, it follows from the facts in Section 1 that the classes of the closures of these cells give an additive basis for the cohomology of $\mathcal{F}_m$.

Now, if $w_0 = m \quad m-1 \quad \dots \quad 1$ in S_m, we have the following:

Lemma 2.1. *For $w \in S_m$, $[\Omega_w] = [X_{w^\vee}]$, where $w^\vee = w_0 w$ (equivalently, $w^\vee(i) = m+1-w(i)$ for $1 \leq i \leq m$).*

Combining this with the above, we see that the *Schubert classes*

$$\sigma_w = [\Omega_w] = [X_{w^\vee}] = [X_{w_0 w}] \in H^{2l(w)}(\mathcal{F}_m)$$

form an additive basis of $H^*(\mathcal{F}_m)$. Note, in particular, that the odd cohomology of $\mathcal{F}_m$ vanishes. In the next section, we will explore the multiplicative structure of the cohomology ring.

3. The cohomology ring of the flag variety

We have seen in Section 2 that the Schubert classes σ_w, $w \in S_m$ form an additive basis for the cohomology ring of the flag variety. We would now like to examine the multiplicative structure of this ring. In particular, we will give a presentation in term of generators and relations and express the Schubert classes in terms of these generators.

Recall that the intersection pairing is a bilinear map

$$H^{d_1}(\mathcal{F}_m) \times H^{d_2}(\mathcal{F}_m) \to H^{d_1+d_2}(\mathcal{F}_m).$$

Using the facts in Sections 1 and 2, one can show that for partitions v and w of equal length,

$$\langle [X_w], [\Omega_v] \rangle = \delta_{vw}.$$

In terms of the descriptions given in Section 2, the varieties $[X_w]$ and $[\Omega_w]$ intersect transversely at the point corresponding to the flag $E_\bullet$ with E_p spanned by the first p rows of the matrix with 1 in the entries $(p, w(p))$ and zeros elsewhere. Thus the basis consisting of the $[X_w]$ is dual to the basis given by the Schubert classes.

The cohomology ring of $\mathcal{F}_m$ is generated by some basic classes $x_1, \dots, x_m$ in $H^2(\mathcal{F}_m)$ which we now describe. There is a natural vector bundle U_i, $1 \leq i \leq m$, over $\mathcal{F}_m$ of rank i. The fiber of the bundle U_i over a point in $\mathcal{F}_m$ corresponding to a flag $E_\bullet$ is the vector space E_i of the flag. These bundles form a *universal* or *tautological* filtration

$$0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_m = E_{\mathcal{F}_m}.$$

Here $E_{\mathcal{F}_m} = E \times \mathcal{F}_m$ is the trivial bundle. We then form the line bundles

$$L_i = U_i / U_{i-1}.$$

Now, a line bundle L on a nonsingular projective variety X has a *first Chern class* $c_1(L)$ in $H^2(X)$. This class is equal to $[D]$ where D is the subvariety of X consisting of the zeros of a nice section of the line bundle L. We set

$$x_i = -c_1(L_i),\ 1 \leq i \leq m.$$

Recall that the k^{th} *elementary symmetric polynomial* $e_k(X_1, \dots, X_m)$ is the sum of all monomials $X_{i_1} \dots X_{i_k}$ for all strictly increasing sequences $1 \le i_1 < \dots < i_k \le m$.

Proposition 3.1. *The cohomology ring $H^*(\mathcal{F}_m)$ is generated by the basic classes $x_1, \dots, x_m$ subject to the relations $e_i(x_1, \dots, x_m) = 0$, $1 \le i \le m$. That is,*

$$H^*(\mathcal{F}_m) \cong R_m := \mathbb{Z}[X_1, \dots, X_m]/(e_1(X_1, \dots, X_m), \dots, e_m(X_1, \dots, X_m)).$$

The classes $x_1^{i_1} x_2^{i_2} \dots x_m^{i_m}$ with exponents $i_j \le m - j$ form a $\mathbb{Z}$-basis for $H^*(\mathcal{F}_m)$.

We now have two bases for the cohomology ring $H^*(\mathcal{F}_m)$. The first is the geometric basis $\{\sigma_w \mid w \in S_m\}$ and the second is the algebraic basis $\{x_1^{i_1} x_2^{i_2} \dots x_m^{i_m} \mid i_j \le m - j\}$. We would like to express the geometric basis in terms of the algebraic one.

There is a natural embedding $\iota : \mathcal{F}_m \hookrightarrow \mathcal{F}_{m+1}$ that sends a flag $E_\bullet$ in $E = \mathbb{C}^m$ to the following flag in $E' = E \oplus \mathbb{C} = \mathbb{C}^{m+1}$:

$$E_1 \oplus 0 \subset E_2 \oplus 0 \subset \dots \subset E_m \oplus 0 \subset E_m \oplus \mathbb{C} = E \oplus \mathbb{C} = E'.$$

This is a closed embedding and identifies $\mathcal{F}_m$ with the set of flags in E' with m^{th} member $E \oplus 0$. If we regard S_m as the subgroup of S_{m+1} fixing $m+1$, we see that for all $w \in S_m$, ι maps the Schubert cell X_w° in $\mathcal{F}_m$ isomorphically onto the Schubert cell X_w° in $\mathcal{F}_{m+1}$ and that $\iota(X_w)$ is the Schubert variety corresponding to w in $\mathcal{F}_m$. We also have the pullback homomorphism

$$\iota^* : H^{2d}(\mathcal{F}_{m+1}) \to H^{2d}(\mathcal{F}_m).$$

When we consider $\mathcal{F}_m$ for different m, we denote the element $\sigma_w \in H^{2l(w)}(\mathcal{F}_m)$ by $\sigma_w^{(m)}$.

Proposition 3.2. (1) *For $w \in S_m$, we have $\iota^*(\sigma_w^{(m+1)}) = \sigma_w^{(m)}$.*
(2) *We have $\iota^*(x_i) = x_i$ for $1 \le i \le m$ and $\iota^*(x_{m+1}) = 0$.*

Define a map from R_{m+1} to R_m by $X_i \mapsto X_i$ for $1 \le i \le m$ and $X_{m+1} \mapsto 0$. Then by the above proposition, the diagram

$$\begin{array}{ccc} R_{m+1} & \xrightarrow{\cong} & H^*(\mathcal{F}_{m+1}) \\ \downarrow & & \downarrow \\ R_m & \xrightarrow{\cong} & H^*(\mathcal{F}_m) \end{array}$$

commutes.

Proposition 3.3. *Let $w \in S_k$. There is a unique homogeneous polynomial of degree $l(w)$ in $\mathbb{Z}[X_1, \dots, X_k]$ that maps to $\sigma_w^{(m)}$ in $H^{2l(w)}(\mathcal{F}_m)$ for all $m \ge k$. We denote this polynomial by $\mathfrak{S}_w = \mathfrak{S}_w(X_1, \dots, X_k)$. It is called the* Schubert polynomial *corresponding to w.*

Thus the Schubert polynomials tell us how to write the geometric basis of the cohomology of the flag variety (given by the classes of Schubert varieties) in terms of the algebraic basis.

Schubert polynomials will be discussed in more detail in a later talk in this workshop. We mention here only a few examples.

$$\mathfrak{S}_{\text{id}} = 1,$$
$$\mathfrak{S}_{s_i} = X_1 + X_2 + \dots + X_i,$$
$$\mathfrak{S}_{w_0} = X_1^{m-1} X_2^{m-2} \dots X_{m-1}, \ w_0 = m \quad m-1 \quad \dots \quad 1 \in S_m.$$

Here id is the identity permutation and s_i is the permutation interchanging i and $i+1$.

If one multiplies two Schubert polynomials, the result can be written as a linear combination of Schubert polynomials:

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_w c_{u,v}^w \mathfrak{S}_w.$$

One can see from the geometry of flag varieties that the coefficients $c_{u,v}^w$ are nonnegative integers. While there are algorithms for computing these coefficients, there is not yet any combinatorial formula for these numbers, such as the Littlewood-Richardson rule for multiplying Schur polynomials which are the analogue of Schubert polynomials for the cohomology of the Grassmannian.

References

[1] W. Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.

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