Symmetric Functions

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Contents

Preface

These are notes for the topics course *Symmetric Functions* (MAT 4995/5327) at the University of Ottawa, taking place in Winter 2022. They will be updated as the course progresses.

We assume that the reader has a basic knowledge of the theory of rings and modules as covered, for example, in Rings and Modules (MAT 3143), or its French equivalent Anneaux et Modules (MAT 3543), at the University of Ottawa.

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Course website: `https://alistairsavage.ca/SymFunc/`

Chapter 1

Preliminaries

In this first chapter we cover some background that will be used throughout the course. In particular, we will discuss the symmetric group, partitions, compositions, graded rings, and formal power series. Our discussion will be brief. Throughout these notes we let

$$\mathbb{N} := \{0, 1, 2, \dots\}$$

denote the set of natural numbers, including zero.

1.1 The symmetric group

We begin recalling a few important facts about the symmetric group. For $n \in \mathbb{N}$, let $\mathfrak{S}_n$ denote the *symmetric group* on n letters. Thus $\mathfrak{S}_n$ is the group of permutations of

$$\{1, 2, \dots, n\}.$$

By convention $\mathfrak{S}_0$ is the trivial group. The group $\mathfrak{S}_n$ has order $n!$.

If $\pi \in \mathfrak{S}_n$, we can write it in *one-line notation as*

$$\pi = \pi(1)\,\pi(2)\,\cdots\,\pi(n).$$

Alternatively, we may write it in *cycle notation*, which describes the effect of repeatedly applying the permutation to the elements of $\{1, 2, \dots, n\}$.

Example 1.1.1. Consider the element $\pi \in \mathfrak{S}_6$ given by

$$\pi(1) = 6, \quad \pi(2) = 1, \quad \pi(3) = 4, \quad \pi(4) = 3, \quad \pi(5) = 5, \quad \pi(6) = 2.$$

Then in one-line notation we have

$$\pi = 6\,1\,4\,3\,5\,2 \tag{1.1}$$

and in cycle notation we have

$$\pi = (1\,6\,2)(3\,4)(5).$$

We sometimes omit cycles of length one in cycle notation, writing

$$\pi = (1\,6\,2)(3\,4).$$

The inverse of an element can be given easily in cycle notation by reversing the order of the cycles. For example,

$$\pi^{-1} = (2\,6\,1)(4\,3). \qquad \triangle$$

Note that we have a natural inclusion $\{1, 2, \dots, n\} \subseteq \{1, 2, \dots, n+1\}$. Thus, we can view $\mathfrak{S}_n$ as the groups of $\mathfrak{S}_{n+1}$ that fixes $n+1$. In this way we have a chain of groups

$$\mathfrak{S}_0 \subseteq \mathfrak{S}_1 \subseteq \mathfrak{S}_2 \subseteq \mathfrak{S}_3 \subseteq \cdots .$$

Define

$$\mathfrak{S}_\infty := \bigcup_{n \in \mathbb{N}} \mathfrak{S}_n.$$

Thus, $\mathfrak{S}_\infty$ is the set of all bijections $\pi \colon \mathbb{N} \to \mathbb{N}$ such that there exists $n \in \mathbb{N}$ with $\pi(m) = m$ for $m \geq n$.

Exercises.

1.1.1. Give an example of a bijection $\pi \colon \mathbb{N} \to \mathbb{N}$ such that $\pi \notin \mathfrak{S}_\infty$.

1.1.2. Suppose $\pi \in \mathfrak{S}_k$ and that, for $i \in \mathbb{N}$, π has m_i cycles of length i. Prove that the number of permutations $\sigma \in \mathfrak{S}_k$ that commute with π (that is, such that $\sigma\pi = \pi\sigma$) is

$$\prod_{i \geq 1} i^{m_i} m_i!.$$

Hint: Write π in cycle notation. What are the cycles of $\sigma\pi\sigma^{-1}$?

1.2 Partitions and compositions

In the theory of symmetric functions, many mathematical objects are labelled by partitions and compositions. In this section we define these concepts and prove some basic results concerning them that will be used in the sequel.

Definition 1.2.1 (Partition)**.** A *partition* λ of a nonnegative integer $k \in \mathbb{N}$ is a weakly decreasing sequence $(\lambda_j)_{j=1}^\infty$ of nonnegative integers such that $\sum_{j=1}^\infty \lambda_j = k$. If λ is a partition of k, then we often write $\lambda \vdash k$ and $|\lambda| = k$, and we sometimes call k the *size* of λ. We let $\mathrm{Par}(k)$ denote the set of partitions of k and let $\mathrm{Par} := \bigcup_{k \in \mathbb{N}} \mathrm{Par}(k)$ denote the set of all partitions. We denote the unique partition $(0, 0, \dots)$ of 0 by $\varnothing$.

A partition has finitely many nonzero terms, and we typically omit zero terms when writing λ. For example

$$(5,2,2,1), \qquad (5,2,2,1,0), \qquad (5,2,2,1,0,0), \qquad \text{and} \quad (5,2,2,1,0,0,\dots)$$

are all the same partition of 10, which we will usually write as $(5,2,2,1)$. The nonzero terms in a partition λ are called the *parts* of λ. The *length* of λ, denoted $\ell(\lambda)$, is the number of parts of λ. For $n \in \mathbb{N}$, we let

$$\mathrm{Par}_n(k) := \{\lambda \in \mathrm{Par}(k) : \ell(\lambda) \le n\} \qquad \text{and} \qquad \mathrm{Par}_n := \bigcup_{k\in\mathbb{N}} \mathrm{Par}_n(k).$$

We will often view elements λ of Par_n as weakly decreasing sequences $(\lambda_1, \dots, \lambda_n)$ of length n, since we have $\lambda_j = 0$ for $j > n$ by definition. It follows immediately from the definitions that

$$\mathrm{Par}_n(k) \subseteq \mathrm{Par}_m(k) \qquad \text{and} \qquad \mathrm{Par}_n \subseteq \mathrm{Par}_m \qquad \text{for} \quad n \le m$$

and that

$$\mathrm{Par}(k) = \bigcup_{n\in\mathbb{N}} \mathrm{Par}_n(k) \qquad \text{and} \qquad \mathrm{Par} = \bigcup_{n\in\mathbb{N}} \mathrm{Par}_n.$$

For any partition λ and $i \in \mathbb{N}$, the *multiplicity of i in λ*, denoted $m_i(\lambda)$, is the number of parts of λ equal to i. In other words,

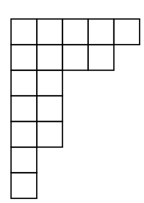
$$m_i(\lambda) = \#\{j \in \mathbb{N} : \lambda_j = i\},$$

where $\#X$ denotes the cardinality of a finite set X. We will sometimes write a partition by writing each of its distinct parts once, with an exponent indicating the multiplicity of that part. For example we might write

$$(8,8,7,5,5,5,5,3,2,2,1,1,1) \qquad \text{as} \qquad (8^2, 7^1, 5^4, 3^1, 2^2, 1^3).$$

We will often think about partitions geometrically as follows. The *Young diagram*, also known as the *Ferrers diagram*, of a partition λ is a stack of boxes in which each row is left-justified and the j-th row has λ_j boxes. We will use the English convention, in which the first row is the top row. Thus, the Young diagram of the partition $(5,4,2,2,2,1,1)$ is

(1.2)



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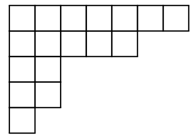
Some references use the French convention, where the first row is the bottom row. (There is also a Russian convention!) We will often identify a partition with its Young diagram.

The *conjugate* of a partition λ is the partition λ' whose Young diagram is the transpose of the Young diagram of λ, that is, the Young diagram obtained by reflection in the main diagonal. For example if $\lambda = (5, 4, 2, 2, 2, 1, 1)$, whose Young diagram is given in (1.2), then the conjugate partition $\lambda' = (7, 5, 2, 2, 1)$ has Young diagram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It follows that λ'_j is the number of boxes in the j-th column of λ, or equivalently

$$\lambda'_j = \#\{i : \lambda_i \geq j\}. \tag{1.3}$$

In particular, we have $\lambda'_1 = \ell(\lambda)$ and $\lambda_1 = \ell(\lambda')$. We also have $\lambda'' = \lambda$.

For $k \in \mathbb{N}$, we define the *dominance partial ordering* (sometimes called the *natural partial ordering*) $\leq$ on $\mathrm{Par}(k)$ as follows: for $\lambda, \mu \in \mathrm{Par}$,

$$\lambda \leq \mu \iff \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i \qquad \text{for all } i \geq 1. \tag{1.4}$$

We write $\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$.

Example 1.2.2. If $k = 6$, we have

$$(1^6) < (2, 1^4) \qquad \text{and} \qquad (2, 2, 2) < (4, 2).$$

However, $(3, 1^3)$ and (2^3) are not comparable. △

Definition 1.2.3 (Weak composition)**.** A *weak composition* α of $k \in \mathbb{N}$ is a sequence $(\alpha_j)_{j=1}^{\infty}$ of nonnegative integers such that $\sum_{j=1}^{\infty} \alpha_j = k$. (By contrast, a *composition* of $k \in \mathbb{N}$ is a finite sequence $(\alpha_j)_{j=1}^{n}$ of *positive* integers such that $\sum_{j=1}^{n} \alpha_j = k$.) We let $\mathrm{WComp}(k)$ denote the set of weak compositions of k and let $\mathrm{WComp} = \bigcup_{k \in \mathbb{N}} \mathrm{WComp}(k)$ denote the set of all weak compositions. We denote the unique composition $(0, 0, \dots)$ of 0 by $\varnothing$. For $\alpha \in \mathrm{WComp}$, we define

$$|\alpha| = \sum_{j=1}^{\infty} \alpha_j.$$

For $k, n \in \mathbb{N}$, we define

$$\mathrm{WComp}_n(k) := \{\alpha \in \mathrm{WComp}(k) : \alpha_j = 0 \text{ for } j > n\} \qquad \text{and}$$
$$\mathrm{WComp}_n := \bigcup_{k \in \mathbb{N}} \mathrm{WComp}_n(k).$$

As for partitions, we will often view elements α of WComp_n as sequences $(\alpha_1, \dots, \alpha_n)$ of length n. We have

$$\mathrm{WComp}_m(k) \subseteq \mathrm{WComp}_n(k) \qquad \text{and} \qquad \mathrm{WComp}_m \subseteq \mathrm{WComp}_n \qquad \text{for} \quad m \leq n$$

and

$$\mathrm{WComp}(k) = \bigcup_{n \in \mathbb{N}} \mathrm{WComp}_n(k) \qquad \text{and} \qquad \mathrm{WComp} = \bigcup_{n \in \mathbb{N}} \mathrm{WComp}_n.$$

Note that every partition is a weak composition but not the other way around since we require partitions to be *weakly decreasing* sequences. For example $(2, 0, 1, 1)$ is a weak composition of 5, but is not a partition of 5.

The symmetric group $\mathfrak{S}_n$ acts on the set $\mathrm{WComp}_n(k)$ as follows. For $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\pi \in \mathfrak{S}_n$, define

$$\alpha^\pi := (\alpha_{\pi(1)}, \alpha_{\pi(2)}, \dots, \alpha_{\pi(n)}). \tag{1.5}$$

This defines a right action of $\mathfrak{S}_n$ on $\mathrm{WComp}_n(k)$; see Exercise 1.2.1. For $\alpha \in \mathrm{WComp}_n(k)$, we define

$$\alpha\mathfrak{S}_n := \{\alpha^\pi : \pi \in \mathfrak{S}_n\}. \tag{1.6}$$

(We use the notation $\alpha\mathfrak{S}_n$ instead of $\mathfrak{S}_n\alpha$ since we have a *right* action of $\mathfrak{S}_n$.) For every $\alpha \in \mathrm{WComp}_n(k)$, there exists a permutation $\pi \in \mathfrak{S}_n$ such that $\alpha^\pi \in \mathrm{Par}_n(k)$. (This permutation π is unique if and only if α has no repeated terms.) It follows that

$$\bigcup_{\alpha \in \mathrm{Par}_n(k)} \alpha\mathfrak{S}_n = \{\alpha^\pi : \alpha \in \mathrm{Par}_n(k),\ \pi \in \mathfrak{S}_n\} = \mathrm{WComp}_n(k).$$

Similarly, the set $\mathfrak{S}_\infty$ acts on the set $\mathrm{WComp}(k)$. Precisely, for $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathrm{WComp}(k)$ and $\pi \in \mathfrak{S}_\infty$, we define

$$\alpha^\pi := (\alpha_{\pi(1)}, \alpha_{\pi(2)}, \dots) \qquad \text{and} \qquad \alpha\mathfrak{S}_\infty = \{\alpha^\pi : \pi \in \mathfrak{S}_\infty\}.$$

For every $\alpha \in \mathrm{WComp}(k)$, there exists a (not unique!) permutation $\pi \in \mathfrak{S}_\infty$ such that $\alpha^\pi \in \mathrm{Par}(k)$ and we have

$$\bigcup_{\alpha \in \mathrm{Par}(k)} \alpha\mathfrak{S}_\infty = \{\alpha^\pi : \alpha \in \mathrm{Par}(k),\ \pi \in \mathfrak{S}_\infty\} = \mathrm{WComp}(k).$$

The dominance partial ordering (1.4) can be naturally extended to a partial order on weak compositions. (Note that we will still reserve the term *dominance partial ordering* to refer to partitions only.) Namely, for $\alpha, \beta \in \mathrm{WComp}$, we define

$$\alpha \le \beta \iff \alpha_1 + \cdots + \alpha_i \le \beta_1 + \cdots + \beta_i \qquad \text{for all} \quad i \in \mathbb{N}. \tag{1.7}$$

Lemma 1.2.4. *Suppose* $\alpha \in \mathrm{WComp}$. *Then*

$$\alpha \in \mathrm{Par} \iff \alpha^\pi \le \alpha \quad \textit{for all} \quad \pi \in \mathfrak{S}_\infty.$$

Proof. Suppose that $\alpha \in \mathrm{Par}$. Then $\alpha_1 \ge \alpha_2 \ge \cdots$. Suppose $\beta = \alpha^\pi$ for some $\pi \in \mathfrak{S}_\infty$. For $i \in \mathbb{N}$, $\alpha_1 + \cdots + \alpha_i$ is the sum of the i greatest elements of α, which are the same as the i greatest elements of β. Thus

$$\beta_1 + \cdots + \beta_i \le \alpha_1 + \cdots + \alpha_i.$$

Hence $\alpha^\pi = \beta \le \alpha$.

Conversely, suppose $\alpha^\pi \le \alpha$ for all $\pi \in \mathfrak{S}_\infty$. Then, for $i \in \mathbb{N}$, we have

$$(\alpha_1, \alpha_2, \dots, \alpha_{i-i}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \alpha_{i+3}, \dots) \le \alpha.$$

It follows that, for $i \ge 2$,

$$\alpha_1 + \cdots + \alpha_{i-1} + \alpha_{i+1} \le \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i$$

and so $\alpha_{i+1} \le \alpha_i$. Hence $\alpha \in \mathrm{Par}$. $\square$

Exercises.

1.2.1. If G is a group, let G^{op} denote the opposite group. Thus,

$$G^{\text{op}} = \{g^{\text{op}} : g \in G\}, \quad g^{\text{op}}h^{\text{op}} = (hg)^{\text{op}}, \quad g, h \in G.$$

A *right action* of a group G on a set X is a group homomorphism $G^{\text{op}} \to \mathfrak{S}_X$, where $\mathfrak{S}_X$ denotes the group of permutations of X. By contrast, a *left action* of G on R is a group homomorphism $G \to \mathfrak{S}_X$.

(a) Show that (1.5) defines a right action of $\mathfrak{S}_k$ on $\text{WComp}_k(n)$. In other words, show that

$$\alpha^{\pi_1\pi_2} = (\alpha^{\pi_1})^{\pi_2}$$

for all $\alpha \in \text{WComp}_k(n)$ and $\pi_1, \pi_2 \in \mathfrak{S}_n$.

(b) Show that, for $k \geq 3$, (1.5) does *not* define a left action of $\mathfrak{S}_k$ on $\text{WComp}_k(n)$.

(c) How could you modify (1.5) to give a natural left action of $\mathfrak{S}_k$ on $\text{WComp}_k(n)$?

1.2.2. Prove that, for $k \leq 5$, the dominance partial ordering on $\text{Par}(k)$ is a *total* ordering.

1.2.3. A partition λ is *self-conjugate* if $\lambda' = \lambda$. Prove that, for $k \in \mathbb{N}$, the number of self-conjugate partitions of k is equal to the number of partitions of k into distinct, odd parts. *Hint*: Think about Young diagrams.

1.2.4. For $k \in \mathbb{N}$, define the *lexicographic ordering* $<_{\text{lex}}$ on $\text{Par}(k)$ as follows: For $\lambda, \mu \in \text{Par}(k)$, we write $\lambda <_{\text{lex}} \mu$ if and only if there exists some $i \in \mathbb{N}$ such that

$$\lambda_j = \mu_j \text{ for all } j < i, \quad \text{and } \lambda_i < \mu_i.$$

This is a total order on $\text{Par}(k)$. Prove that, for $\lambda, \mu \in \text{Par}(k)$,

$$\lambda < \mu \implies \lambda <_{\text{lex}} \mu.$$

Thus, the lexicographic order is a refinement of the dominance ordering.

1.3 Graded rings

In these notes we will assume all rings are unital (i.e. they posses a multiplicative identity). In this section we discuss the notion of a graded ring, which plays a crucial role in theory of symmetric functions.

Definition 1.3.1 (Graded ring)**.** A *graded ring* is a ring R with a direct sum decomposition

$$R = \bigoplus_{k \in \mathbb{N}} R_k \tag{1.8}$$

such that each R_k is an additive subgroup of R, and such that

$$R_k R_l \subseteq R_{k+l} \quad \text{for all } k, l \in \mathbb{N}.$$

A nonzero element of R_k is said to be *homogeneous of degree k*. If R is any ring, we say that the decomposition (1.8) is a *gradation* on R if this decomposition makes it a graded ring.

Remarks 1.3.2. One can consider rings graded by any abelian monoid. Since our main example of interest is grading by $\mathbb{N}$, we stick to that case here. Later we will use superscripts to denote the gradation, since our rings will already have subscripts. △

Examples 1.3.3. (a) Any ring R can be given a gradation by letting $R_0 = R$ and $R_i = 0$ for $i > 0$. This is called the *trivial gradation* on R.

(b) The polynomial ring $\mathbb{Z}[x]$ is graded by degree. More precisely, we have a gradation $\mathbb{Z}[x] = \bigoplus_{k\in\mathbb{N}} \mathbb{Z}[x]_k$, where $\mathbb{Z}[x]_k$ consists of the homogeneous polynomials of degree k, together with the zero polynomial.

(c) The polynomial ring $\mathbb{Z}[x, y]$ is also graded by total degree. △

If R is a graded ring, then by the definition of direct sum, every element $r \in R$ can be written *uniquely* as a sum

$$r = \sum_{i\in\mathbb{N}} r_i, \qquad r_i \in R_i,$$

where only *finitely* many of the r_i are nonzero. The r_i are called the *homogeneous components of r*.

Definition 1.3.4 (Graded subring)**.** Suppose R is a graded ring. A *graded subring* of R is a subring S of R such that

$$S = \bigoplus_{k\in\mathbb{N}} S_k, \qquad S_k = S \cap R_k. \tag{1.9}$$

Equivalently, S is a graded subring of R if it is a subring of R and the homogeneous components of all elements of S are again elements of S. If S is a graded subring of R, then it is itself a graded ring with gradation (1.9).

Example 1.3.5. Recall that, if R is a ring and $X \subseteq R$, then the subring of R generated by X is the smallest subring of R containing X. Consider the case $R = \mathbb{Z}[x, y]$, graded by total degree. Then the subring generated by $x + y$, which consists of all polynomials in $x + y$ with integer coefficients, is a graded subring of R, while the subring generated by $x + y^2$ is *not* a graded subring of R. See Exercise 1.3.1. △

Definition 1.3.6. Suppose R is a graded ring. A *graded ideal* of R is an ideal I of R such that

$$I = \bigoplus_{k\in\mathbb{N}} I_k, \qquad I_k = I \cap R_k.$$

Equivalently, I is a graded ideal of R if it is an ideal of R and the homogeneous components of all elements of I are again elements of I.

Example 1.3.7. Consider the ring $\mathbb{Z}[x,y]$, graded by total degree. Then the ideal $I = x\mathbb{Z}[x,y]$ generated by x is a graded ideal of $\mathbb{Z}[x,y]$. Precisely, we have

$$I = \bigoplus_{k\in\mathbb{N}} I_k, \qquad I_k = I \cap \mathbb{Z}[x,y]_k = x\mathbb{Z}[x,y]_{k-1}.$$

However, the ideal $(x+1)\mathbb{Z}[x,y]$ is *not* a graded ideal. See Exercise 1.3.2. △

Definition 1.3.8 (Homomorphism of graded rings)**.** Suppose R and S are graded rings. A *homomorphism of graded rings* from R to S is a ring homomorphism $\sigma\colon R \to S$ such that

$$\sigma(R_k) \subseteq S_k \quad \text{for all } k \in \mathbb{N}.$$

If, in addition, σ is an isomorphism, then we say it is an *isomorphism of graded rings*. A homomorphism of graded rings $\sigma\colon R \to R$ is an *endomorphism of graded rings*; it is an *automorphism of graded rings* if it is also an isomorphism.

If $\sigma\colon R \to S$ is a homomorphism of graded rings, then we have its *homogeneous components*

$$\sigma_k\colon R_k \to S_k, \qquad k \in \mathbb{N},$$

obtained from the restriction of σ to R_k. These are homomorphisms of additive groups. We often write $\sigma = \sum_{k\in\mathbb{N}} \sigma_k$ since

$$\sigma(r) = \sum_{k\in\mathbb{N}} \sigma_k(r_k), \qquad r \in R.$$

(Here $r = \sum_{k\in\mathbb{N}} r_k$ is the decomposition of r into its homogeneous components.) This sum is always well defined since only finitely many terms are nonzero.

Lemma 1.3.9. *If $\sigma\colon R \to S$ is a homomorphism of graded rings, then its kernel* $\ker\sigma$ *is a graded ideal of R.*

Proof. We know from ring theory that $\ker\sigma$ is an ideal of R. It remains to show that the homogeneous components of $\ker\sigma$ are again elements of $\ker\sigma$. Suppose $r = \sum_{k\in\mathbb{N}} r_k \in \ker\sigma$. Then we have

$$0 = \sigma(r) = \sum_{k\in\mathbb{N}} \sigma(r_k).$$

Since σ is a homomorphism of graded rings, we have $\sigma(r_k) \in S_k$ for all $k \in \mathbb{N}$. Since the sum $S = \bigoplus_{k\in\mathbb{N}} S_k$ is direct, this implies that $\sigma(r_k) = 0$ for all $k \in \mathbb{N}$. In other words $r_k \in \ker\sigma$ for all $k \in \mathbb{N}$, as desired. □

Example 1.3.10. Suppose R is a commutative ring and consider the polynomial ring $R[x]$, graded by degree. For any $a \in R$, we have the evaluation homomorphism

$$R[x] \to R, \quad f(x) \mapsto f(a).$$

Since x has degree 1 and a has degree zero if $a \neq 0$, the evaluation homomorphism is graded if and only if $a = 0$. △

Proposition 1.3.11. *If R is a graded ring and X is a set of graded ring automorphisms of R, then R^X*

$$R^X := \{r \in R : \sigma(r) = r \text{ for all } \sigma \in X\}$$

is a graded subring of R.

Proof. We leave it as an exercise (Exercise 1.3.3) to verify that R^X is a subring of R. It remains to show that the homogenous components of elements of R^X are again elements of R^X. Suppose $r \in R^X$. Then, for $\sigma \in X$,

$$\sum_{k \in \mathbb{N}} r_k = r = \sigma(r) = \sum_{k \in \mathbb{N}} \sigma(r_k).$$

Since $\sigma(r_k) \in R_k$, it follows from the uniqueness of the decomposition of an element of R as a sum of homogeneous components that $\sigma(r_k) = r_k$. Hence $r_k \in R^X$ for all $k \in \mathbb{N}$, as desired. □

Many of the important results on rings can be generalized to the setting of graded rings. For example, the quotient of a graded ring by a graded ideal is naturally a graded ring (see Exercise 1.3.7) and one has graded versions of the isomorphism theorems for rings.

We conclude this section with a brief discussion of associative algebras, which are more general than rings. We fix a commutative ring $\Bbbk$, which is often called the *ground ring*. An *associative* $\Bbbk$*-algebra* R is a ring that is also a $\Bbbk$-module in such a way that the two additions (the ring addition and the module addition) are the same operation and the scalar multiplication satisfies

$$a(xy) = (ax)y = x(ay) \qquad \text{for all} \quad a \in \Bbbk,\ x, y \in R.$$

In these notes, the term $\Bbbk$*-algebra* will always mean associative $\Bbbk$-algebra. Since a $\mathbb{Z}$-module is the same as an abelian group, a $\mathbb{Z}$-algebra is simply a ring. This is the sense in which associative algebras generalize rings.

Examples 1.3.12.
(a) The ring of polynomials $\Bbbk[x]$ with coefficients in $\Bbbk$ is a $\Bbbk$-algebra.
(b) For $n \geq 1$, the ring of matrices $\mathrm{Mat}_n(\Bbbk)$ with entries in $\Bbbk$ is a $\Bbbk$-algebra.
(c) If G is a finite group, the group ring $\Bbbk G$ is a $\Bbbk$-algebra. △

A *graded* $\Bbbk$*-algebra* is a $\Bbbk$-algebra that is also graded as a ring. All of the results of this section on graded rings have natural analogues for $\Bbbk$-algebras.

Examples 1.3.13. We can repeat Examples 1.3.3 with $\Bbbk$-algebras in place of rings.

(a) Any $\Bbbk$-algebra R can be given a gradation by letting $R_0 = R$ and $R_i = 0$ for $i > 0$. This is called the *trivial gradation* on R.
(b) The $\Bbbk$-algebras $\Bbbk[x]$ and $\Bbbk[x, y]$ are graded by degree. △

Exercises.

1.3.1. Consider the ring $\mathbb{Z}[x, y]$ graded by total degree.

(a) Show that the subring generated by $x + y$ is a graded subring of R.

(b) Show that the subring generated by $x + y^2$ is not a graded subring of R.

1.3.2. Prove that $(x + 1)\mathbb{Z}[x, y]$ is not a graded ideal of $\mathbb{Z}[x, y]$, where the gradation is given by total degreee.

1.3.3. Suppose R is a ring and X is a set of ring automorphisms. Show that

$$R^X := \{r \in R : \sigma(r) = r \text{ for all } \sigma \in X\}$$

is a subring of R.

1.3.4. Prove that the multiplicative unit in a graded ring is homogeneous of degree zero.

1.3.5. Suppose u is a homogeneous invertible element in a graded ring R. Show that u has degree zero.

1.3.6. Suppose u is a homogeneous invertible element in a graded ring R. Show that the map

$$\rho_u \colon R \to R, \qquad \rho_u(r) = uru^{-1}, \quad r \in R,$$

is a graded ring automorphism of R.

1.3.7. Suppose that I is a graded ideal of a graded ring R. Show that $\bigoplus_{k \in \mathbb{N}} R_k / I_k$ has a natural induced structure of a graded ring.

1.4 Formal power series

In this section we briefly introduce the concept of formal power series. We will use this formalism in the sequel to concisely encode relations between various sequences of symmetric functions. Throughout this section R denotes a commutative ring.

The *ring of formal power series with coefficients in R* is

$$R[\![t]\!] := \left\{ \sum_{n=0}^{\infty} a_n t^n : a_n \in R \text{ for all } n \in \mathbb{N} \right\}.$$

More precisely, elements of $R[\![t]\!]$ are functions $f \colon \mathbb{N} \to R$, where we write such a function as $\sum_{n=0}^{\infty} f(r) t^n$. The unit element of R is 1. (Here we are writing 1 to denote the formal power series $\sum a_n t^n$, where $a_0 = 1$, the multiplicative identity of R, and $a_n = 0$ for $r > 0$.) Addition and multiplication are given by

$$\sum_{n=0}^{\infty} a_n t^n + \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} (a_n + b_n) t^n,$$
$$\left(\sum_{n=0}^{\infty} a_n t^n\right)\left(\sum_{n=0}^{\infty} b_n t^n\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_i b_{n-i}\right) t^n. \tag{1.10}$$

Note that, for the multiplication of two formal power series, the coefficient of each t^n involves a *finite* sum and hence is well defined.

The use of the word *formal* in our terminology comes from the fact that we do not worry ourselves with issues of convergence. The formal power series $\sum_{n=0}^{\infty} a_n t^n$ is sometimes called the *generating function* of the sequence $(a_n)_{n=0}^{\infty}$. Note that we have a natural inclusion of rings

$$R[t] \subseteq R[\![t]\!].$$

We denote by $R^\times$ the group of units of R:

$$R^\times = \{a \in R : ab = 1 \text{ for some } b \in R\}.$$

The following proposition describes the units of $R[\![t]\!]$.

Proposition 1.4.1. *We have*

$$R[\![t]\!]^\times = \left\{\sum_{n=0}^{\infty} a_n t^n \in R[\![t]\!] : a_0 \in R^\times\right\}.$$

Proof. Let $a(t) = \sum_{n=0}^{\infty} a_n t^n \in R[\![t]\!]$. If $a(t) \in R[\![t]\!]^\times$, then there exists $\sum_{n=0}^{\infty} b_n t^n \in R[\![t]\!]$ such that

$$1 = \left(\sum_{n=0}^{\infty} a_n t^n \in R[\![t]\!]\right)\left(\sum_{n=0}^{\infty} b_n t^n \in R[\![t]\!]\right) = a_0 b_0 + \sum_{n=1}^{\infty}\left(\sum_{i=0}^{n} a_i b_{n-i}\right) t^n.$$

It follows that $a_0 b_0 = 1$, and hence $a_0 \in R^\times$.

Conversely, suppose that $a_0 \in R^\times$. Then we define a sequence $(b_n)_{n=0}^{\infty}$ recursively by

$$b_0 = a_0^{-1}, \qquad b_n = -a_0^{-1} \sum_{i=1}^{n} a_i b_{n-i}, \quad n \geq 1.$$

It is then straightforward to verify that

$$\left(\sum_{n=0}^{\infty} a_n t^n\right)\left(\sum_{n=0}^{\infty} b_n t^n\right) = 1.$$ □

Example 1.4.2. Suppose $a \in R$. Then, by Proposition 1.4.1, $1 - at \in R[\![t]\!]^\times$. We leave it as an exercise (Exercise 1.4.3) to show that

$$(1 - at)^{-1} = \sum_{n=0}^{\infty} a^n t^n. \tag{1.11}$$ △

For $n \in \mathbb{N}$, let $p(n)$ denote the number of partitions of n. No simple explicit formula for $p(n)$ is known. However, the sequence $(p(n))_{n=0}^{\infty}$ satisfies some recurrence relations that can be used to efficiently compute $p(n)$. As the following result shows, these recurrence relations can be encoded as a product expression for the generating function of the sequence.

Proposition 1.4.3. *If $p(n;k)$ denotes the number of partitions $\lambda \vdash n$ with largest part $\lambda_1 \leq k$, then*

$$\sum_{n=0}^{\infty} p(n;k)t^n = \prod_{j=1}^{k} \frac{1}{1-t^j}. \tag{1.12}$$

Proof. We prove the result by induction on k. When $k = 0$, both sides of (1.12) are equal to 1. Now, for $k \geq 1$, we have

$$p(n;k) = p(n;k-1) + p(n-k,k-1) + p(n-2k,k-1) + \cdots ,$$

where $p(n-lk,k-1)$ counts the number of partitions $\lambda \vdash n$ with $m_k(\lambda) = l$. Thus

$$p(n;k)t^n = p(n;k-1)t^n + t^k p(n-k,k-1)t^{n-k} + t^{2k} p(n-2k,k-1)t^{n-2k} + \cdots .$$

Summing over n gives

$$\sum_{n=0}^{\infty} p(n;k)t^n = \left(1 + t^k + t^{2k} + \cdots\right) \sum_{n=0}^{\infty} p(n;k-1)t^n = \prod_{j=1}^{k} \frac{1}{1-t^j},$$

where we used the induction hypothesis in the last step. □

Taking the limit at $k \to \infty$ then gives a generating function for the sequence $(p(n))_{n=0}^{\infty}$.

Corollary 1.4.4. *We have*

$$\sum_{n=0}^{\infty} p(n)t^n = \prod_{j=1}^{\infty} \frac{1}{1-t^j}. \tag{1.13}$$

Proof. First note that the right-hand side of (1.13) a well-defined formal power series, even though it involves an infinite product. This is because, to compute the coefficient of t^k in $\prod_{j=1}^{\infty} \frac{1}{1-t^j}$, only the factors $\frac{1}{1-t^j}$ for $j \leq k$ contribute. Then the result follows from (1.12) and the fact that $p(n;k) = p(n)$ for $k \geq n$. □

Our discussion of formal power series can be generalized to any number of indeterminates, even infinitely many. Since the case of countably many indeterminates $x_1, x_2, \ldots$ will be important for us, let us discuss it here. It is useful to use *multi-index notation* for monomials. For $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathrm{WComp}$, we define the monomial

$$x^{\alpha} := \prod_{i=1}^{\infty} x_i^{\alpha_i}. \tag{1.14}$$

Note that since all but finitely many of the exponents α_i are equal to zero, this is actually a finite product. The degree of this monomial is $|\alpha|$. For example

$$x^{(5,3,0,4)} = x_1^5 x_2^3 x_4^4 \quad \text{has degree} \quad |(5,3,0,4)| = 12.$$

Define

$$R[\![x_1, x_2, \dots]\!] := \left\{ \sum_{\alpha \in \mathrm{WComp}} a_\alpha x^\alpha : a_\alpha \in R \text{ for all } \alpha \in \mathrm{WComp} \right\}. \tag{1.15}$$

This is a ring with multiplication given by

$$\left(\sum_{\alpha \in \mathrm{WComp}} a_\alpha x^\alpha \right) \left(\sum_{\beta \in \mathrm{WComp}} b_\beta x^\beta \right) = \sum_{\alpha,\beta \in \mathrm{WComp}} a_\alpha b_\beta x^{\alpha+\beta}, \tag{1.16}$$

where $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots)$. The infinite sum on the right-hand side of (1.16) is well defined since, for any $\gamma \in \mathrm{WComp}$, there exists only finitely many pairs (α, β) such that $\alpha + \beta = \gamma$, and so the coefficient of x^γ is finite.

For $k \in \mathbb{N}$, define

$$R[\![x_1, x_2, \dots]\!]^k := \left\{ \sum_{\alpha \in \mathrm{WComp}(k)} a_\alpha x^\alpha : a_\alpha \in R \text{ for all } \alpha \in \mathrm{WComp}(k) \right\}. \tag{1.17}$$

This is the R-submodule of $R[\![x_1, x_2, \dots]\!]$ consisting of zero and formal power series of degree k. Note that

$$R[\![x_1, x_2, \dots]\!] \neq \bigoplus_{k \in \mathbb{N}} R[\![x_1, x_2, \dots]\!]^k, \tag{1.18}$$

since the right-hand side consists of formal power series of bounded total degree. For example,

$$\sum_{n=0}^{\infty} x_1^n \in R[\![x_1, x_2, \dots]\!] \qquad \text{but} \qquad \sum_{n=0}^{\infty} x_1^n \notin \bigoplus_{k \in \mathbb{N}} R[\![x_1, x_2, \dots]\!]^k.$$

Exercises.

Throughout these exercises, R denotes a commutative ring.

1.4.1. The ring of *formal Laurent series with coefficients in R* is

$$R(\!(t)\!) = \left\{ \sum_{n=N}^{\infty} a_n t^n : N \in \mathbb{Z},\ a_n \in R \text{ for all } n \in \mathbb{N} \right\}.$$

Note that the exponents n appearing in any given element of $R(\!(t)\!)$ are bounded below (by some $N \in \mathbb{Z}$), but we allow arbitrarily low bounds.

(a) Show that $R((t))$ is a ring with the usual addition and multiplication (that is, as in (1.10), but with lower bound N in the sums).

(b) Show that

$$\left\{ \sum_{n=-\infty}^{\infty} a_n t^n : a_n \in R \text{ for all } n \in \mathbb{Z} \right\}$$

is *not* a ring under the usual addition and multiplication.

1.4.2. Is the ring $R[\![t]\!]$ graded by degree?

1.4.3. Prove (1.11).

1.4.4. Recall that the *Fibonacci sequence* $(f_n)_{n=0}^{\infty}$ is defined recursively by

$$f_0 = f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2.$$

(a) Use the recursion relation to find a generating function for the Fibonacci sequence.

(b) Show that, for $n \in \mathbb{N}$,

$$f_n = \frac{a^n - b^n}{\sqrt{5}}, \qquad \text{where} \quad a = \frac{1+\sqrt{5}}{2},\ b = \frac{1-\sqrt{5}}{2}.$$

The next four exercises aim to justify some manipulations we will do with power series in the sequel. For the remainder of the exercises we work over a field $\Bbbk$ and fix a formal power series $A(t) = \sum_{n \geq 0} a_n t^n \in \Bbbk[\![t]\!]$.

1.4.5. If $a_0 = 1$, show that

$$A(t)^{-1} = \sum_{n=0}^{\infty} (1 - A(t))^n.$$

Hint: Write $A(t) = 1 - (1 - A(t))$ and compare to (1.11).

1.4.6. Recall the Taylor series for the natural logarithm: $\log(1+x) = \sum_{n \geq 1} (-1)^{n-1} \frac{x^n}{n}$. If $a_0 = 1$ (so that the constant term of $A(t)$ is 1), we define

$$\log A(t) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(A(t)-1)^n}{n}.$$

Prove that this is indeed a well-defined formal power series.

1.4.7. Define the formal derivative of $A(t)$ to be

$$A'(t) = \frac{\mathrm{d}}{\mathrm{d}t} A(t) := \sum_{n \geq 1} n a_n t^{n-1}.$$

If $a_1 = 1$, prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} \log A(t) = A'(t)/A(t).$$

1.4.8. (a) Suppose $a_0 = 0$. Prove that

$$\exp A(t) := \sum_{n\geq 0} \frac{1}{n!} A(t)^n$$

is a well-defined formal power series and that its constant term is 1. Furthermore, show that

$$\exp \sum_{n\geq 1} a_n t^n = \prod_{n\geq 1} \exp\left(a_n t^n\right).$$

(b) Show that, if $a_0 = 0$, then

$$\log \exp A(t) = A(t).$$

(c) Show that if $a_0 = 1$, then

$$\exp \log A(t) = A(t).$$

Chapter 2

The ring of symmetric functions

In this chapter we introduce the main object of study in the course: the ring of symmetric functions. We will give two descriptions of this ring, one as a certain type of limit and another more direct definition. Both definitions will be useful in what follows. After defining the ring of symmetric functions, we will introduce several important bases and generating sets for this ring.

2.1 Symmetric polynomials

Let

$$\mathrm{Pol}_n := \mathbb{Z}[x_1, \dots, x_n]$$

denote the ring of polynomials in n indeterminates with integer coefficients. This is a unique factorization domain. It is also a graded ring with gradation

$$\mathrm{Pol}_n = \bigoplus_{k \geq 0} \mathrm{Pol}_n^k,$$

where Pol_n^k consists of homogeneous polynomials of degree k, together with the zero polynomial. (Note that we use superscripts to indicate the grading, instead of subscripts as in Section 1.3.)

It is useful to use *multi-index notation* for monomials. For $\alpha = \mathrm{WComp}_n$, we define the monomial

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

The degree of this monomial is

$$|\alpha| := \sum_{i=1}^{n} \alpha_i.$$

The monomials x^α, $\alpha \in \mathrm{WComp}_n$, form a $\mathbb{Z}$-basis for Pol_n. Similarly, the monomials x^α, $\alpha \in \mathrm{WComp}_n(k)$, form a basis for Pol_n^k.

For $\pi \in \mathfrak{S}_n$, the map

$$f \mapsto f^\pi := f(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)}), \qquad f \in \mathrm{Pol}_n, \tag{2.1}$$

is a ring automorphism of Pol_n. This defines a right action of $\mathfrak{S}_n$ on Pol_n; see Exercise 2.1.1. We say that a polynomial $f \in \mathrm{Pol}_n$ is *symmetric* if it is invariant under the action of $\mathfrak{S}_n$. In other words, f is symmetric if

$$f^\pi = f \quad \text{for all } \pi \in \mathfrak{S}_n.$$

The *ring of symmetric polynomials* is the subring of Pol_n consisting of symmetric polynomials:

$$\mathrm{Sym}_n := \mathrm{Pol}_n^{\mathfrak{S}_n} := \{f \in \mathrm{Pol}_n \text{ such that } f^\pi = f \text{ for all } \pi \in \mathfrak{S}_n\}.$$

By Proposition 1.3.11, Sym_n is a graded ring:

$$\mathrm{Sym}_n := \bigoplus_{k \in \mathbb{N}} \mathrm{Sym}_n^k, \qquad \mathrm{Sym}_n^k := (\mathrm{Pol}_n^k)^{\mathfrak{S}_n}. \tag{2.2}$$

Let $\lambda \in \mathrm{Par}_n$ and recall the definition of $\lambda\mathfrak{S}_n$ from (1.6). The corresponding *monomial symmetric polynomial* is

$$m_\lambda(x_1, \dots, x_n) := \sum_{\alpha \in \lambda\mathfrak{S}_n} x^\alpha. \tag{2.3}$$

Remark 2.1.1. Note that (2.3) is *not* equal to $\sum_{\pi \in \mathfrak{S}_n} x^{\alpha^\pi}$ in general since there may be $\pi_1, \pi_2 \in \mathfrak{S}_n$ with $\alpha^{\pi_1} = \alpha^{\pi_2}$, and we only want to count these terms once. For example

$$\begin{aligned} m_{(3,2,2)}(x_1, x_2, x_3, x_4) = {} & x_1^3x_2^2x_3^2 + x_1^3x_2^2x_4^2 + x_1^3x_3^2x_4^2 + x_1^2x_2^3x_3^2 + x_1^2x_2^3x_4^2 + x_2^3x_3^2x_4^2 \\ & + x_1^2x_2^2x_3^3 + x_1^2x_3^3x_4^2 + x_2^2x_3^3x_4^2 + x_1^2x_2^2x_4^3 + x_1^2x_3^2x_4^3 + x_2^2x_3^2x_4^3 \end{aligned}$$

but

$$\sum_{\pi \in \mathfrak{S}_4} x^{(3,2,2)^\pi} = 2m_{(3,2,2)}(x_1, x_2, x_3, x_4).$$

△

We have

$$m_\lambda(x_1, \dots, x_n) \in \mathrm{Sym}_n^k \qquad \text{for} \quad \lambda \in \mathrm{Par}_n(k).$$

Define

$$m_\lambda(x_1, \dots, x_n) = 0 \qquad \text{for} \quad \ell(\lambda) > n.$$

Proposition 2.1.2. *The monomial symmetric polynomials*

$$m_\lambda(x_1, \dots, x_n), \qquad \lambda \in \mathrm{Par}_n(k), \tag{2.4}$$

form a $\mathbb{Z}$*-basis of* Sym_n^k.

Proof. We must show that the elements (2.4) are linearly independent and span Sym_n^k. To see that they are linearly independent, note that, for any monomial x^α, there is only one partition λ that is a permutation of α, and x^α is a term in $m_\lambda(x_1, \dots, x_n)$. It follows that, for partitions $\lambda \neq \mu$, the polynomials $m_\lambda(x_1, \dots, x_n)$ and $m_\mu(x_1, \dots, x_n)$ have no terms in common. It follows that

$$\sum_{\lambda \in \mathrm{Par}_n(k)} a_\lambda m_\lambda(x_1, \dots, x_n) = 0 \iff (a_\lambda = 0 \text{ for all } \lambda).$$

Hence the elements (2.4) are linearly independent.

Next we prove that the elements (2.4) span Sym_n^k. Suppose $f \in \mathrm{Sym}_n^k$. We prove that f lies in the span of (2.4) by induction on the number of nonzero terms in f. The base case $f = 0$ is clear, so we suppose $f \neq 0$. Choose a term ax^α, $a \in \mathbb{Z} \setminus \{0\}$, of f. Since f is symmetric, the terms ax^β also appear in f for all permutations β of α. Let λ be the unique permutation of α that is a partition. Then

$$f - am_\lambda \in \mathrm{Sym}_n^k$$

has fewer terms than f. By the induction hypothesis, $f - am_\lambda$ can be written as a $\mathbb{Z}$-linear combination of the elements (2.4); hence the same is true for f. □

Corollary 2.1.3. (a) *The monomial symmetric polynomials*

$$m_\lambda(x_1, \dots, x_n), \qquad \lambda \in \mathrm{Par}_n,$$

form a $\mathbb{Z}$*-basis of* Sym_n.

(b) *If* $n \geq k$, *then monomial symmetric polynomials*

$$m_\lambda(x_1, \dots, x_n), \qquad \lambda \in \mathrm{Par}(k),$$

form a $\mathbb{Z}$*-basis of* Sym_n^k. *In particular,* Sym_n^k *is a free* $\mathbb{Z}$*-module of rank* $p(k)$, *the number of partitions of* k.

Proof. Part (a) follows from Proposition 2.1.2 and (2.2). Part (b) follows from Proposition 2.1.2 and the fact that, when $n \geq k$, we have $\mathrm{Par}_n(k) = \mathrm{Par}(k)$ since $\ell(\lambda) \leq n$ for all partitions λ of k. □

Exercises.

2.1.1. For a group G, recall the definition of oppose group G^{op} from Exercise 1.2.1. A *right action* of a group G on a ring R is a group homomorphism $G^{\mathrm{op}} \to \mathrm{Aut}\, R$, where $\mathrm{Aut}\, R$ denotes the group of ring automorphisms of R. By contrast, a *left action* of G on R is a group homomorphism $G \to \mathrm{Aut}\, R$.

(a) Show that (2.1) defines a right action of $\mathfrak{S}_n$ on Pol_n. In other words, show that

$$f^{\pi_1 \pi_2} = (f^{\pi_1})^{\pi_2}$$

for all $f \in \mathrm{Pol}_n$ and $\pi_1, \pi_2 \in \mathfrak{S}_n$.

(b) Show that, for $n \geq 3$, (2.1) does *not* define a left action of $\mathfrak{S}_n$ on Pol_n.

(c) How could you modify (2.1) to give a natural left action of $\mathfrak{S}_n$ on Pol_n?

The next three problems use the following notation. For a subset $X \subseteq \mathfrak{S}_n$, define

$$\mathrm{Pol}_n^X = \{f \in \mathrm{Pol}_n : f^\pi = f \text{ for all } \pi \in X\}.$$

For a subset $P \subseteq \mathrm{Pol}_n$, define

$$G(P) = \{\pi \in \mathfrak{S}_n : f^\pi = f \text{ for all } f \in P\}.$$

2.1.2. Prove that $G(P)$ is a subgroup of $\mathfrak{S}_n$ for all subsets $P \subseteq \mathrm{Pol}_n$.

2.1.3. Is it true that $\mathrm{Pol}_n^{G(P)} = P$ for all subrings $P \subseteq \mathrm{Pol}_n$, $n \in \mathbb{N}$? Prove or give a counterexample.

2.1.4. It is true that $G(\mathrm{Pol}_n^H) = H$ for all subgroups $H \subseteq \mathfrak{S}_n$, $n \in \mathbb{N}$? Prove or give a counterexample.

2.2 Symmetric functions

It turns out that, in many ways, the symmetric polynomials behaves better in the limit where we have an infinite number of indeterminates $x_1, x_2, \ldots$. This limit, which we must be careful to define precisely, leads to the concept of a symmetric function.

For $m \geq n \geq 0$, consider the ring homomorphism

$$\mathrm{Pol}_m \to \mathrm{Pol}_n, \quad f(x_1, \ldots, x_m) \mapsto f(x_1, \ldots, x_n, 0, \ldots, 0).$$

Restriction to Sym_m gives a ring homomorphism

$$\rho_{m,n} \colon \mathrm{Sym}_m \to \mathrm{Sym}_n. \tag{2.5}$$

Note that, for $l \geq m \geq n \geq 0$, we have

$$\rho_{m,n} \circ \rho_{l,m} = \rho_{l,n}.$$

Proposition 2.2.1. *For $m \geq n \geq 0$ and $\lambda \in \mathrm{Par}$, we have*

$$\rho_{m,n}(m_\lambda(x_1, \ldots, x_m)) = m_\lambda(x_1, \ldots, x_n).$$

In particular,

$$\rho_{m,n}(m_\lambda(x_1, \ldots, x_m)) = 0 \quad \text{if} \quad \ell(\lambda) > n.$$

Proof. Let A be the (possibly empty) set of permutations $\alpha = (\alpha_1, \ldots, \alpha_m)$ of $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\alpha_i = 0$ for all $n < i \leq m$. Let B be the set of permutations $\alpha = (\alpha_1, \ldots, \alpha_m)$ of $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\alpha_i \neq 0$ for some $n < i \leq m$. Then

$$m_\lambda(x_1, \ldots, x_m) = \sum_{\alpha \in A} x^\alpha + \sum_{\alpha \in B} x^\alpha,$$

and so

$$\rho_{m,n}\left(m_\lambda(x_1,\dots,x_m)\right) = \sum_{\alpha\in A}\rho_{m,n}(x^\alpha) + \sum_{\alpha\in B}\rho_{m,n}(x^\alpha) = \sum_{\alpha\in A} x^\alpha = m_\lambda(x_1,\dots,x_n).$$

Note that, if $\ell(\lambda) > n$, then $A = \varnothing$, and so $\rho_{m,n}\left(m_\lambda(x_1,\dots,x_m)\right) = 0$. □

Corollary 2.2.2. *For all $m \ge n \ge 0$, the ring homomorphism $\rho_{m,n}$ is surjective.*

Proof. This follows from Propositions 2.1.2 and 2.2.1. □

Restricting $\rho_{m,n}$ to Sym_n^k, we obtain its homogeneous components (see Section 1.3)

$$\rho_{m,n}^k\colon \mathrm{Sym}_m^k \twoheadrightarrow \mathrm{Sym}_n^k, \qquad k \ge 0,\ m \ge n.$$

Lemma 2.2.3. *The map $\rho_{m,n}^k$ is bijective if $m \ge n \ge k$.*

Proof. The proof of this lemma is left as Exercise 2.2.1. □

We now take the inverse limit

$$\mathrm{Sym}^k = \varprojlim_n \mathrm{Sym}_n^k \tag{2.6}$$

of the $\mathbb{Z}$-modules Sym_n^k with respect to the homomorphisms $\rho_{m,n}^k$. By definition,

$$\mathrm{Sym}^k := \left\{ f = (f_n)_{n=0}^\infty \in \prod_{n=0}^\infty \mathrm{Sym}_n^k : \rho_{m,n}^k(f_m) = f_n \text{ for all } m \ge n \right\}. \tag{2.7}$$

For $k, n \ge 0$, consider the projection

$$\rho_n^k\colon \mathrm{Sym}^k \to \mathrm{Sym}_n^k, \quad \rho_n^k(f) = f_n.$$

Proposition 2.2.4. *The projection ρ_n^k is an isomorphism for all $n \ge k$.*

Proof. By Lemma 2.2.3, any element $f \in \mathrm{Sym}^k$ is uniquely determined by f_k since we must have

$$f_n = \rho_{k,n}^k(f_k) \text{ for } n \le k \qquad \text{and} \qquad f_n = (\rho_{n,k}^k)^{-1}(f_k) \text{ for } n > k.$$ □

By Proposition 2.2.1, for any $\lambda \in \mathrm{Par}(k)$, the sequence of monomial symmetric polynomials $(m_\lambda(x_1,\dots,x_n))_{n=0}^\infty$ is an element of Sym^k, which we call the *monomial symmetric function corresponding to* λ and denote by m_λ. We think of it as a formal sum

$$m_\lambda = \sum_{\alpha\in\lambda\mathfrak{S}_\infty} x^\alpha \in \mathrm{Sym}^k. \tag{2.8}$$

Note that, in contrast with (2.3), the above sum is infinite as long as $\lambda \ne \varnothing$. This is because we are consdering permutations of the infinite sequence $(\lambda_1, \lambda_2, \dots)$ rather than the finite sequence $(\lambda_1,\dots,\lambda_n)$. For example,

$$m_{(2,1)} = \sum_{i\ne j} x_i^2 x_j = x_1^2x_2 + x_1x_2^2 + x_1x_3^2 + x_1^2x_3 + x_2x_3^2 + x_2^2x_3 + \cdots.$$

However, when we apply ρ_n^k for some $n \in \mathbb{N}$, only finitely many terms survive. More precisely, for $\lambda \in \operatorname{Par}(k)$,

$$\rho_n^k(m_\lambda) = m_\lambda(x_1, \dots, x_n) \qquad \text{for all } n \geq 0.$$

Now define

$$\operatorname{Sym} := \bigoplus_{k=0}^{\infty} \operatorname{Sym}^k. \tag{2.9}$$

Thus, Sym is the free $\mathbb{Z}$-module generated by the monomial symmetric functions m_λ for *all* partitions λ. We have surjective $\mathbb{Z}$-module homomorphisms

$$\rho_n := \bigoplus_{k=0}^{\infty} \rho_n^k \colon \operatorname{Sym} \to \operatorname{Sym}_n, \qquad n \geq 0. \tag{2.10}$$

If we define, for $n, k \geq 0$,

$$\operatorname{Sym}^{\leq k} := \bigoplus_{l=0}^{k} \operatorname{Sym}^l \qquad \text{and} \qquad \operatorname{Sym}_n^{\leq k} := \bigoplus_{l=0}^{k} \operatorname{Sym}_n^l, \tag{2.11}$$

then the restriction of ρ_n to $\operatorname{Sym}^{\leq k}$ yields a homomorphism of $\mathbb{Z}$-modules

$$\rho_n^{\leq k} \colon \operatorname{Sym}^{\leq k} \to \operatorname{Sym}_n^{\leq k}$$

which is an isomorphism when $k \leq n$.

Proposition 2.2.5. (a) *The monomial symmetric functions m_λ, $\lambda \in \operatorname{Par}(k)$, form a basis of* Sym^k. *In particular,* Sym^k *is a free $\mathbb{Z}$-module of rank $p(k)$, the number of partitions of k.*

(b) *The monomial symmetric functions m_λ, $\lambda \in \operatorname{Par}$, form a basis of* Sym.

Proof. Part (a) follows from Proposition 2.2.4 and Corollary 2.1.3(a). Then part (b) follows from (2.9). □

So far we have only described Sym as a $\mathbb{Z}$-module. However, as the next result shows, it is naturally a ring.

Proposition 2.2.6. *There is a unique structure of a graded ring on* Sym *such that the ρ_n are ring homomorphisms.*

Proof. We first prove uniqueness. Suppose we have defined a multiplication on Sym such that the ρ_n are ring homomorphisms. Let $f, g \in \operatorname{Sym}$. Then there exists some $k \geq 0$ such that $f, g \in \operatorname{Sym}^{\leq k}$ and we must have

$$\rho_{2k}(fg) := \rho_{2k}(f)\rho_{2k}(g) \in \operatorname{Sym}^{\leq 2k}.$$

Now, since ρ_{2k} is an isomorphism when restricted to $\operatorname{Sym}^{\leq 2k}$, we must have

$$fg = \left(\rho_{2k}^{\leq 2k}\right)^{-1} \left(\rho_{2k}(f)\rho_{2k}(g)\right). \tag{2.12}$$

This completes the proof of uniqueness. Existence is then seen by taking (2.12) as the *definition* of the multiplication in Sym. We leave it as exercise Exercise 2.2.2 to verify that the structure we have defined satisfies the axioms of a ring. □

The graded ring Sym is called the *ring of symmetric functions*. The term 'function' is used here in a formal sense, to distinguish this ring from the ring of symmetric *polynomials*. We do not view elements of Sym as actual functions.

It was important that we first took the inverse limit in (2.6) of the graded pieces Sym_n^k and then took the direct sum in (2.9). This construction ensures that elements of Sym are infinite sums of monomials with bounded degree. What we are doing is taking the inverse limit of the Sym_n in the category of graded rings. (General inverse limits in a category are defined in terms of a universal property.) If we had instead taken the inverse limit of the Sym_n directly (the inverse limit in the category of rings) our ring would include elements such as the infinite product $\prod_{i=1}^{\infty}(1+x_i)$.

To illustrate this difference further, let us discuss another approach to the definition of the ring of symmetric functions. Recall the definitions of $\mathbb{Z}[\![x_1, x_2, \dots]\!]$ and $\mathbb{Z}[\![x_1, x_2, \dots]\!]^k$ from (1.15) and (1.17). The group $\mathfrak{S}_\infty$ acts on each $\mathbb{Z}[\![x_1, x_2, \dots]\!]^k$ in the usual way:

$$f(x_1, x_2, \dots)^\pi := f(x_{\pi(1)}, x_{\pi(2)}, \dots), \qquad f(x_1, x_2, \dots) \in \mathbb{Z}[\![x_1, x_2, \dots]\!]^k, \quad \pi \in \mathfrak{S}_\infty.$$

Then

$$\mathrm{Sym}^k = (\mathbb{Z}[\![x_1, x_2, \dots]\!]^k)^{\mathfrak{S}_\infty}, \qquad \mathrm{Sym} = \bigoplus_{k\in\mathbb{N}} (\mathbb{Z}[\![x_1, x_2, \dots]\!]^k)^{\mathfrak{S}_\infty}.$$

Note that

$$\mathrm{Sym} \neq \mathbb{Z}[\![x_1, x_2, \dots]\!]^{\mathfrak{S}_\infty}. \tag{2.13}$$

(Compare to (1.18).) For example,

$$\prod_{i=1}^{\infty}(1+x_i) \in \mathbb{Z}[\![x_1, x_2, \dots]\!]^{\mathfrak{S}_\infty} \qquad \text{but} \qquad \prod_{i=1}^{\infty}(1+x_i) \notin \mathrm{Sym}.$$

Warning 2.2.7. The definition of the ring of symmetric functions given in [Egg19, Def. 1.19] seems to be incorrect. The ring defined there is $\mathbb{Z}[\![x_1, x_2, \dots]\!]^{\mathfrak{S}_\infty}$. See Warning 2.4.8.

Remark 2.2.8. We have worked in this section over the coefficient ring $\mathbb{Z}$. We could have instead worked over any other commutative ring. In particular, we will denote by $\mathrm{Sym}_\mathbb{Q}$ the ring of symmetric functions over $\mathbb{Q}$, which will be important for us later on. We can also obtain $\mathrm{Sym}_\mathbb{Q}$ by an extension of scalars: $\mathrm{Sym}_\mathbb{Q} \cong \mathbb{Q} \otimes_\mathbb{Z} \mathrm{Sym}$. (Students not familiar with tensor products can find the basics in [Sav, Ch. 3].) △

Exercises.

2.2.1. Prove Lemma 2.2.3. Furthermore, prove that $\rho_{m,n}^k$ is *not* bijective if $m > n$ and $k > n$.

2.2.2. Show that the multiplication on Sym defined in Proposition 2.2.6 satisfies the defining axioms for a ring.

2.2.3. For $n \in \mathbb{N}$, let

$$R_n = \mathbb{Z}[x]/x^n\mathbb{Z}[x]$$

be the quotient of $\mathbb{Z}[x]$ by the ideal generated by x^n. Then, for all $m \geq n \geq 0$, we have a projection

$$\varphi_{m,n} \colon R_m \to R_n, \qquad \varphi_{m,n}(f + x^m\mathbb{Z}[x]) = f + x^n\mathbb{Z}[x], \quad f \in \mathbb{Z}[x].$$

It is clear that $\varphi_{m,n} \circ \varphi_{l,m} = \varphi_{l,n}$ for all $l \geq m \geq n \geq 0$. Identify the inverse limit

$$\varprojlim_n R_n := \left\{ f = (f_n)_{n=0}^\infty \in \prod_{n=0}^\infty R_n : \varphi_{m,n}(f_m) = f_n \text{ for all } m \geq n \right\}$$

with a ring we studied in Chapter 1.

2.2.4. Suppose $k, l \in \mathbb{N}$ with $k > l$. Write $m_{(k)}m_{(l)}$ as a linear combination of m_λ, $\lambda \in \mathrm{Par}$.

2.2.5. Write $m_{(4)}m_{(3)}m_{(2)}m_{(1)}$ as a linear combination of m_λ, $\lambda \in \mathrm{Par}$.

2.3 Tableaux

Before we continue our discussion of symmetric functions, we introduce a combinatorial tool that will prove useful in our discussions.

For a partition λ, a *tableau* (plural *tableaux*) of *shape* λ is a filling of a Young diagram of λ with positive integers. (These are also called *Young tableaux*.) For example

$$T_1 = \begin{array}{|c|c|c|c|}\hline 5 & 1 & 3 & 2 \\ \hline 1 & 1 & 2 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}\ , \qquad T_2 = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 1 & 2 & 8 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array}\ , \qquad \text{and} \qquad T_3 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array} \tag{2.14}$$

are tableaux of shape $(4, 3, 1)$. We let $\mathrm{shape}(T)$ denote the shape of a tableau T. The *weight* $\mathrm{wt}(T)$ of a tableau T (also called its *content*) is the composition $\mu = (\mu_1, \dots, \mu_r)$ where μ_i is the number of entries of T equal to i. For example, for the tableaux in (2.14), we have

$$\mathrm{wt}(T_1) = (3, 2, 2, 0, 1), \qquad \mathrm{wt}(T_2) = (2, 1, 1, 1, 1, 1, 0, 1), \qquad \mathrm{wt}(T_3) = (8).$$

Note that $|\,\mathrm{wt}(T)| = |\,\mathrm{shape}(T)|$.

We say that a tableau is *row strict* if its entries are strictly increasing along each row. For $\lambda \in \mathrm{Par}$, we let $\mathrm{RowStrict}(\lambda)$ denote the set of row-strict tableaux of shape λ, and we let $\mathrm{RowStrict}(\lambda, n)$ denote the set of row-strict tableaux of shape λ with entries chosen from the set $\{1, 2, \dots, n\}$. For example, for the tableaux of (2.14) and $\lambda = (4, 3, 1)$, we have

$$T_1 \notin \mathrm{RowStrict}(\lambda), \qquad T_2 \in \mathrm{RowStrict}(\lambda, 8), \qquad T_2 \in \mathrm{RowStrict}(\lambda),$$
$$T_2 \notin \mathrm{RowStrict}(\lambda, 5), \qquad T_3 \notin \mathrm{RowStrict}(\lambda).$$

We say that a tableau is *row nondecreasing* if its entries are weakly increasing along each row. For $\lambda \in \mathrm{Par}$, let $\mathrm{RowWeak}(\lambda)$ denote the set of row-nondecreasing tableaux of shape

λ. We let $\mathrm{RowWeak}(\lambda, n)$ denote the set of such tableaux with entries chosen from the set $\{1, 2, \ldots, n\}$. For example, for the tableaux of (2.14) and $\lambda = (4, 3, 1)$, we have

$$T_1 \notin \mathrm{RowWeak}(\lambda), \qquad T_2 \in \mathrm{RowWeak}(\lambda, 8), \qquad T_3 \in \mathrm{RowWeak}(\lambda).$$

We say that a tableau is *semistandard* if it is row nondecreasing and its entries are strictly increasing down each column. For example

$$T_4 = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 7 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad T_5 = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 7 \\ \hline 2 & 6 & 9 \\ \cline{1-3} 8 \\ \cline{1-1}\end{array} \tag{2.15}$$

are semistandard tableaux. For $\lambda \in \mathrm{Par}$, we let $\mathrm{SST}(\lambda)$ denote the set of semistandard tableau of shape λ and we let $\mathrm{SST}(\lambda, n)$ denote the set of such tableaux with entries chosen from the set $\{1, 2, \ldots, n\}$.

We say that a tableau is *standard* if it is row strict, its entries are strictly increasing down each column, and it contains each integer $1, \ldots, n$ exactly once, where n is the number of boxes in T. Thus, a standard tableaux has weight $\mathrm{wt}(T) = (1^n)$. For example, for the tableaux in (2.15), T_5 is standard, but T_4 is not. For $\lambda \in \mathrm{Par}$, we let $\mathrm{ST}(\lambda)$ denote the set of standard tableaux of shape λ.

If T is a tableau, we define the monomial

$$x^T := \prod_{j \in T} x_j = x^{\mathrm{wt}(T)} \tag{2.16}$$

where the first product is over the entries in T. For example, for the tableaux of (2.14), we have

$$x^{T_1} = x_1^3 x_2^2 x_3^2 x_5, \qquad x^{T_2} = x_1^2 x_2 x_3 x_4 x_5 x_6 x_8, \qquad \text{and} \qquad x^{T_3} = x_1^8.$$

We will use this convention to give explicit descriptions of important symmetric functions as sums of monomials x^T over certain tableaux.

Exercises.

2.3.1. Find all elements of $\mathrm{SST}((2, 2, 1), 3)$. Which are standard?

2.3.2. Find all elements of $\mathrm{SST}((2, 1), 3)$. Which are standard?

2.4 Elementary symmetric functions

In this section we introduce our first important generating set for the ring of symmetric functions: the elementary symmetric functions. The fact that these give an algebraically independent set of generators for Sym is the fundamental theorem of symmetric functions (Theorem 2.4.6).

For $r \in \mathbb{N}$, the r-th *elementary symmetric function* e_r is the sum of all products of r distinct indeterminates x_i. Thus,

$$e_0 = 1, \qquad e_r := \sum_{1 \le i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} = m_{(1^r)}, \quad r \ge 1. \tag{2.17}$$

We adopt the convention that

$$e_r = 0 \qquad \text{for} \qquad r < 0.$$

For $r, n \in \mathbb{N}$, the r-th *elementary symmetric polynomial in n indeterminates* is

$$e_r(x_1, \dots, x_n) = \rho_n(e_r) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r} = m_{(1^r)}(x_1, \dots, x_n). \tag{2.18}$$

It follows immediately from the definition that

$$e_r(x_1, \dots, x_n) = 0 \qquad \text{for } r > n.$$

Convention 2.4.1. From now on, whenever we define a symmetric function $f \in \mathrm{Sym}$, we adopt the convention that

$$f(x_1, \dots, x_n) = \rho_n(f).$$

Lemma 2.4.2. *The generating function for the elementary symmetric polynomials in n indeterminates is*

$$\sum_{r=0}^{n} e_r(x_1, \dots, x_n) t^r = \prod_{i=1}^{n} (1 + x_i t).$$

Proof. We prove the result by induction on n. When $n = 0$, the result is clear when we recall the convention that the product $\prod_{i=1}^{n}(1 + x_i t) = 1$ when $n = 0$. Now suppose the result holds for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
\prod_{i=1}^{n+1}(1 + x_i t) &= \left(\sum_{r=0}^{n} e_r(x_1, \dots, x_n) t^r \right) (1 + x_{n+1} t) \\
&= 1 + \sum_{r=1}^{n+1} \left(e_r(x_1, \dots, x_n) + e_{r-1}(x_1, \dots, x_n) x_{n+1} \right) t^r \\
&= \sum_{r=0}^{n+1} e_r(x_1, \dots, x_{n+1}) t^r,
\end{aligned}$$

where we have use the fact that, for $r \ge 1$,

$$\begin{aligned}
&e_{r-1}(x_1, \dots, x_n) x_{n+1} + e_r(x_1, \dots, x_n) \\
&\qquad = \sum_{1 \le i_1 < i_2 < \cdots < i_r = n+1} x_{i_1} x_{i_2} \cdots x_{i_r} + \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r} \\
&\qquad = e_r(x_1, \dots, x_{n+1}).
\end{aligned}$$

□

It follows from Lemma 2.4.2 that the generating function for the elementary symmetric functions is

$$E(t) := \sum_{r=0}^{\infty} e_r t^r = \prod_{i=1}^{\infty} (1 + x_i t). \tag{2.19}$$

Remark 2.4.3. Let us take a moment to discuss how to interpret the infinite product occurring on the right-hand side of (2.19). We want to interpret it as an element of $\mathrm{Sym}[\![t]\!]$. So it should be a formal power series with coefficients in Sym. Elements of Sym are sequences $f = (f_n)_{n=0}^{\infty}$ such that $\rho_{m,n}(f_m) = f_n$ for all $m \geq n \geq 0$. For each $r \in \mathbb{N}$, the coefficient of t^r in $\prod_{i=1}^{\infty}(1+x_i t)$ is the symmetric function $f = (f_n)_{n=0}^{\infty}$ where f_n is the coefficient of t^r in the *finite* product $\prod_{i=1}^{n}(1+x_i t)$. By Lemma 2.4.2, we have $f_n = e_r(x_1, \dots, x_n)$, and so $f = e_r$. This justifies the equality (2.19). △

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, define

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots . \tag{2.20}$$

Note that this is a finite product since λ has finitely many nonzero terms. Our next goal is to show that the e_λ, $\lambda \in \mathrm{Par}$, form a basis of Sym. To do this, we work out how to express the e_λ in terms of the monomial symmetric functions, which we know form a basis of Sym by Proposition 2.2.5. Recall the definition of the conjugate partition (1.3) and the dominance ordering (1.4).

Proposition 2.4.4. *Let $\lambda \in \mathrm{Par}(k)$ and let λ' be the conjugate partition. Then*

$$e_\lambda = m_{\lambda'} + \sum_{\mu < \lambda'} M_{\lambda,\mu}(e,m) m_\mu. \tag{2.21}$$

where, for $\mu \in \mathrm{Par}(k)$, $\mu < \lambda'$, $M_{\lambda,\mu}(e,m)$ is the number of row-strict Young tableaux of shape λ and weight μ.

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_k)$. We have

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}, \qquad e_{\lambda_j} = \sum_{i_{j,1} < i_{j,2} < \cdots < i_{j,\lambda_j}} x_{i_{j,1}} x_{i_{j,2}} \cdots x_{i_{j,\lambda_j}}, \quad 1 \leq j \leq k.$$

Fix a particular choice of the indices

$$i_{j,1} < i_{j,2} < \cdots < i_{j,\lambda_j}, \qquad 1 \leq j \leq k,$$

and consider the corresponding term

$$x^\alpha = \prod_{j=1}^{k} x_{i_{j,1}} x_{i_{j,2}} \cdots x_{i_{j,\lambda_j}} \tag{2.22}$$

in the product $e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_k}$. For each $j \in \{1,\dots,k\}$, write the indices $i_{j,1} < \cdots < i_{j,\lambda_j}$ in the j-th row of the Young diagram of λ:

$$\begin{array}{cccccc}
i_{1,1} & i_{1,2} & \cdots & \cdots & \cdots & i_{1,\lambda_1} \\
i_{2,1} & i_{2,2} & \cdots & \cdots & \cdots & i_{2,\lambda_2} \\
\vdots & \vdots & & & & \\
i_{j,1} & \cdots & i_{j,\lambda_j} & & & \\
\vdots & & & & & \\
\vdots & i_{\lambda_2',2} & & & & \\
\vdots & & & & & \\
i_{\lambda_1',1} & & & & &
\end{array} \tag{2.23}$$

Since the entries are strictly increasing along each row, all entries $\le r$ must occur in the leftmost r columns of the diagram. Hence the total number of entries $\le r$ is less than or equal to the total number of boxes in the first r columns. Since α_r is the number of entries equal to r, this implies that

$$\alpha_1 + \cdots + \alpha_r \le \lambda_1' + \cdots + \lambda_r'.$$

Thus we have $\alpha \le \lambda'$.

Now, since $\alpha \le \lambda'$ for all terms (2.22) and e_λ is $\mathfrak{S}_\infty$ invariant, it follows from Lemma 1.2.4 that only m_μ for $\mu \le \lambda'$ appear when we write e_λ in the basis of monomial symmetric functions. Thus

$$e_\lambda = \sum_{\mu \le \lambda'} M_{\lambda,\mu}(e,m) m_\mu,$$

where $M_{\lambda,\mu}(e,m)$ is the number of row-strict Young tableaux of shape λ and weight μ. If $\mu = \lambda'$, then the only such filling is the one in which each entry in (2.23) is equal to its column number, that is, $i_{j,l} = l$ for all j, l. Thus $M_{\lambda,\lambda'}(e,m) = 1$. □

The proof of Proposition 2.4.4 gives us an explicit combinatorial description of the elementary symmetric functions in terms of Young tableaux.

Proposition 2.4.5. *For any partition λ and $n \ge 1$, we have*

$$e_\lambda = \sum_{T \in \mathrm{RowStrict}(\lambda)} x^T \quad \textit{and} \quad e_\lambda(x_1,\dots,x_n) = \sum_{T \in \mathrm{RowStrict}(\lambda,n)} x^T. \tag{2.24}$$

Proof. The first equation follows from the proof of Proposition 2.4.4. Since $e_\lambda(x_1,\dots,x_n) = \rho_n(e_\lambda)$, the second equation then follows from the fact that $\rho_n(x^T) = 0$ if T has any entries greater than n. □

Recall that elements of a commutative ring R are *algebraically independent* over a subring $S \subseteq R$ if the elements do not satisfy any polynomial equation with coefficients in S.

Theorem 2.4.6 (Fundamental theorem of symmetric functions)**.** *The elementary symmetric functions* e_λ, $\lambda \in \mathrm{Par}$*, form a basis of* Sym*. Furthermore,*

$$\mathrm{Sym} = \mathbb{Z}[e_1, e_2, \dots]$$

and the e_r, $r \in \mathbb{N}$*, are algebraically independent over* $\mathbb{Z}$*.*

Proof. It suffices to prove that, for $k \in \mathbb{N}$, the e_λ, $\lambda \in \mathrm{Par}(k)$, form a basis of Sym^k. Fix a total order on the basis m_λ, $\lambda \in \mathrm{Par}(k)$, of Sym^k that refines the dominance ordering on Sym^k. (For instance, we can take the lexicographic ordering; see Exercise 1.2.4.) Then Proposition 2.4.4 implies that matrix expressing the $e_{\lambda'}$, $\lambda \in \mathrm{Par}(k)$, in terms of this ordered basis is triangular with ones on the diagonal, and hence invertible. It follows every monomial symmetric function can be written as a $\mathbb{Z}$-linear combination of elementary symmetric functions. Thus the elementary symmetric functions e_λ, $\lambda \in \mathrm{Par}$, form a basis of Sym. This implies that every element of Sym is uniquely expressible as a polynomial in the e_r, proving the second statement of the theorem. □

Corollary 2.4.7 (Fundamental theorem of symmetric polynomials)**.** *For* $n \in \mathbb{N}$*, the* $e_\lambda(x_1, \dots, x_n)$, $\lambda \in \mathrm{Par}_n$*, form a basis of* Sym_n*. Furthermore,*

$$\mathrm{Sym}_n = \mathbb{Z}[e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n)]$$

and the elementary symmetric polynomials $e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n)$ *are algebraically independent over* $\mathbb{Z}$*.*

Proof. This follows from Theorem 2.4.6 and Proposition 2.2.4. □

Theorem 2.4.6 implies that the ring Sym of symmetric functions is isomorphic, as a ring, to a polynomial ring in infinitely many indeterminates. You might be tempted to think that this implies Sym is not so interesting! However, it is the *relationship between various elements* of Sym (for example, different bases) that makes the theory of symmetric functions so rich. For example, in the next section will introduce another set of generators for Sym, and then study their relation to the elementary symmetric functions.

Warning 2.4.8. It is crucial for Theorem 2.4.6 that we defined symmetric functions as we did, instead of defining them to be elements of $\mathbb{Z}[\![x_1, x_2, \dots]\!]^{\mathfrak{S}_\infty}$. The elementary symmetric functions $e_1, e_2, \dots$ do *not* generate $\mathbb{Z}[\![x_1, x_2, \dots]\!]^{\mathfrak{S}_\infty}$ as a ring; see Exercise 2.4.1. Thus, the statement at the top of [Egg19, p. 38] is incorrect (since the definition of symmetric functions in that reference is incorrect); see Warning 2.2.7.

Exercises.

2.4.1. Prove that the elementary symmetric functions $e_1, e_2, \dots$ do *not* generate $\mathbb{Z}[\![x_1, x_2, \dots]\!]^{\mathfrak{S}_n}$ as a ring. *Hint*: Find an element of $\mathbb{Z}[\![x_1, x_2, \dots]\!]^{\mathfrak{S}_n}$ that cannot be written as a polynomial in the $e_1, e_2, \dots$.

2.4.2. Write $e_{(2,2)}$ and $e_{(2,1,1)}$ as linear combinations of monomial symmetric functions.

2.4.3. Find and prove a formula for $M_{\lambda,(n)}(e,m)$, where $\lambda \vdash n$.

2.4.4. Find and prove a formula for $M_{\lambda,(1^n)}(e,m)$, where $\lambda \vdash n$.

2.4.5. Find and prove a formula for $M_{\lambda,(n-1,1)}(e,m)$, where $\lambda \vdash n$.

2.5 Complete homogeneous symmetric functions

For $r \in \mathbb{N}$, the *r-th complete homogeneous symmetric function* h_r is the sum of all monomials of total degree r in the indeterminates $x_1, x_2, \dots,$ so that

$$h_0 = 1, \qquad h_r := \sum_{1 \le i_1 \le i_2 \le \cdots \le i_r} x_{i_1} x_{i_2} \cdots x_{i_r} = \sum_{\lambda \in \mathrm{Par}(r)} m_\lambda, \quad r \ge 1. \tag{2.25}$$

Note that

$$h_1 = e_1.$$

We adopt the convention that

$$h_r = 0 \qquad \text{for} \qquad r < 0.$$

As usual, for $n \in \mathbb{N}$, we then define the *r-th complete homogeneous symmetric polynomial in n indeterminates* to be

$$h_r(x_1, \dots, x_n) := \rho_n(h_r).$$

Proposition 2.5.1. *The generating function for the complete homogeneous symmetric functions is*

$$H(t) := \sum_{r \ge 0} h_r t^r = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}. \tag{2.26}$$

Proof. The proof of this result is left as Exercise 2.6.1. □

It follows from (2.19) and (2.26) that

$$H(t)E(-t) = 1. \tag{2.27}$$

Comparing coefficients of powers of t, this implies that

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0 \qquad \text{for all} \quad n \ge 1. \tag{2.28}$$

(We see here the power of the approach of generating functions!)

By Theorem 2.4.6, we may define a homomorphism of graded rings

$$\omega \colon \mathrm{Sym} \to \mathrm{Sym}, \qquad \omega(e_r) = h_r, \quad r \in \mathbb{N}. \tag{2.29}$$

Recall that an endomorphism is an *involution* if its square is the identity map (equivalently, it is an automorphism and equal to its own inverse.)

Proposition 2.5.2. *We have*

$$\omega(h_r) = e_r, \qquad \text{for all} \quad r \in \mathbb{N}.$$

In particular, the map ω is an involution.

Proof. Once we prove the first statement, we have $\omega^2(e_n) = e_n$ for all $n \in \mathbb{N}$, and hence the second follows statement follows from Theorem 2.4.6.

We prove that $\omega(h_n) = e_n$ for all $n \in \mathbb{N}$ by induction on n. Since ω is a ring homomorphism, we have

$$\omega(h_0) = \omega(1) = 1 = e_0,$$

and so the result holds for $n = 0$. Now suppose that $n \geq 1$ and $\omega(h_r) = e_r$ for $r < n$. By (2.28), we have

$$h_n = -\sum_{r=1}^{n}(-1)^r e_r h_{n-r} \qquad \text{and} \qquad e_n = (-1)^{n+1}\sum_{r=0}^{n-1}(-1)^r e_r h_{n-r}.$$

Thus

$$\begin{aligned}\omega(h_n) = -\sum_{r=1}^{n}(-1)^r \omega(e_r)\omega(h_{n-r}) &= -\sum_{r=1}^{n}(-1)^r h_r e_{n-r} \\ &= -\sum_{r=0}^{n-1}(-1)^{n-r} h_{n-r} e_r = (-1)^{n+1}\sum_{r=0}^{n-1}(-1)^r e_r h_{n-r} = e_n,\end{aligned}$$

where, in the third equality, we replaced r by $n - r$. □

Theorem 2.5.3. *We have*

$$\mathrm{Sym} = \mathbb{Z}[h_1, h_2, \dots]$$

and the complete homogeneous symmetric functions h_r, $r \in \mathbb{N}$, are algebraically independent over $\mathbb{Z}$.

Proof. By Proposition 2.5.2, ω is an automorphism. Thus the result follows from Theorem 2.4.6. □

Remark 2.5.4. If we work with symmetric polynomials in n indeterminates, we still have a mapping

$$\omega \colon \mathrm{Sym}_n \to \mathrm{Sym}_n, \qquad \omega(e_r) = h_r, \quad 1 \leq r \leq n.$$

Applying ρ_n to (2.28) shows that

$$\sum_{r=0}^{n}(-1)^r e_r(x_1, \dots, x_n) h_{n-r}(x_1, \dots, x_n) = 0.$$

Thus, one can repeat the argument in the proofs of Proposition 2.5.2 and Theorem 2.5.3 to see that

$$\mathrm{Sym}_n = \mathbb{Z}[h_1(x_1, \dots, x_n), \dots, h_n(x_1, \dots, x_n)],$$

with $h_1, \dots, h_n$ algebraically independent. However, even though $e_r(x_1, \dots, x_n) = 0$ for $r > n$, the complete homogeneous symmetric polynomials $h_{n+1}(x_1, \dots, x_n), h_{n+2}(x_1, \dots, x_n), \dots$ are *nonzero*. For example,

$$h_3(x_1, x_2) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_3^2 = 2h_1(x_1, x_2)h_2(x_1, x_2) - h_1(x_1, x_2)^3.$$

In particular, we do *not* have $\omega(e_r) = h_r$ for $r > n$. This is example of how symmetric functions are often simpler than symmetric polynomials. △

For $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathrm{Par}$, we define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots . \tag{2.30}$$

By Theorem 2.5.3, the h_λ, $\lambda \in \mathrm{Par}$, form a $\mathbb{Z}$-basis of Sym. Define the *forgotten symmetric functions*

$$f_\lambda = \omega(m_\lambda), \qquad \lambda \in \mathrm{Par}. \tag{2.31}$$

The name "forgotten" comes from the fact that these symmetric functions have no simple direct description and thus tend not to appear much in applications. We now have four bases of Sym:

$$\{m_\lambda : \lambda \in \mathrm{Par}\}, \quad \{e_\lambda : \lambda \in \mathrm{Par}\}, \quad \{h_\lambda : \lambda \in \mathrm{Par}\}, \quad \{f_\lambda : \lambda \in \mathrm{Par}\}.$$

The automorphism ω interchanges the first two bases and the last two bases.

It follows from Proposition 2.2.5 that we can write each h_λ in terms of the monomial symmetric functions. The next result gives an explicit formula.

Proposition 2.5.5. *For $\lambda \in \mathrm{Par}(k)$, we have*

$$h_\lambda = \sum_{\mu \vdash |\lambda|} M_{\lambda,\mu}(h, m) m_\mu. \tag{2.32}$$

where, for $\mu \vdash |\lambda|$, $M_{\lambda,\mu}(h, m)$ is the number of row-nondecreasing Young tableaux of shape λ and weight μ.

Proof. The proof of this proposition is left as Exercise 2.5.1. □

We can now give an explicit combinatorial description of the complete homogeneous symmetric functions in terms of Young tableaux. (Compare to Proposition 2.4.5.)

Proposition 2.5.6. *For any partition λ and $n \geq 1$, we have*

$$h_\lambda = \sum_{T \in \mathrm{RowWeak}(\lambda)} x^T \quad \text{and} \quad h_\lambda(x_1, \dots, x_n) = \sum_{T \in \mathrm{RowWeak}(\lambda, n)} x^T. \tag{2.33}$$

Proof. The proof of this proposition is analogous to that of Proposition 2.4.5. In particular, to prove the first equality, one alters the proof of Proposition 2.4.4 to work with the h_λ instead of the e_λ. The first equality also follows from Proposition 2.5.5. Then applying ρ_n gives the second equality. □

Exercises.

2.5.1. Prove Proposition 2.5.5.

2.5.2. Write $h_{(2,2)}$ and $h_{(2,1,1)}$ as linear combinations of monomial symmetric functions.

2.5.3. Find and prove a formula for $M_{\lambda,(n)}(h,m)$, where $\lambda \vdash n$.

2.5.4. Find and prove a formula for $M_{\lambda,(1^n)}(h,m)$, where $\lambda \vdash n$.

2.5.5. Find and prove a formula for $M_{\lambda,(n-1,1)}(h,m)$, where $\lambda \vdash n$.

2.5.6. Prove that, for all $n \geq 1$,

$$e_n = \det(h_{1-j+k})_{1 \leq j,k \leq n},$$

where $(h_{1-j+k})_{1 \leq j,k \leq n}$ is the matrix with (j,k) entry equal to h_{1-j+k}.

2.5.7. Prove that, for all $n \geq 1$,

$$h_n = \det(e_{1-j+k})_{1 \leq j,k \leq n},$$

where $(e_{1-j+k})_{1 \leq j,k \leq n}$ is the matrix with (j,k) entry equal to e_{1-j+k}.

2.6 Power sums

In this section we introduce another important set of symmetric functions. As we will see, these new symmetric functions are most well-behaved when we work over the rational numbers instead of the integers. We will make use is this section of the results of Exercises 1.4.5 to 1.4.8 on formal power series.

For $r \geq 1$, the *r-th power sum* is

$$p_r := \sum_{i=1}^{\infty} x_i^r = m_{(r)}. \tag{2.34}$$

As usual, for $n \in \mathbb{N}$, we define

$$p_r(x_1, \dots, x_n) := \rho_n(p_r) = x_1^r + x_2^r + \cdots + x_n^r.$$

It turns out that many formulas are more natural when we consider the *scaled power sums* $\frac{p_r}{r}$.

Proposition 2.6.1. *The generating function for the sequence of scaled powers sums in n indeterminates is*

$$\sum_{r=1}^{\infty} \frac{p_r(x_1, \dots, x_n)}{r} t^r = \log \left(\prod_{j=1}^{\infty} \frac{1}{1 - x_j t} \right).$$

Proof. Recall that $\log\left(\frac{1}{1-x}\right) = -\log(1-x) = \sum_{r=1}^{\infty} \frac{1}{r}x^r$. Thus we have

$$\sum_{r=1}^{\infty} \frac{p_r(x_1,\dots,x_n)}{r} t^r = \sum_{r=1}^{\infty}\sum_{j=1}^{n} \frac{1}{r}(x_j t)^r = \sum_{j=1}^{n} \log\left(\frac{1}{1-x_j t}\right) = \log\left(\prod_{j=1}^{n} \frac{1}{1-x_j t}\right). \qquad \square$$

Now, taking the limit as $n \to \infty$, we obtain the generating function for the power sums p_n:

$$P(t) := \sum_{r\geq 1} \frac{p_r}{r} t^r = \log\left(\prod_{j=1}^{\infty} \frac{1}{1-x_j t}\right) \overset{(2.26)}{=} \log H(t). \tag{2.35}$$

(See Remark 2.4.3 for a discussion on how we interpret this type of infinite product of generating functions.)

Remark 2.6.2. Note that, in [Mac15, p. 23], $P(t)$ denotes the generating function $\sum_{r\geq 1} p_r t^{r-1}$, which is the formal derivative of our generating function (2.35). △

Note that

$$P'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log H(t) = H'(t)/H(t). \tag{2.36}$$

Also, by (2.27), $E(t) = H(-t)^{-1}$, and so

$$E'(t) = H(-t)^{-2} H'(-t)$$

Thus

$$E'(t)/E(t) = H'(-t)/H(-t) = P'(-t). \tag{2.37}$$

Proposition 2.6.3. *For $n \geq 1$, we have*

$$nh_n = \sum_{r=1}^{n} p_r h_{n-r}, \tag{2.38}$$

$$ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r}. \tag{2.39}$$

(The equations (2.38) and (2.39) are known as Newton's identities.*)*

Proof. From (2.36), we have $H'(t) = P'(t)H(t)$. Thus

$$\sum_{n\geq 1} nh_n t^{n-1} = \left(\sum_{n\geq 1} p_n t^{n-1}\right)\left(\sum_{m\geq 1} h_m t^m\right) = \sum_{n\geq 1}\sum_{r=1}^{n} p_r h_{n-r} t^{n-1}.$$

Equating coefficients yields (2.38). The proof of (2.39) is analogous. $\square$

For a partition $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathrm{Par}$, we define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots . \tag{2.40}$$

Recall, from 2.2.8, that $\mathrm{Sym}_\mathbb{Q}$ denotes the symmetric functions with coefficients in $\mathbb{Q}$.

Proposition 2.6.4. *We have*

$$\mathrm{Sym}_\mathbb{Q} = \mathbb{Q}[p_1, p_2, \dots]$$

and the power sums $p_1, p_2, \dots$ are algebraically independent over $\mathbb{Q}$. Furthermore the p_λ, $\lambda \in \mathrm{Par}$, form a basis for $\mathrm{Sym}_\mathbb{Q}$.

Proof. The identity (2.38) allows us to solve for the power sums recursively in terms of the complete homogeneous symmetric functions and vice versa:

$$p_n = nh_n - \sum_{r=1}^{n-1} p_r h_{n-r} \qquad \text{and} \qquad h_n = \frac{1}{n} \sum_{r=1}^{n} p_r h_{n-r}. \tag{2.41}$$

It follows that

$$\mathrm{Sym}_\mathbb{Q} = \mathbb{Q}[h_1, h_2, \dots] = \mathbb{Q}[p_1, \dots, p_n],$$

where the first equality is Theorem 2.5.3. Furthermore, if the p_r satisfied a polynomial equation over $\mathbb{Q}$ then we can replace them by their expressions in terms of the h_r to see that the h_r satisfy a polynomial equation over $\mathbb{Q}$. Then, clearing denominators, we would have that the h_r are algebraically dependent. But this contradicts Theorem 2.5.3. Hence the p_r are algebraically independent over $\mathbb{Q}$. □

Remark 2.6.5. It is crucial in Proposition 2.6.4 that we work over $\mathbb{Q}$ instead of $\mathbb{Z}$. For example,

$$h_2 = \frac{1}{2}\left(p_1^2 + p_2\right)$$

does not have integral coefficients when expressed in terms of the p_λ. For this reason, when we really want to work over the integers, the powers sums can be problematic. △

Proposition 2.6.6. *For $n \in \mathbb{N}$ and $\lambda \in \mathrm{Par}$, we have*

$$\omega(p_n) = (-1)^{n-1} p_n \qquad \textit{and} \qquad \omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda, \tag{2.42}$$

where ω is the automorphism of Sym *defined in* (2.29).

Proof. We prove the first equality by induction on n. Since

$$\omega(p_1) = \omega(e_1) = h_1 = p_1,$$

the result holds for $n = 1$. Then, for $n \ge 2$, we have, using (2.41),

$$\omega(p_n) \overset{(2.41)}{=} n\omega(h_n) - \sum_{r=1}^{n-1} \omega(p_r)\omega(h_{n-r}) = ne_n - \sum_{r=1}^{n-1} (-1)^{r-1} p_r e_{n-r} \overset{(2.39)}{=} (-1)^{n-1} p_n,$$

completing the proof of the induction step. Then, for $\lambda \in \mathrm{Par}$, we have

$$\omega(p_\lambda) = \omega(p_{\lambda_1}) \cdots \omega(p_{\lambda_{\ell(\lambda)}}) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda.$$ □

It follows from Proposition 2.2.5 that we can write each p_λ in terms of the monomial symmetric functions. The next result gives an explicit formula.

Proposition 2.6.7. *For $k \geq 1$ and $\lambda \in \mathrm{Par}(k)$, we have*

$$p_\lambda = \sum_{\mu \vdash k} M_{\lambda,\mu}(p,m) m_\mu, \tag{2.43}$$

where $M_{\lambda,\mu}(p,m)$ is the number of Young tableaux of shape λ and weight μ in which the entries in each row are constant.

Proof. The proof of this result is left as Exercise 2.6.2. □

We conclude this section by deducing explicit formulas for the h_n and e_n as linear combinations of the p_λ. Recall that, for $\lambda \in \mathrm{Par}$, $m_i(\lambda)$ denotes the multiplicity of i in λ. Define

$$z_\lambda := \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)! \tag{2.44}$$

(See Exercise 1.1.2 for a combinatorial interpretation of z_λ.)

Proposition 2.6.8. *We have*

$$H(t) = \sum_{\lambda \in \mathrm{Par}} \frac{p_\lambda}{z_\lambda} t^{|\lambda|} \qquad \text{and} \qquad E(t) = \sum_{\lambda \in \mathrm{Par}} (-1)^{|\lambda| - \ell(\lambda)} \frac{p_\lambda}{z_\lambda} t^{|\lambda|}. \tag{2.45}$$

Proof. By (2.36), we have

$$H(t) = \exp P(t) = \exp \sum_{r \geq 1} \frac{p_r}{r} t^r = \prod_{r \geq 1} \exp\left(\frac{p_r}{r} t^r\right) = \prod_{r \geq 1} \sum_{m_r = 0}^{\infty} \frac{(p_r t^r)^{m_r}}{m_r! r^{m_r}} = \sum_{\lambda \in \mathrm{Par}} \frac{p_\lambda}{z_\lambda} t^{|\lambda|}.$$

We now obtain the second equation in (2.45) by applying ω and using (2.42). □

Corollary 2.6.9. *For $n \in \mathbb{N}$, we have*

$$h_n = \sum_{\lambda \in \mathrm{Par}(n)} \frac{p_\lambda}{z_\lambda} \qquad \text{and} \qquad e_n = (-1)^n \sum_{\lambda \in \mathrm{Par}(n)} (-1)^{\ell(\lambda)} \frac{p_\lambda}{z_\lambda}. \tag{2.46}$$

Proof. This follows from equating coefficients in (2.45). □

Exercises.

2.6.1. Prove (2.26) by following the method of the proof of Lemma 2.4.2 and (2.19).

2.6.2. Prove Proposition 2.6.7.

2.6.3. Write $p_{(2,2,1)}$ and $p_{(2,2,2)}$ as linear combinations of monomial symmetric functions.

2.6.4. Find and prove a formula for $M_{\lambda,(n)}(p,m)$, where $\lambda \vdash n$.

2.6.5. Find and prove a formula for $M_{\lambda,(1^n)}(p,m)$, where $\lambda \vdash n$.

2.6.6. Find and prove a formula for $M_{\lambda,(n-1,1)}(p,m)$, where $\lambda \vdash n$.

Index of notation

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