Abstract

Relations between algebraic and geometric constructions in representation theory

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Geometric methods have been utilized to explore many problems in representation theory. In particular, Lusztig and Nakajima have used the geometry of quiver varieties to examine Kac-Moody Lie algebras and their representations. However, few concrete examples exist of the applications of these methods. In this dissertation, some such examples are examined and the geometric interpretation of some classical and recent algebraic constructions are given.

Using the tensor product variety introduced by Malkin and Nakajima, the complete structure of the tensor product of a finite number of integrable highest weight modules of $U_q(\mathfrak{sl}_2)$ is recovered. In particular, the elementary basis, Lusztig's canonical basis, and the basis adapted to the decomposition of the tensor product into simple modules are all exhibited as distinguished elements of certain spaces of constructible functions on the tensor product variety. For the latter two bases, these distinguished elements are closely related to the irreducible components of this variety, which turns out to be closely related to Grassmannians and flag varieties. The space of intertwining operators is also interpreted geometrically. A similar construction is used to produce the *fusion* tensor products, certain truncations of the usual tensor products, which are not covered by the theory of Lusztig and Nakajima.

In addition, two apparently different bases in the representations of affine Lie algebras of type A, one arising from statistical mechanics, the other from quiver theory with origins in gauge theory, are compared. It is shown that the two are governed by the same combinatorics that also respects the weight space decomposition of the representations. In particular, an alternative and much simpler geometric proof of a result of Date, Jimbo, Kuniba, Miwa and Okado on the construction of bases of affine Lie algebra representations is given. At the same time, a simple parametrization of the irreducible components of Nakajima quiver varieties associated to infinite and cyclic quivers is presented. Finally, new varieties whose irreducible components are in one to one correspondence with highest weight representations of $\widehat{\mathfrak{gl}}_{n+1}$ are defined.

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To my father, Russell Savage,

who has always been my academic inspiration.

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Introduction

This dissertation examines some applications of geometric methods to algebraic constructions in representation theory. These techniques have produced many results such as the proof of the Kazhdan-Lusztig conjecture and irreducible representations of Weyl groups which mathematicians have been unable to obtain by direct algebraic methods. Geometric representation theory is also particularly well suited to proving positivity and integrality results as these are often easy consequences of the geometric nature of the objects involved. Such facts can be hard to verify from a purely algebraic viewpoint.

The basic idea underlying many constructions in geometric representation theory is the following. For an irreducible representation of an associative algebra A, one defines an algebraic variety M and a subvariety Z of $M \times M$ called a correspondence. Under convolution, the homology of Z has the structure of an algebra and we have a surjective homomorphism from A to the homology of Z. The convolution also yields an action of the homology of Z on the homology of M and via the above homomorphism, this endows the homology of M with the structure of an A module. The top dimensional homology is then shown to have the structure of the representation in question. This method is thoroughly reviewed in [2], which includes a geometric construction of the Weyl group and the universal enveloping algebra of $\mathfrak{sl}_n$. Here the varieties involved are Grassmannians and flag varieties. Instead of homology one can use an equivalent language of constructible functions, which is the primary tool in this dissertation.

One area of current interest is to obtain results in the representation theory of Lie algebras in the above way via varieties attached to quivers. These techniques allow one to prove old results in different ways as well as discover new results. Quivers and varieties associated with them have a long history. The study of the Jordan quiver consisting of one vertex connected by an edge with itself is simply the study of Jordan normal forms. The study of quiver varieties is also related to such fields as moduli spaces of instantons and geometric Langlands correspondence. The field has recently flourished with the work of Lusztig and Nakajima who defined a new class of quiver varieties which yield a geometric construction of universal enveloping algebras of Kac-Moody algebras and their representations. Among other things, their results defined a canonical basis in these spaces with some remarkable properties. These bases were discovered algebraically in a completely different manner by Kashiwara via the theory of crystal bases. However, certain results such as positivity can only be proved by geometric methods. While the geometric theory of Lusztig and Nakajima is very beautiful, few concrete examples of applications of the theory exist. One of the central goals of this dissertation is to provide some such examples and to give an explicit realization of the geometric constructions for certain cases where additional algebraic structures have been developed.

In a remarkable series of papers originally motivated by gauge theory, Nakajima has given a geometric realization of integrable highest weight representations V_λ of a Kac-Moody algebra $\mathfrak{g}$ in the homology of a certain Lagrangian subvariety $\mathcal{L}(\lambda)$ of a symplectic variety $\mathcal{M}(\lambda)$ constructed from the Dynkin diagram of $\mathfrak{g}$ (the *quiver variety*). More recently, he (and independently Malkin) realized the tensor product $V_\lambda \otimes V_\mu$ as the homology of a tensor product variety $\mathcal{L}(\lambda, \mu) \subset \mathcal{M}(\lambda+\mu)$ and endowed the irreducible components of this variety with the structure of a crystal isomorphic to that of the q-analogue of the tensor product.

In the first chapter of this dissertation the specific case of $U_q(\mathfrak{sl}_2)$ is considered.

Here the quiver varieties involved turn out to be closely related to Grassmannians and flag varieties. The complete structure of the tensor product of a finite number of integrable highest weight modules of $U_q(\mathfrak{sl}_2)$ is realized geometrically. In particular, the elementary basis, Lusztig's canonical basis, and the basis adapted to the decomposition of the tensor product into simple modules are all exhibited as distinguished elements of certain spaces of constructible functions on the tensor product variety. For the latter two bases, these distinguished elements are closely related to the irreducible components of the tensor product variety. The space of intertwining operators $\mathrm{Hom}_{\mathfrak{sl}_2}(V_\lambda \otimes V_\mu, V_\rho)$ is also interpreted geometrically. This yields much more structure than that obtained in the general case by Nakajima and Malkin where only the crystal structure ($q = 0$ case) is recovered. It was known that the algebraic structure of tensor products of $\mathfrak{sl}_2$-modules can be described by the graphical calculus developed by Rumer, Teller and Weyl and later rediscovered and further developed by Penrose and others. Chapter 1 gives a natural geometric interpretation of this graphical calculus and its q-deformation. The results of this chapter also appear in [26].

Another important product in representation theory is the fusion product. Fusion products appear both in conformal field theory and quantum group theory. The natural ring associated to these products, which are certain truncations of tensor products, is the Grothendieck ring of both the modular category of integrable $\widehat{\mathfrak{sl}}_2$-modules of level l and of a suitable quotient of the category of tilting modules over $U_\epsilon(\mathfrak{sl}_2)$ when ϵ is an $(l+2)$th root of unity.

Chapter 2 consists of realizing the fusion products $V_\lambda \otimes_l V_\mu$ as both the homology of the most natural subvarieties $\mathcal{L}_l(\lambda,\mu) \subset \mathcal{L}(\lambda,\mu)$ and a space of constructible functions on these subvarieties. The case of the product of an arbitrary (finite) number of $\mathfrak{sl}_2$-modules is also considered. A graphical calculus for the fusion product is developed by modifying that of $U_q(\mathfrak{sl}_2)$ and is used to give a combinatorial description of the irreducible components of the fusion product varieties. This chapter is a first step in an attempt to define a general fusion tensor product variety for more general Lie algebras, which is an important open problem. The results of this chapter also appear in [27].

We see that the explicit realization of Nakajima's theory in the case of $\mathfrak{sl}_2$ leads to further developments such as a geometric description of the fusion product. This realization also leads to conjectures concerning the extension of some results to more general types of Lie algebras. One conjecture of this type is made in Chapter 1. Also, equipped with the geometrization of the tensor product $V_\lambda \otimes V_\mu$ using homology, K-theory or constructible functions on quiver varieties, we can then pass to the category of sheaves on these varieties. Then intertwining operators $V_\lambda \otimes V_\mu \to V_\rho$ yield functors. We can obtain a categorical version of the representation theory of $\mathfrak{sl}_2$ whose Grothendieck ring yields the tensor product structure. This is related to a program of Khovanov to obtain structures such as the Temperley-Lieb algebra, the Jones polynomial and representations of Lie algebras from categories and to develop combinatorial constructions of invariants of two knots in four dimensional space (see [12] and [13]). One should also mention that the correspondence between geometry and algebra can be utilized in reverse. With the above mentioned results, methods in the representation theory of $\mathfrak{sl}_2$ can be used as a useful tool in the study of the geometry of Grassmannians and flag varieties that appear in our geometric realization.

Thus for $\mathfrak{sl}_2$, the geometric structure of the tensor product is quite rich and leads to many interesting results. However, for $\mathfrak{sl}_n$ and its associated affine Lie algebra $\widehat{\mathfrak{sl}}_n$, the representation theory alone is much more involved than for $\mathfrak{sl}_2$. This is the focus of the second part of this dissertation.

A remarkable relation between the representation theory of affine Lie algebras and models of statistical mechanics based on the Yang-Baxter equation has been discovered and intensively studied by E. Date, M. Jimbo, A. Kuniba, T. Miwa and

M. Okado. Their results yield certain explicit bases in the representations that admit pure combinatorial descriptions and imply various identities for the characters.

Chapter 3 relates the bases of representations of Lie algebras of type A obtained from statistical mechanics with those obtained geometrically via Nakajima quiver varieties. It is shown that the two are governed by the same combinatorics that also respects the weight space decomposition of the representations. An alternative and much simpler geometric proof of the statistical mechanics results is presented while at the same time giving simple parametrizations of the irreducible components of Nakajima quiver varieties associated to infinite and cyclic quivers. In particular, one sees Young diagrams arising as commutative diagrams related to the quiver varieties. We believe that this work reflects a more general principle that the combinatorial structures of integrable models in statistical mechanics have a profound geometric or gauge theoretic origin. This chapter is an important step toward this vast program. The results of this chapter also appear in [6].

Thus, as for the graphical calculus for $\mathfrak{sl}_2$, we have obtained a geometric realization of Young diagrams. The graphical calculus and the theory of Young diagrams have both been studied intensively and are considered basic notions of representation theory. The current work provides, among other results, a new geometric interpretation of these classical subjects.

Chapter 1

Tensor Products

Introduction

The purpose of this chapter is to obtain a geometric description of the tensor product of a finite number of integrable highest weight representations of $U_q(\mathfrak{sl}_2)$ using quiver varieties. The definition of a *tensor product variety* corresponding to the tensor product of a finite number of integrable highest weight representations of a Lie algebra $\mathfrak{g}$ of ADE type was introduced in [19] and [25] (see also [28] for a geometric description of the tensor product). There it is demonstrated that the set of irreducible components of the tensor product variety can be equipped with the structure of a $\mathfrak{g}$-crystal isomorphic to the crystal of the canonical basis in the tensor product representation.

In this chapter, we consider the specific case $\mathfrak{g} = \mathfrak{sl}_2$ and recover the entire structure (as opposed to the crystal structure alone) of $U_q(\mathfrak{sl}_2)$ via the tensor product variety. Our definition of the tensor product variety differs slightly from that of [19] and [25] in that we consider our varieties over the finite field $\mathbb{F}_{q^2}$ with q^2 elements (or its algebraic closure $\bar{\mathbb{F}}_{q^2}$) rather than over $\mathbb{C}$. The reader who is only interested in representations of $\mathfrak{sl}_2$, rather than its associated quantum group, may replace $\mathbb{F}_{q^2}$ by $\mathbb{C}$ and set $q = 1$ everywhere. With a few obvious modifications, the arguments still hold. Let $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^k$. We find three distinct spaces, $\mathcal{T}_0(\mathbf{d})$, $\mathcal{T}_c(\mathbf{d})$, and $\mathcal{T}_s(\mathbf{d})$, of constructible (with respect to a natural stratification) functions on the tensor product variety $\mathfrak{T}(\mathbf{d})$, each isomorphic to $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$. In each space we define a natural basis. These three bases, $\mathcal{B}_e$, $\mathcal{B}_c$, and $\mathcal{B}_s$, correspond respectively to the elementary basis, Lusztig's canonical basis [17], and a basis compatible with the decomposition of $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ into a direct sum of irreducible modules. The two bases $\mathcal{B}_c$ and $\mathcal{B}_s$ are characterized by their relation to the irreducible components of $\mathfrak{T}(\mathbf{d})$. We define the irreducible components of $\mathfrak{T}(\mathbf{d})$ (defined over $\mathbb{F}_{q^2}$) to be the $\mathbb{F}_{q^2}$ points of the irreducible components of $\mathfrak{T}(\mathbf{d})'$ (the corresponding variety defined over $\bar{\mathbb{F}}_{q^2}$). We then define the *dense points* of an irreducible component of $\mathfrak{T}(\mathbf{d})$ to be the $\mathbb{F}_{q^2}$ points of a certain dense subset of the corresponding irreducible component of $\mathfrak{T}(\mathbf{d})'$. Distinct elements of the basis $\mathcal{B}_c$ and $\mathcal{B}_s$ are supported on distinct irreducible components of $\mathfrak{T}(\mathbf{d})$ and equal to a non-zero constant on the set of dense points of that irreducible component (see Theorems 1.2.I and 1.3.F). However, the supports of the elements of $\mathcal{B}_s$ are disjoint whereas the supports of the elements of $\mathcal{B}_c$ are not. We also find a geometric description of the space of intertwining operators $H^{\mu}_{\mathbf{d}_1,\ldots,\mathbf{d}_k} = \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}, V_\mu)$. A natural basis $\mathcal{B}_I$ of this space is again characterized by its relation to the irreducible components of $\mathfrak{T}(\mathbf{d})$.

An important tool used in the development and proof of the results of this chapter is the graphical calculus of intertwining operators of $U_q(\mathfrak{sl}_2)$ introduced by Penrose, Kauffman and others. This graphical calculus is expanded in [5] and used to prove various results concerning Lusztig's canonical basis. The present chapter can be considered a "geometrization" of these results.

In Section 1.2.7 we conjecture a characterization of the basis $\mathcal{B}_c$ as the image of certain intersection cohomology sheaves of $\mathfrak{T}(\mathbf{d})$ under a particular functor from the space of constructible semisimple perverse sheaves on $\mathfrak{T}(\mathbf{d})$ to the space of constructible functions on $\mathfrak{T}(\mathbf{d})$. Since the definition of $\mathcal{T}_c(\mathbf{d})$ relies on the graphical calculus of intertwining operators of $U_q(\mathfrak{sl}_2)$ (and no such graphical calculus exists for more general Lie algebras), this conjecture should play a key role in the possible extension of the results of this chapter to a more general set of Lie algebras (for instance, those of type ADE).

This chapter is organized as follows. Section 1.1 contains a review of $U_q(\mathfrak{sl}_2)$ and its representations, Nakajima's quiver varieties, and the graphical calculus of intertwining operators of $U_q(\mathfrak{sl}_2)$. The tensor product variety is defined in Section 1.2 where the spaces $\mathcal{T}_0(\mathbf{d})$ and $\mathcal{T}_c(\mathbf{d})$ are introduced, an isomorphism between the two is given, and various results concerning these spaces and their distinguished bases $\mathcal{B}_e$ and $\mathcal{B}_c$ are proved. Section 1.3 is concerned with a geometric realization of the space of intertwining operators and the decomposition of the tensor product representation into a direct sum of irreducible modules (via the space $\mathcal{T}_s(\mathbf{d})$ and the distinguished basis $\mathcal{B}_s$). It is concluded with the discussion of an isomorphism between the spaces $\mathcal{T}_c(\mathbf{d})$ and $\mathcal{T}_s(\mathbf{d})$.

The notation used in the description of quiver varieties is not standardized. Lusztig denotes the fixed vector space by D and the subpace by V while Nakajima denotes these objects by W and V respectively. Since we wish to use the notation V_n for certain $U_q(\mathfrak{sl}_2)$ modules (to agree with the notation of [5]), we denote the fixed vector space by D and the subspace by W. We hope that this will not cause confusion among those readers familiar with the work of Lusztig and Nakajima.

Throughout this chapter the topology is the Zariski topology and the ground field is $\bar{\mathbb{F}}_{q^2}$ unless otherwise specified. However, we will usually deal with varieties defined over $\mathbb{F}_{q^2}$ and consider the corresponding set of $\mathbb{F}_{q^2}$-rational points. Thus, for instance, $\mathbb{P}^n = \mathbb{P}^n\mathbb{F}_{q^2}$ and a vector space is an $\mathbb{F}_{q^2}$ vector space. A function on an algebraic variety is a function into $\mathbb{C}(q)$, the field of rational functions in an indeterminate q. The span of a set of such functions is their $\mathbb{C}(q)$-span. The support of a function f is defined to be the set $\{x \mid f(x) \neq 0\}$ and *not* the closure of this set.

1.1 The Quantum Group $U_q(\mathfrak{sl}_2)$ and its Representations

1.1.1 The Hopf Algebra Structure of $U_q(\mathfrak{sl}_2)$

Let $\mathbb{C}(q)$ be the field of rational functions in an indeterminate q and define $^{-} : \mathbb{C}(q) \to \mathbb{C}(q)$ to be the $\mathbb{C}$-algebra involution such that $\overline{q^n} = q^{-n}$ for all n. The quantum group $U_q(\mathfrak{sl}_2)$ (which we will denote by $\mathbf{U}_q$) is the associative algebra over $\mathbb{C}(q)$ with generators E, F, K, K^{-1} and relations

$$KK^{-1} = K^{-1}K$$

$$KE = q^2 EK$$

$$KF = q^{-2} FK$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

The comultiplication and counit of the Hopf algebra structure of $\mathbf{U}_q$ are given by

$$\Delta K^{\pm 1} = K^{\pm 1} \otimes K^{\pm 1}$$

$$\Delta E = E \otimes 1 + K \otimes E$$

$$\Delta F = F \otimes K^{-1} + 1 \otimes F$$

and

$$\eta(K^{\pm 1}) = 1$$

$$\eta(E) = \eta(F) = 0$$

respectively. Although an explicit expression for the antipode exists, we will not need it.

Let us introduce two involutions of $\mathbf{U}_q$. The first one is the *Cartan involution*, denoted by ω, which acts as follows:

$$\omega(E) = F, \quad \omega(F) = E, \quad \omega(K^{\pm 1}) = K^{\pm 1}, \quad \omega(q^{\pm 1}) = q^{\pm 1}$$

$$\omega(xy) = \omega(y)\omega(x), \quad x, y \in \mathbf{U}_q.$$

The second, denoted by σ, is called the *"bar" involution* and is defined by

$$\sigma(E) = E, \quad \sigma(F) = F, \quad \sigma(K^{\pm 1}) = K^{\mp 1}, \quad \sigma(q^{\pm 1}) = q^{\mp 1}$$
$$\sigma(xy) = \sigma(x)\sigma(y), \quad x, y \in \mathbf{U}_q.$$

Using σ we can define a second comultiplication $\bar{\Delta}$ by

$$\bar{\Delta}(x) = (\sigma \otimes \sigma)\Delta(\sigma(x)), \quad x \in \mathbf{U}_q$$

which implies

$$\bar{\Delta}K^{\pm 1} = K^{\pm 1} \otimes K^{\pm 1}$$
$$\bar{\Delta}E = E \otimes 1 + K^{-1} \otimes E$$
$$\bar{\Delta}F = F \otimes K + 1 \otimes F.$$

1.1.2 Irreducible Representations of $U_q(\mathfrak{sl}_2)$

Any finite dimensional irreducible $\mathbf{U}_q$-module V is generated by a highest weight vector, v, of weight εq^d where $\varepsilon = \pm 1$ and $d = \dim(V) - 1$ [11]. We consider those representations with $\varepsilon = +1$. Let $v_{d-2k} = F^k v/[k]!$ where

$$[k] = (q^k - q^{-k})/(q - q^{-1}) = q^{-k+1} + q^{-k+3} + \cdots + q^{k-1},$$
$$[k]! = [1][2] \cdots [k].$$

Then $v_{d-2k} = 0$ for $k > d$ and $\{v = v_d, v_{d-2}, \ldots, v_{-d}\}$ is a basis of V. We denote this representation by V_d. The action of $\mathbf{U}_q$ on V_d is given by

$$\begin{aligned} K^{\pm 1} v_m &= q^{\pm m} v_m \\ E v_m &= \left[\frac{d+m}{2} + 1\right] v_{m+2} \\ F v_m &= \left[\frac{d-m}{2} + 1\right] v_{m-2}. \end{aligned} \tag{1.1}$$

Define a bilinear symmetric pairing on V_d by requiring

$$\langle xu, v\rangle = \langle u, \omega(x)v\rangle\,, \quad \langle v_d, v_d\rangle = 1, \quad u, v \in V_d \text{ and } x \in \mathbf{U}_q.$$

It follows that

$$\langle v_{d-2k}, v_{d-2l}\rangle = \delta_{k,l}\begin{bmatrix} d \\ k \end{bmatrix}$$

where

$$\begin{bmatrix} d \\ k \end{bmatrix} = \frac{[d]!}{[k]![d-k]!}.$$

Let $\{v^{d-2k}\}_{k=0}^{d}$ be the basis dual to $\{v_{d-2k}\}_{k=0}^{d}$ with respect to the form $\langle\ ,\ \rangle$. Then

$$v^{d-2k} = \begin{bmatrix} d \\ k \end{bmatrix}^{-1} v_{d-2k}$$

and the action of $\mathbf{U}_q$ in the dual basis is

$$K^{\pm 1}v_m = q^{\pm m}v_m$$
$$Ev^m = \begin{bmatrix} \frac{d-m}{2} \end{bmatrix} v^{m+2}$$
$$Fv^m = \begin{bmatrix} \frac{d+m}{2} \end{bmatrix} v^{m-2}.$$

1.1.3 Geometric Realization of Irreducible Representations of $U_q(\mathfrak{sl}_2)$

We recall here Nakajima's quiver variety construction of finite dimensional irreducible representations of Kac-Moody algebras associated to symmetric Cartan matrices [21, 23] in the specific case of $U_q(\mathfrak{sl}_2)$. In order to introduce the quantum parameter q, some of our definitions differ slightly from those in [21, 23]. Since the Dynkin diagram of $\mathfrak{sl}_2$ consists of a single vertex and no edges, the definition of the quiver variety simplifies considerably. Fix vector spaces W and D of dimensions w and d respectively and consider the variety

$$\mathbf{M}(w, d) = \mathrm{Hom}(D, W) \oplus \mathrm{Hom}(W, D).$$

The two components of an element of $\mathbf{M}(w,d)$ will be denoted by f_1 and f_2 respectively. $GL(W)$ acts on $\mathbf{M}(w,d)$ by

$$(f_1,f_2)\mapsto g(f_1,f_2)\stackrel{\text{def}}{=}(gf_1,f_2g^{-1}),\ g\in GL(W).$$

Define the map $\mu:\mathbf{M}(w,d)\to \operatorname{End} W$ by

$$\mu(f_1,f_2)=f_1f_2.$$

Let $\mu^{-1}(0)$ be the algebraic variety defined as the zero set of μ. We say a point (f_1,f_2) of $\mu^{-1}(0)$ is *stable* if f_2 is injective. The *quiver variety* is then given by

$$\{(f_1,f_2)\in\mu^{-1}(0)\,|\,(f_1,f_2)\text{ is stable}\}/GL(W).$$

Via the map $(f_1,f_2)\mapsto(\operatorname{im} f_2, f_2f_1)$, this variety is seen to be isomorphic to the variety

$$\mathfrak{M}(w,d)=\{(W,t)\,|\,W\subset D,\ \dim W=w,\ t\in\operatorname{End} D,\ \operatorname{im} t\subset W\subset\ker t\}.$$

Note that the condition $\operatorname{im} t\subset W\subset\ker t$ implies $t^2=0$. Let

$$\mathfrak{M}(d)=\bigcup_w\mathfrak{M}(w,d)=\{(W,t)\,|\,W\subset D,\ t\in\operatorname{End} D,\ \operatorname{im} t\subset W\subset\ker t\}.$$

and

$$\mathfrak{M}(w,w+1,d)=\{(U,W,t)\,|\,t\in\operatorname{End} D,\ \operatorname{im} t\subset U\subset W\subset\ker t,$$
$$\dim U=w,\ \dim W=w+1\}.$$

We then have the projections

$$\mathfrak{M}(d)\xleftarrow{\pi_1}\bigcup_w\mathfrak{M}(w,w+1,d)\xrightarrow{\pi_2}\mathfrak{M}(d)$$

given by $\pi_1(U,W,t)=(U,t)$ and $\pi_2(U,W,t)=(W,t)$.

For a subset Y of a variety A, let $\mathbf{1}_Y$ denote the function on A which takes the value 1 on Y and 0 elsewhere. Note that since our varieties are defined over $\mathbb{F}_{q^2}$, they consist of a finite number of ($\mathbb{F}_{q^2}$- rational) points. Let $\chi_q(Y)$ denote the Euler characteristic of the algebraic variety Y, which is merely the number of points in Y. For a map π between algebraic varieties A and B, let $\pi_!$ [18] denote the map between the abelian groups of functions on A and B given by

$$\pi_!(f)(x) = \sum_{y \in \pi^{-1}(x)} f(y)$$
$$\Rightarrow \pi_!(\mathbf{1}_Y)(x) = \chi_q(\pi^{-1}(x) \cap Y),\ Y \subset A$$

and let π^* be the pullback map from functions on B to functions on A acting as $\pi^* f(x) = f(\pi(x))$.

We then define the action of E, F and $K^{\pm 1}$ on the set of functions on $\mathfrak{M}(d)$ by

$$\begin{aligned} Ef &= q^{-\dim(\pi_1^{-1}(\cdot))}(\pi_1)_! \pi_2^* f \\ Ff &= q^{-\dim(\pi_2^{-1}(\cdot))}(\pi_2)_! \pi_1^* f \\ K^{\pm 1} f &= q^{\pm(d - 2\dim(\cdot))} f \end{aligned} \tag{1.2}$$

where the notation means that for a function f on $\mathfrak{M}(d)$ and $(W, t) \in \mathfrak{M}(d)$,

$$\begin{aligned} Ef(W,t) &= q^{-\dim(\pi_1^{-1}(W,t))}(\pi_1)_! \pi_2^* f(W,t) \\ Ff(W,t) &= q^{-\dim(\pi_2^{-1}(W,t))}(\pi_2)_! \pi_1^* f(W,t) \\ K^{\pm 1} f(W,t) &= q^{\pm(d - 2\dim W)} f(W,t). \end{aligned} \tag{1.3}$$

Let

$$\mathfrak{M}^r(d) = \{(W,t) \in \mathfrak{M}(d) \mid \operatorname{rank} t = r\}$$
$$\mathfrak{M}^r(w,d) = \{(W,t) \in \mathfrak{M}(w,d) \mid \operatorname{rank} t = r\}$$
$$\mathcal{M}^r(w,d) = \mathbb{C}(q)\mathbf{1}_{\mathfrak{M}^r(w,d)}$$
$$\mathcal{M}^r(d) = \bigoplus_w \mathcal{M}^r(w,d)$$
$$\mathcal{M}(w,d) = \bigoplus_r \mathcal{M}^r(w,d)$$
$$\mathcal{M}(d) = \bigoplus_w \mathcal{M}(w,d).$$

Also, let us introduce the following notation for Grassmannians:

$$Gr_w^d = \{W \subset (\mathbb{F}_{q^2})^d \,|\, \dim W = w\}.$$

Proposition 1.1.A. *The action of* $\mathbf{U}_q$ *defined by* (1.2) *endows* $\mathcal{M}^r(d)$ *(and hence* $\mathcal{M}(d)$*) with the structure of a* $\mathbf{U}_q$*-module and the map* $\mathbf{1}_{\mathfrak{M}^r(w,d)} \mapsto v_{d-2w}$ *(extended by linearity) is an isomorphism* $\mathcal{M}^r(d) \cong V_{d-2r}$ *of* $\mathbf{U}_q$*-modules.*

To prove this proposition, we will need the following lemmas.

Lemma 1.1.B. *For vector spaces* $W \subset D$*,* $\{U \,|\, W \subset U \subset D, \dim U = u\} \cong Gr_{u-\dim W}^{\dim D - \dim W}$.

Proof. This follows immediately from the fact that

$$\{U \,|\, W \subset U \subset D, \dim U = u\} \cong \{U' \,|\, U' \subset D/W, \dim U' = u - \dim W\}$$

via the map $U \mapsto U' = U/W$. □

Lemma 1.1.C. $\chi_q(\mathbb{P}^n) = \sum_{i=0}^n q^{2i}$

Proof. This follows from simply counting the number of possible one dimensional subspaces of $\mathbb{P}^n$. □

Proof of Proposition 1.1.A. If $(W,t) \in \mathfrak{M}^r(w,d)$ then

$$\begin{aligned}
E\mathbf{1}_{\mathfrak{M}^r(w+1,d)}(W,t) &= q^{-\dim(\pi_1^{-1}(W,t))}(\pi_1)_!\pi_2^*\mathbf{1}_{\mathfrak{M}^r(w+1,d)}(W,t) \\
&= q^{-\dim(\{U \,|\, W \subset U \subset \ker t, \dim U = w+1\})}(\pi_1)_!\mathbf{1}_{\mathfrak{M}^r(w,w+1,d)}(W,t) \\
&= q^{-\dim(Gr_1^{d-w-r})}\chi_q(\pi_1^{-1}(W,t) \cap \mathfrak{M}^r(w,w+1,d)) \\
&= q^{-\dim(\mathbb{P}^{d-w-r-1})}\chi_q(\{U \,|\, W \subset U \subset \ker t, \dim U = w+1\}) \\
&= q^{-(d-w-r-1)}\chi_q(Gr_1^{d-w-r}) \\
&= q^{-(d-w-r-1)}\chi_q(\mathbb{P}^{d-w-r-1}) \\
&= q^{-(d-w-r-1)}\sum_{i=0}^{d-w-r-1} q^{2i}
\end{aligned}$$

$$= q^{-(d-w-r-1)} + q^{-(d-w-r-1)+2} + \cdots + q^{d-w-r-1}$$

$$= [d-w-r]$$

and $E\mathbf{1}_{\mathfrak{M}^r(w+1,d)}(W,t) = 0$ otherwise. So $E\mathbf{1}_{\mathfrak{M}^r(w+1,d)} = [d-w-r]\mathbf{1}_{\mathfrak{M}^r(w,d)}$. Similarly, if $(W,t) \in \mathfrak{M}^r(w+1,d)$,

$$\begin{aligned}
F\mathbf{1}_{\mathfrak{M}^r(w,d)}(W,t) &= q^{-\dim(\pi_2^{-1}(W,t))}(\pi_2)_!\pi_1^*\mathbf{1}_{\mathfrak{M}^r(w,d)}(W,t) \\
&= q^{-\dim(\{U \mid \operatorname{im} t \subset U \subset W, \dim U = w\})}(\pi_2)_!\mathbf{1}_{\mathfrak{M}^r(w,w+1,d)}(W,t) \\
&= q^{-\dim(Gr^{w+1-r}_{w-r})}\chi_q(\pi_2^{-1}(W,t) \cap \mathfrak{M}^r(w,w+1,d)) \\
&= q^{-\dim(\mathbb{P}^{w-r})}\chi_q(\{U \mid \operatorname{im} t \subset U \subset W,\ \dim U = w\}) \\
&= q^{-(w-r)}\chi_q(\mathbb{P}^{w-r}) \\
&= q^{-(w-r)}\sum_{i=0}^{w-r} q^{2i} \\
&= q^{-(w-r)} + q^{-(w-r)+2} + \cdots + q^{w-r} \\
&= [w+1-r]
\end{aligned}$$

and $F\mathbf{1}_{\mathfrak{M}^r(w,d)}(W,t) = 0$ otherwise. So $F\mathbf{1}_{\mathfrak{M}^r(w,d)} = [w+1-r]\mathbf{1}_{\mathfrak{M}^r(w+1,d)}$. It is obvious that

$$K^{\pm 1}\mathbf{1}_{\mathfrak{M}^r(w,d)} = q^{\pm(d-2w)}\mathbf{1}_{\mathfrak{M}^r(w,d)}. \tag{1.4}$$

Now, $\mathfrak{M}^r(w,d) = \emptyset$ unless $r \le w \le d-r$ due to the requirement $\operatorname{im} t \subset W \subset \ker t$ in the definition of $\mathfrak{M}^r(w,d)$. Thus $\mathcal{M}^r(d) = \bigoplus_{w=r}^{w=d-r} \mathcal{M}^r(w,d)$.

Comparing the above calculations to (1.1), the result follows. □

So $\mathcal{M}(d)$ is isomorphic to the direct sum of the irreducible representations of highest weight $d-2r$ where $0 \le r \le d/2$ since these are the possible ranks of t (recall that $t^2 = 0$).

Let $\mathfrak{L}(d) = \mathfrak{M}^0(d)$. Then $\mathfrak{L}(d)$ is isomorphic to the algebraic variety of all subspaces $W \subset D$, which is a union of Grassmannians. Let

$$\mathfrak{L}(w,d) = \mathfrak{M}^0(w,d) = \{W \subset D \mid \dim W = w\} \cong Gr^d_w.$$

and

$$\mathcal{L}(w,d) = \mathcal{M}^0(w,d) = \mathbb{C}(q)\mathbf{1}_{\mathfrak{L}(w,d)}, \quad \mathcal{L}(d) = \mathcal{M}^0(d) = \bigoplus_{w=1}^{d} \mathcal{L}(w,d).$$

We see from Proposition 1.1.A that the action of $\mathbf{U}_q$ defined by (1.2) endows $\mathcal{L}(d)$ with the structure of the irreducible module V_d via the isomorphism $\mathbf{1}_{\mathfrak{L}(w,d)} \mapsto v_{d-2w}$ (extended by linearity). Note that for $(W,t) \in \mathfrak{M}(d)$, we can think of t as belonging to $\mathrm{Hom}(D/W, W)$ and thus $\mathfrak{M}(d)$ is the cotangent bundle of $\mathfrak{L}(d)$.

1.1.4 Tensor Products and the Graphical Calculus of Intertwining Operators

We define the bilinear pairing of $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ with $V_{\mathbf{d}_k} \otimes \cdots \otimes V_{\mathbf{d}_1}$ by

$$\left\langle v_{i_1} \otimes \cdots \otimes v_{i_k}, v^{l_k} \otimes \cdots \otimes v^{l_1} \right\rangle = \delta_{i_1}^{l_1} \ldots \delta_{i_k}^{l_k}.$$

Then

$$\left\langle \Delta^{n-1}(x) v_{i_1} \otimes \cdots \otimes v_{i_k}, v^{l_k} \otimes \cdots \otimes v^{l_1} \right\rangle \\ = \left\langle v_{i_1} \otimes \cdots \otimes v_{i_k}, \bar{\Delta}^{n-1}(\omega(x)) v^{l_k} \otimes \cdots \otimes v^{l_1} \right\rangle.$$

Lusztig's canonical basis of the tensor product is described in [17]. We refer the reader to this article or the overview in [5], Section 1.5, for the definition of this basis. As in [5] and [17], we denote the elements of Lusztig's canonical basis by $v_{i_1} \diamondsuit \cdots \diamondsuit v_{i_k}$ and their dual by $v_{i_1} \heartsuit \cdots \heartsuit v_{i_k}$. The dual is defined with respect to the form $\langle\ ,\ \rangle$:

$$\left\langle v_{i_1} \diamondsuit \cdots \diamondsuit v_{i_k}, v^{l_k} \heartsuit \cdots \heartsuit v^{l_1} \right\rangle = \delta_{i_1}^{l_1} \ldots \delta_{i_k}^{l_k}.$$

When we wish to make explicit to which representation a vector belongs, we use the notation ${}^d v_k$, ${}^d v^k \in V_d$

To simplify notation, we make the following definitions

$$\otimes^{\mathbf{d}} v_{\mathbf{w}} = {}^{\mathbf{d}_1} v_{\mathbf{d}_1 - 2\mathbf{w}_1} \otimes \cdots \otimes {}^{\mathbf{d}_k} v_{\mathbf{d}_k - 2\mathbf{w}_k}$$

$$\diamondsuit^{\mathbf{d}} v_{\mathbf{w}} = {}^{\mathbf{d}_1} v_{\mathbf{d}_1 - 2\mathbf{w}_1} \diamondsuit \ldots \diamondsuit^{\mathbf{d}_k} v_{\mathbf{d}_k - 2\mathbf{w}_k}$$

$$\otimes^{\mathbf{d}} v^{\mathbf{w}} = {}^{\mathbf{d}_1} v^{\mathbf{d}_1 - 2\mathbf{w}_1} \otimes \cdots \otimes {}^{\mathbf{d}_k} v^{\mathbf{d}_k - 2\mathbf{w}_k}$$

$$\heartsuit^{\mathbf{d}} v^{\mathbf{w}} = {}^{\mathbf{d}_1} v^{\mathbf{d}_1 - 2\mathbf{w}_1} \heartsuit \ldots \heartsuit^{\mathbf{d}_k} v^{\mathbf{d}_k - 2\mathbf{w}_k}$$

where $\mathbf{d}, \mathbf{w} \in (\mathbb{Z}_{\geq 0})^k$.

We can extend the bar involution σ to tensor products of irreducible representations as follows. Define

$$\sigma\left(f(q)\left(\otimes^{\mathbf{d}} v_{\mathbf{w}}\right)\right) = f(q^{-1})\left(\otimes^{\mathbf{d}} v_{\mathbf{w}}\right)$$

and extend by $\mathbb{C}$-linearity. Then σ is an isomorphism from $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ to itself and

$$\sigma(\Delta^{(k-1)}(x)(v)) = ((\sigma \otimes \cdots \otimes \sigma)(\Delta^{(k-1)} x))(\sigma v) \tag{1.5}$$

for $x \in \mathbf{U}_q$ and $v \in V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$.

We now recall some results on the graphical calculus of tensor products and intertwining operators. For a more complete treatment, see [5]. In the graphical calculus, V_d is depicted by a box marked d with d vertices. To depict $CM^{\mathbf{a}_1,\ldots,\mathbf{a}_l}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$, we place the boxes representing the $V_{\mathbf{d}_i}$ on a horizontal line and the boxes representing the $V_{\mathbf{a}_i}$ on another horizontal line lying above the first one. $CM^{\mathbf{a}_1,\ldots,\mathbf{a}_l}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ is then the set of non-intersecting curves (up to isotopy) connecting the vertices of the boxes such that the following conditions are satisfied:

1. Each curve connects exactly two vertices.

2. Each vertex is the endpoint of exactly one curve.

3. No curve joins a box to itself.

4. The curves lie inside the box bounded by the two horizontal lines and the vertical lines through the extreme right and left points.

An example is given in Figure 1.1. We call the curves joining two lower boxes *lower curves*, those joining two upper boxes *upper curves* and those joining a lower

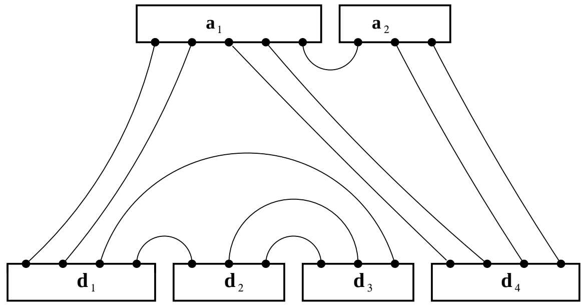

Figure 1.1: A crossingless match

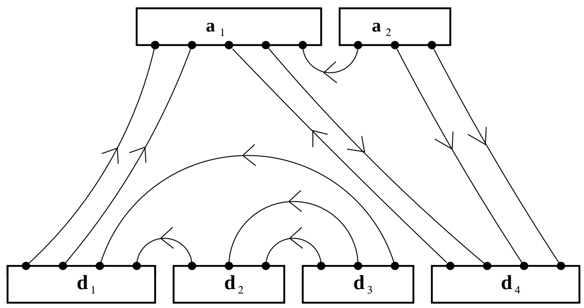

Figure 1.2: An oriented crossingless match

and an upper box *middle curves*. We define the set of oriented crossingless matches $OCM^{\mathbf{a}_1,\ldots,\mathbf{a}_l}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ to be the set of elements of $CM^{\mathbf{a}_1,\ldots,\mathbf{a}_l}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ along with an orientation of the curves such that all upper and lower curves are oriented to the left and all middle curves are oriented so that those oriented down are to the right of those oriented up. See Figure 1.2.

As shown in [5], the set of crossingless matches $CM^{\mathbf{a}_1,\ldots,\mathbf{a}_l}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ is in one to one correspondence with a basis of the set of intertwining operators

$$H^{\mathbf{a}_1,\ldots,\mathbf{a}_l}_{\mathbf{d}_1,\ldots,\mathbf{d}_k} = \mathrm{Hom}_{\mathbf{U}_q}\left(V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}, V_{\mathbf{a}_1} \otimes \cdots \otimes V_{\mathbf{a}_l}\right).$$

The matrix coefficients of the intertwining operator associated to a particular crossingless match are given by Theorem 2.1 of [5]. Note that these are intertwining operators in the dual basis and thus commute with the action of $\mathbf{U}_q$ on the tensor product given by $\bar{\Delta}^{(k-1)}$. Let $\tilde{\gamma}$ be such an intertwining operator and define $\gamma = \sigma\tilde{\gamma}\sigma$.

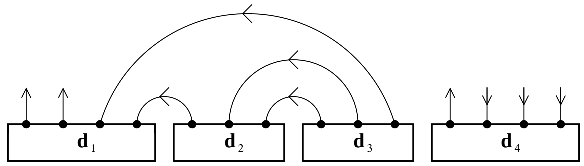

Figure 1.3: An oriented lower crossingless match

Then for $x \in \mathbf{U}_q$ and $v \in V_{\mathbf{d_1}} \otimes \cdots \otimes V_{\mathbf{d}_k}$,

$$\begin{aligned}
\gamma\Delta^{(k-1)}(x)(v) &= \sigma\tilde{\gamma}\sigma\Delta^{(k-1)}(x)(v) \\
&= \sigma\tilde{\gamma}((\sigma \otimes \cdots \otimes \sigma)\Delta^{(k-1)}(x))(\sigma v) \\
&= \sigma\tilde{\gamma}\bar{\Delta}^{(k-1)}(\sigma x)(\sigma v) \\
&= \sigma\bar{\Delta}^{(k-1)}(\sigma x)\tilde{\gamma}(\sigma v) \\
&= \sigma((\sigma \otimes \cdots \otimes \sigma)\Delta^{(k-1)}(x))\sigma\gamma(v) \\
&= \Delta^{(k-1)}(x)\gamma(v).
\end{aligned}$$

Thus γ is an intertwining operator in the usual basis commuting with the action of $\mathbf{U}_q$ given by $\Delta^{(k-1)}$.

We will also need to define the set of *lower crossingless matches* $LCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ and *oriented lower crossingless matches* $OLCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$. Elements of $LCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ and $OLCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ are obtained from elements of $CM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$and $LCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ (respectively) by removing the upper boxes (thus converting lower endpoints of upper curves to unmatched vertices). For the case of $OLCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$, unmatched vertices will still have an orientation (indicated by an arrow attached to the vertex). As for middle curves in the case of $OCM^{\mathbf{a}_1,\ldots,\mathbf{a}_l}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$, the unmatched vertices in an element of $OLCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ must be arranged to that those oriented down are to the right of those oriented up. See Figure 1.3.

Let $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^k$ be such that $\mathbf{a}_i \leq \mathbf{d}_i$ for $i = 1, 2, \ldots, k$. We associate an oriented lower crossingless match to $\mathbf{a}$ as follows. For each i, place down arrows on the rightmost $\mathbf{a}_i$ vertices of the box representing $V_{\mathbf{d}_i}$. Place up arrows on the remaining

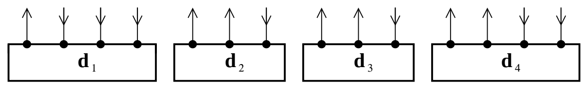

Figure 1.4: $\mathbf{d} = (4,3,3,4)$, $\mathbf{a} = (3,1,1,2)$

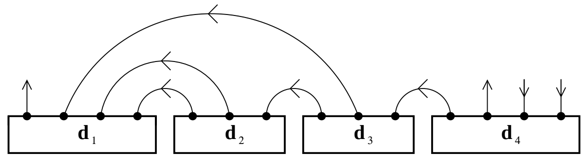

Figure 1.5: Oriented lower crossingless match associated to $\mathbf{d} = (4,3,3,4)$, $\mathbf{a} = (3,1,1,2)$

vertices. See Figure 1.4. There is a unique way to form an oriented lower crossingless match such that the orientation of any curve agrees with the direction of the arrows at its endpoints. Namely, starting from the right connect each down arrow to the first unmatched up arrow to its right (if there is any). Note that this produces an oriented lower crossingless match where the unmatched vertices are arranged so that all those with down arrows are to the right of those with up arrows (otherwise, we could have matched more vertices). See Figure 1.5. So to each $\mathbf{a}$ there is an associated element of $OLCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$. Conversely, given an element of $OLCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$, there is exactly one $\mathbf{a}$ which produces it. So we have a one to one correspondence between the set of elements $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^k$ such that $\mathbf{a}_i \leq \mathbf{d}_i$ and oriented lower crossingless matches $OLCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$. We will denote the oriented lower crossingless match associated to $\mathbf{a}$ by $M(\mathbf{d}, \mathbf{a})$.

We can put a partial ordering on the sets $CM^{\mathbf{a}_1,\ldots,\mathbf{a}_l}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$, $OCM^{\mathbf{a}_1,\ldots,\mathbf{a}_l}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$, $LCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ and $OLCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ as follows. For any two elements S_1 and S_2 of one of these sets, $S_1 \leq S_2$ if the set of lower curves of S_1 is a subset of the set of lower curves of S_2.

Given the geometrization of irreducible representations of $\mathbf{U}_q$ (Section 1.1.3), it is natural to seek a geometrization of the tensor product and the space of intertwining operators. This geometric realization is the focus of Sections 1.2 and 1.3.

1.2 Geometric Realization of The Tensor Product

1.2.1 Definition of the Tensor Product Variety $\mathfrak{T}(\mathbf{d})$

We now describe a variety (introduced in [19] and [25]) corresponding to the tensor product of the irreducible representations $V_{\mathbf{d}_1}, V_{\mathbf{d}_2}, \dots, V_{\mathbf{d}_k}$. This construction will yield three distinct bases of the tensor product in a natural way.

Fix a d-dimensional vector space D and let $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^k$ be such that $\sum_{i=1}^k \mathbf{d}_i = d$. Define

$$\mathfrak{T}_0(\mathbf{d}) = \{(\mathbf{D} = \{\mathbf{D}_i\}_{i=0}^k, W) \mid 0 = \mathbf{D}_0 \subset \mathbf{D}_1 \subset \cdots \subset \mathbf{D}_k = D, \\ W \subset D, \dim \mathbf{D}_i/\mathbf{D}_{i-1} = \mathbf{d}_i\}. \quad (1.6)$$

$\mathfrak{T}_0(\mathbf{d})$ admits a natural $GL(D)$ action. Namely

$$g \cdot (\{\mathbf{D}_i\}_{i=0}^k, W) = (\{g\mathbf{D}_i\}_{i=0}^k, gW)$$

for $g \in GL(D)$ and $(\mathbf{D}, W) \in \mathfrak{T}_0(\mathbf{d})$. Now let

$$\mathfrak{T}(\mathbf{d}) \stackrel{\text{def}}{=} \{(\mathbf{D} = \{\mathbf{D}_i\}_{i=0}^k, W, t) \mid 0 = \mathbf{D}_0 \subset \mathbf{D}_1 \subset \cdots \subset \mathbf{D}_k = D, W \subset D, \\ t \in \operatorname{End} D, t(\mathbf{D}_i) \subset \mathbf{D}_{i-1}, \dim \mathbf{D}_i/\mathbf{D}_{i-1} = \mathbf{d}_i, \operatorname{im} t \subset W \subset \ker t\}. \quad (1.7)$$

We call $\mathfrak{T}(\mathbf{d})$ the *tensor product variety*. We say a flag $\mathbf{D} = (0 = \mathbf{D}_0 \subset \mathbf{D}_1 \subset \cdots \subset \mathbf{D}_k = D)$ is *t-stable* if $t(\mathbf{D}_i) \subset \mathbf{D}_{i-1}$ for $i = 1, \dots, k$.

If we consider the corresponding varieties $\mathfrak{T}_0(\mathbf{d})'$ and $\mathfrak{T}(\mathbf{d})'$ defined over $\bar{\mathbb{F}}_{q^2}$, a straightforward computation shows that $\mathfrak{T}(\mathbf{d})'$ is the union of the conormal bundles of the orbits of the action of $GL(D)$ on $\mathfrak{T}_0(\mathbf{d})'$.

We define the action of E, F and $K^{\pm 1}$ on the set of functions on $\mathfrak{T}(\mathbf{d})$ just as for the other spaces considered so far. Namely, let

$$\mathfrak{T}(w; \mathbf{d}) = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}) \mid \dim W = w\}$$

$$\mathfrak{T}(w, w+1; \mathbf{d}) = \{(\mathbf{D}, U, W, t) \mid (\mathbf{D}, U, t), (\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}),\ U \subset W,$$
$$\dim U = w,\ \dim W = w+1\}.$$

We then have the projections

$$\mathfrak{T}(\mathbf{d}) \xleftarrow{\pi_1} \bigcup_w \mathfrak{T}(w, w+1; \mathbf{d}) \xrightarrow{\pi_2} \mathfrak{T}(\mathbf{d}). \tag{1.8}$$

where $\pi_1(\mathbf{D}, U, W, t) = (\mathbf{D}, U, t)$ and $\pi_2(\mathbf{D}, U, W, t) = (\mathbf{D}, W, t)$. The action of E, F and $K^{\pm 1}$ is defined by (1.2) as usual. Of course, the notation for the action of $K^{\pm 1}$ now means that

$$(K^{\pm 1} f)(\mathbf{D}, W, t) = q^{\pm(d - 2\dim W)} f(\mathbf{D}, W, t). \tag{1.9}$$

1.2.2 A Set of Basic Functions on the Tensor Product Variety

We now describe a set of basic functions on $\mathfrak{T}(\mathbf{d})$ which will be used to form spaces of functions isomorphic to $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$. As usual, fix a d-dimensional vector space D. For a flag $\mathbf{D} = (0 = \mathbf{D}_0 \subset \cdots \subset \mathbf{D}_k = D)$ and a subspace $W \subset D$, define $\alpha(W, \mathbf{D}) \in (\mathbb{Z}_{\geq 0})^k$ by

$$\alpha(W, \mathbf{D})_i = \dim (W \cap \mathbf{D}_i)/(W \cap \mathbf{D}_{i-1}).$$

For $\mathbf{w}, \mathbf{r}, \mathbf{n} \in (\mathbb{Z}_{\geq 0})^k$, define

$$A_{\mathbf{w},\mathbf{r},\mathbf{n}} = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}) \mid \alpha(W, \mathbf{D}) = \mathbf{w}, \tag{1.10}$$
$$\alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r},\ \alpha(\ker t, \mathbf{D}) = \mathbf{n}\}.$$

Note that the sets $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ are invariant under the action of $GL(D)$ given by

$$g \cdot (\{\mathbf{D}_i\}_{i=0}^k, W, t) = (\{g\mathbf{D}_i\}_{i=0}^k, gW, gtg^{-1}),\ g \in GL(D).$$

Let $\mathcal{T}(\mathbf{d})$ denote the space of functions on $\mathfrak{T}(\mathbf{d})$ constructible with respect to the stratification given by the $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ (that is, the space of functions constant on these sets). From now on the term *constructible* will always mean constructible with respect to this stratification. We will also use the notation

$$\mathbf{a}^{(j,l)} = \sum_{i=j}^{l} \mathbf{a}_i, \qquad |\mathbf{a}| = \sum_{i=1}^{k} \mathbf{a}_i$$

for $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$ and we will let δ^j denote the element of $(\mathbb{Z}_{\geq 0})^n$ such that $\delta^j_j = 1$ and $\delta^j_i = 0$ for all $i \neq j$.

Let

$$k_{\mathbf{w},\mathbf{r},\mathbf{n}} = q^{\sum_{i<j}(\mathbf{r}_i\mathbf{w}_j + \mathbf{w}_i\mathbf{n}_j - \mathbf{w}_i\mathbf{w}_j)} \tag{1.11}$$

and define

$$f_{\mathbf{w},\mathbf{r},\mathbf{n}} = k_{\mathbf{w},\mathbf{r},\mathbf{n}} \mathbf{1}_{A_{\mathbf{w},\mathbf{r},\mathbf{n}}}. \tag{1.12}$$

Then

$$\mathcal{T}(\mathbf{d}) = \mathrm{Span}\{f_{\mathbf{w},\mathbf{r},\mathbf{n}}\}_{\mathbf{w},\mathbf{r},\mathbf{n}}.$$

We will call the $f_{\mathbf{w},\mathbf{r},\mathbf{n}}$ *basic functions.* Note that $f_{\mathbf{w},\mathbf{r},\mathbf{n}} = \mathbf{1}_{A_{\mathbf{w},\mathbf{r},\mathbf{n}}}$ if $q = 1$. As will be seen below, the factor of $k_{\mathbf{w},\mathbf{r},\mathbf{n}}$ is necessary in order for the $f_{\mathbf{w},\mathbf{r},\mathbf{n}}$ to correspond to certain vectors in the tensor product. Note that $f_{\mathbf{w},\mathbf{r},\mathbf{n}} \equiv 0$ unless $\mathbf{r} \leq \mathbf{w} \leq \mathbf{n}$ where we define the partial ordering such that for $\mathbf{a}, \mathbf{b} \in (\mathbb{Z}_{\geq 0})^k$,

$$\begin{aligned} \mathbf{a} \leq \mathbf{b} &\Longleftrightarrow \sum_{i=1}^{j} \mathbf{a}_i \leq \sum_{i=1}^{j} \mathbf{b}_i \text{ for } 1 \leq j \leq k. \\ \mathbf{a} < \mathbf{b} &\Longleftrightarrow \mathbf{a} \leq \mathbf{b},\ \mathbf{a} \neq \mathbf{b} \end{aligned} \tag{1.13}$$

Also, $f_{\mathbf{w},\mathbf{r},\mathbf{n}} \equiv 0$ unless $|\mathbf{r}| + |\mathbf{n}| = |\mathbf{d}| = d$.

Theorem 1.2.A. *The action of* $\mathbf{U}_q$ *described in Section 1.2.1 endows* $\mathcal{T}(\mathbf{d})$ *with the structure of a* $\mathbf{U}_q$*-module and the map*

$$\eta_{\mathbf{r},\mathbf{n}} : \mathrm{Span}\{f_{\mathbf{w},\mathbf{r},\mathbf{n}}\}_{\mathbf{w}} \to V_{\mathbf{n}_1-\mathbf{r}_1} \otimes \cdots \otimes V_{\mathbf{n}_k-\mathbf{r}_k}$$

given by

$$\eta_{\mathbf{r},\mathbf{n}}(f_{\mathbf{w},\mathbf{r},\mathbf{n}}) = \otimes^{\mathbf{n}-\mathbf{r}} v_{\mathbf{w}-\mathbf{r}} \tag{1.14}$$

(and extended by linearity) is a $\mathbf{U}_q$*-module isomorphism.*

Proof. Fix a $(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})$ such that $\alpha(W, \mathbf{D}) = \mathbf{w} - \delta^j$ for some j (it is easy to see that $Ef_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t) = 0$ unless W satisfies this property). Then

$$\begin{aligned} Ef_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t) &= q^{-\dim(\pi_1^{-1}(\mathbf{D},W,t))}(\pi_1)_! \pi_2^* f_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t) \\ &= k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-\dim(\pi_1^{-1}(\mathbf{D},W,t))} \chi_q(\pi_1^{-1}(\mathbf{D}, W, t) \cap \pi_2^{-1}(A_{\mathbf{w},\mathbf{r},\mathbf{n}})). \end{aligned}$$

Now,

$$\begin{aligned} \pi_1^{-1}(\mathbf{D}, W, t) &\cong \{U \mid W \subset U \subset \ker t, \, \dim U = \dim W + 1\} \\ &\cong \mathbb{P}^{\dim(\ker t) - \dim W - 1} \\ &= \mathbb{P}^{|\mathbf{n}| - (|\mathbf{w}| - 1) - 1} \\ &= \mathbb{P}^{|\mathbf{n}| - |\mathbf{w}|}. \end{aligned}$$

So $\dim(\pi_1^{-1}(\mathbf{D}, W, t)) = |\mathbf{n}| - |\mathbf{w}|$ and

$$\begin{aligned} &\pi_1^{-1}(\mathbf{D}, W, t) \cap \pi_2^{-1}(A_{\mathbf{w},\mathbf{r},\mathbf{n}}) \\ &\quad \cong \{U \mid W \subset U \subset \ker t, \, \alpha(U, \mathbf{D}) = \mathbf{w}\} \\ &\quad \cong \{U \mid (W \cap \mathbf{D}_j) \subset U \subset (\ker t \cap \mathbf{D}_j), \\ &\qquad\quad \dim(U \cap \mathbf{D}_{j-1}) = \mathbf{w}^{(1,j-1)}, \, \dim U = \mathbf{w}^{(1,j)}\} \\ &\quad \cong \{U \mid U \subset (\ker t \cap \mathbf{D}_j)/(W \cap \mathbf{D}_j), \\ &\qquad\quad U \not\subset (\ker t \cap \mathbf{D}_{j-1})/(W \cap \mathbf{D}_{j-1}), \, \dim U = 1\} \\ &\quad \cong \mathbb{P}^{\dim(\ker t \cap \mathbf{D}_j)/(W \cap \mathbf{D}_j) - 1} - \mathbb{P}^{\dim(\ker t \cap \mathbf{D}_{j-1})/(W \cap \ker \mathbf{D}_{j-1}) - 1} \\ &\quad = \mathbb{P}^{\mathbf{n}^{(1,j)} - (\mathbf{w} - \delta^j)^{(1,j)} - 1} - \mathbb{P}^{\mathbf{n}^{(1,j-1)} - (\mathbf{w} - \delta^j)^{(1,j-1)} - 1} \\ &\quad = \mathbb{P}^{\mathbf{n}^{(1,j)} - \mathbf{w}^{(1,j)}} - \mathbb{P}^{\mathbf{n}^{(1,j-1)} - \mathbf{w}^{(1,j-1)} - 1}. \end{aligned}$$

Thus

$$\begin{aligned} Ef_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t) &= k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-(|\mathbf{n}| - |\mathbf{w}|)} \left(\sum_{i=0}^{\mathbf{n}^{(1,j)} - \mathbf{w}^{(1,j)}} q^{2i} - \sum_{i=0}^{\mathbf{n}^{(1,j-1)} - \mathbf{w}^{(1,j-1)} - 1} q^{2i} \right) \\ &= k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{|\mathbf{w}| - |\mathbf{n}|} \sum_{\mathbf{n}^{(1,j-1)} - \mathbf{w}^{(1,j-1)}}^{\mathbf{n}^{(1,j)} - \mathbf{w}^{(1,j)}} q^{2i} \end{aligned}$$

$$= k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{|\mathbf{w}|-|\mathbf{n}|+2\left(\mathbf{n}^{(1,j-1)}-\mathbf{w}^{(1,j-1)}\right)} \sum_{i=0}^{\mathbf{n}_j-\mathbf{w}_j} q^{2i}$$
$$= k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-\mathbf{w}^{(1,j-1)}+\mathbf{w}^{(j+1,k)}+\mathbf{n}^{(1,j-1)}-\mathbf{n}^{(j+1,k)}}[\mathbf{n}_j - \mathbf{w}_j + 1].$$

Now,

$$k_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}} = k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-\mathbf{r}^{(1,j-1)}-\mathbf{n}^{(j+1,k)}+\mathbf{w}^{1,j-1}+\mathbf{w}^{j+1,k}}$$

So

$$k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-\mathbf{w}^{(1,j-1)}+\mathbf{w}^{(j+1,k)}+\mathbf{n}^{(1,j-1)}-\mathbf{n}^{(j+1,k)}} = k_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}} q^{\mathbf{r}^{(1,j-1)}+\mathbf{n}^{(1,j-1)}-2\mathbf{w}^{(1,j-1)}}$$

and thus

$$Ef_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t) = k_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}} q^{\mathbf{r}^{(1,j-1)}+\mathbf{n}^{(1,j-1)}-2\mathbf{w}^{(1,j-1)}}[\mathbf{n}_j - \mathbf{w}_j + 1].$$

Therefore,

$$\begin{aligned} Ef_{\mathbf{w},\mathbf{r},\mathbf{n}} &= \sum_{j=1}^{k} q^{\mathbf{r}^{(1,j-1)}+\mathbf{n}^{(1,j-1)}-2\mathbf{w}^{(1,j-1)}}[\mathbf{n}_j - \mathbf{w}_j + 1] k_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}} \mathbf{1}_{A_{\mathbf{w}-\delta^i,\mathbf{r},\mathbf{n}}} \\ &= \sum_{j=1}^{k} q^{\mathbf{r}^{(1,j-1)}+\mathbf{n}^{(1,j-1)}-2\mathbf{w}^{(1,j-1)}}[\mathbf{n}_j - \mathbf{w}_j + 1] f_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}} \\ &= \sum_{j=1}^{k} q^{\sum_{i=1}^{j-1}(\mathbf{n}_i-\mathbf{r}_i-2(\mathbf{w}_i-\mathbf{r}_i))}[\mathbf{n}_j - \mathbf{w}_j + 1] f_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}}. \end{aligned} \tag{1.15}$$

Similarly

$$Ff_{\mathbf{w},\mathbf{r},\mathbf{n}} = \sum_{j=1}^{k} q^{-\sum_{i=j+1}^{k}(\mathbf{n}_i-\mathbf{r}_i-2(\mathbf{w}_i-\mathbf{r}_i))}[\mathbf{w}_j - \mathbf{r}_j + 1] f_{\mathbf{w}+\delta^j,\mathbf{r},\mathbf{n}}. \tag{1.16}$$

It follows immediately from (1.9) that

$$\begin{aligned} K^{\pm 1} f_{\mathbf{w},\mathbf{r},\mathbf{n}} &= q^{\pm(d-2|\mathbf{w}|)} f_{\mathbf{w},\mathbf{r},\mathbf{n}} \\ &= q^{\pm\sum_{i=1}^{k}(\mathbf{n}_i-\mathbf{r}_i-2(\mathbf{w}_i-\mathbf{r}_i))} f_{\mathbf{w},\mathbf{r},\mathbf{n}} \end{aligned} \tag{1.17}$$

since $|\mathbf{r}| + |\mathbf{n}| = |d|$.

Now recall that $x \in \mathbf{U}_q$ acts on $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ as $\Delta^{(k-1)}(x)$. In particular

$$\begin{aligned} \Delta^{(k-1)} E &= \sum_{i=1}^{k} K \otimes \cdots \otimes K \otimes E \otimes 1 \otimes \cdots \otimes 1 \\ \Delta^{(k-1)} F &= \sum_{i=1}^{k} 1 \otimes \cdots \otimes 1 \otimes F \otimes K^{-1} \otimes \cdots \otimes K^{-1} \\ \Delta^{(k-1)} K^{\pm 1} &= K^{\pm 1} \otimes \cdots \otimes K^{\pm 1} \end{aligned} \tag{1.18}$$

where in the first two equations, the E or F appears in the i^{th} position. Comparing (1.18) and (1.1) to (1.15), (1.16) and (1.17) the result follows. $\square$

1.2.3 The Space $\mathcal{T}_0(\mathbf{d})$ and the Elementary Basis $\mathcal{B}_e$

Note that if $t = 0$, then $\mathbf{r} = \mathbf{0}$ and $\mathbf{n} = \mathbf{d}$. Let $\mathcal{B}_e = \{f_{\mathbf{w},\mathbf{0},\mathbf{d}}\}_{\mathbf{w}}$. Then $\operatorname{Span} \mathcal{B}_e$ is the space of constructible functions on $\mathfrak{T}_0(\mathbf{d})$ which we shall denote by $\mathcal{T}_0(\mathbf{d})$. We see from Theorem 1.2.A that the map

$$\eta_{\mathbf{0},\mathbf{d}} : f_{\mathbf{w},\mathbf{0},\mathbf{d}} \mapsto \otimes^{\mathbf{d}} v_{\mathbf{w}}$$

(extended by linearity) is an isomorphism

$$\mathcal{T}_0(\mathbf{d}) \cong V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}.$$

We have therefore exhibited the elementary basis as a set of functions on the variety $\mathfrak{T}_0(\mathbf{d}) \subset \mathfrak{T}(\mathbf{d})$.

1.2.4 The Space $\mathcal{T}_c(\mathbf{d})$

The goal of this section is to develop a natural way to extend constructible functions on $\mathfrak{T}_0(\mathbf{d})$ to constructible functions on $\mathfrak{T}(\mathbf{d})$ with larger supports. Recall that $\mathcal{T}_0(\mathbf{d})$ and $\mathcal{T}(\mathbf{d})$ are the spaces of constructible functions on $\mathfrak{T}_0(\mathbf{d})$ and $\mathfrak{T}(\mathbf{d})$ respectively. The action of E, F and $K^{\pm 1}$ defined by (1.2) gives both $\mathcal{T}_0(\mathbf{d})$ and $\mathcal{T}(\mathbf{d})$ the structure of a $\mathbf{U}_q$-module as can be seen from Theorem 1.2.A. We will call a $\mathbf{U}_q$-module map $\epsilon : f \mapsto f^e$ from $\mathcal{T}_0(\mathbf{d})$ to $\mathcal{T}(\mathbf{d})$ an *extension*. Assuming an extension ϵ exists, $\eta_{\mathbf{r},\mathbf{n}} \circ \epsilon \circ (\eta_{\mathbf{0},\mathbf{d}})^{-1}$ is an intertwining operator from $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ to $V_{\mathbf{n}_1 - \mathbf{r}_1} \otimes \cdots \otimes V_{\mathbf{n}_k - \mathbf{r}_k}$. Conversely, each such set of intertwining operators determines an extension. Namely, given a set of intertwining operators

$$\{\gamma_{\mathbf{r},\mathbf{n}} : V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k} \to V_{\mathbf{n}_1 - \mathbf{r}_1} \otimes \cdots \otimes V_{\mathbf{n}_k - \mathbf{r}_k}\}_{\mathbf{r},\mathbf{n}}$$

we extend a function $f \in \mathcal{T}_0(\mathbf{d})$ to a function $f^e \in \mathcal{T}(\mathbf{d})$ by defining

$$f^e = \sum_{\mathbf{r},\mathbf{n}} (\eta_{\mathbf{r},\mathbf{n}})^{-1} \circ \gamma_{\mathbf{r},\mathbf{n}} \circ \eta_{\mathbf{0},\mathbf{d}}\,(f). \tag{1.19}$$

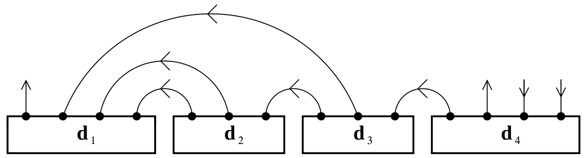

Figure 1.6: Oriented lower crossingless match S. $\mathbf{r}^S = (3,1,1,0)$, $\mathbf{n}^S = (4,1,1,3)$

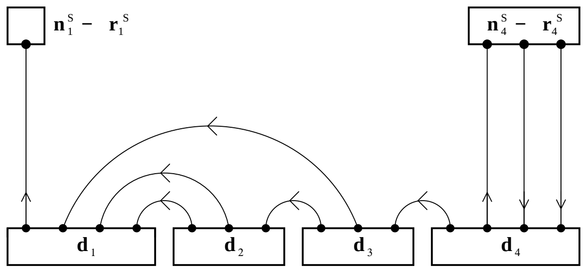

Figure 1.7: Completion of Figure 1.6 to an oriented crossingless match to $V_{\mathbf{n}_1^S-\mathbf{r}_1^S} \otimes \cdots \otimes V_{\mathbf{n}_k^S-\mathbf{r}_k^S} = V_1 \otimes V_0 \otimes V_0 \otimes V_3$

From Section 1.1.4 we know that a basis for the space of intertwining operators between two tensor product representations of $\mathbf{U}_q$ is given by the corresponding crossingless matches. Now, a lower curve represents a particular action of $t \in \operatorname{End} W$. A lower curve connecting $V_{\mathbf{d}_i}$ and $V_{\mathbf{d}_j}$ with $i < j$ represents the fact that t sends a vector in $\mathbf{D}_j - \mathbf{D}_{j-1}$ to a vector in $\mathbf{D}_i - \mathbf{D}_{i-1}$. So for any lower crossingless match S, fix a basis of D compatible with the flag $\mathbf{D}$ and let t be the map whose matrix in this basis has (i,j) component equal to 1 if $i < j$ and S has an curve connecting the i^{th} and j^{th} vertices and is equal to zero otherwise. Then let $\mathbf{r}^S$ and $\mathbf{n}^S$ be defined as $\alpha(\operatorname{im} t, \mathbf{D})$ and $\alpha(\ker t, \mathbf{D})$. Thus, $\mathbf{r}_i^S$ is the number of left endpoints of the lower curves contained in $V_{\mathbf{d}_i}$ and $\mathbf{n}_i^S$ is $\mathbf{d}_i$ minus the number of right endpoints of the lower curves contained in $V_{\mathbf{d}_i}$. See Figure 1.6. Then complete S to a crossingless match to $V_{\mathbf{n}_1^S-\mathbf{r}_1^S} \otimes \cdots \otimes V_{\mathbf{n}_k^S-\mathbf{r}_k^S}$ as in Figure 1.7 (there is a unique way to do this). Let $\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S}$ be the corresponding intertwining operator in the dual basis (that is, commuting with the action of $\mathbf{U}_q$ given by $\bar{\Delta}^{(k-1)}$). Note that $\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S}$ is well-defined since the map $S \mapsto (\mathbf{r}^S, \mathbf{n}^S)$ described above is injective. Now let $\gamma_{\mathbf{r}^S,\mathbf{n}^S} = \sigma\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S}\sigma$. As noted in

Section 1.1.4, $\gamma_{\mathbf{r}^S,\mathbf{n}^S} : V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k} \to V_{\mathbf{n}_1^S - \mathbf{r}_1^S} \otimes \cdots \otimes V_{\mathbf{n}_k^S - \mathbf{r}_k^S}$ is an intertwining operator in the usual basis (that is, it commutes with the action of $\mathbf{U}_q$ given by $\Delta^{(k-1)}$). For all $(\mathbf{r}, \mathbf{n})$ not of the form $(\mathbf{r}^S, \mathbf{n}^S)$ for some lower crossingless match S, let $\gamma_{\mathbf{r},\mathbf{n}} = 0$. Then let $\epsilon : f \mapsto f^e$ be the map defined by (1.19).

Proposition 1.2.B. *The extension ϵ is an isomorphism onto its image and*

$$f^e|_{\mathfrak{T}_0(\mathbf{d})} = f.$$

Proof. This follows immediately from Theorem 1.2.A. □

Let

$$\mathcal{T}_c(\mathbf{d}) = \epsilon(\mathcal{T}_0(\mathbf{d})) \subset \mathcal{T}(\mathbf{d}).$$

It follows from Proposition 1.2.B and Theorem 1.2.A that $\mathcal{T}_c(\mathbf{d}) \cong V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$. And it follows from Proposition 1.2.B that $\epsilon : \mathcal{T}_0(\mathbf{d}) \to \mathcal{T}_c(\mathbf{d})$ is an isomorphism of $\mathbf{U}_q$-modules with inverse given by restriction to $\mathfrak{T}_0(\mathbf{d})$. We will find a distinguished basis of $\mathcal{T}_c(\mathbf{d})$ related to the irreducible components of $\mathfrak{T}(\mathbf{d})'$. Before we do this, we must first examine these irreducible components.

1.2.5 The Irreducible Components of the Tensor Product Variety

For the remainder of this section we consider varieties defined over $\bar{\mathbb{F}}_{q^2}$. To avoid confusion, we denote the corresponding varieties by $\mathfrak{T}_0(\mathbf{d})'$ and $\mathfrak{T}(\mathbf{d})'$. For $\mathbf{w} \in (\mathbb{Z}_{\geq 0})^k$ such that $\mathbf{w}_i \leq \mathbf{d}_i$, let

$$Z'_{\mathbf{w}} = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})' \,|\, \alpha(W, \mathbf{D}) = \mathbf{w}\}.$$

We then have the following.

Theorem 1.2.C. $\{\overline{Z'_{\mathbf{w}}}\}_{\mathbf{w}}$ *are the irreducible components of* $\mathfrak{T}(\mathbf{d})'$.

Proof. It is obvious that $\sqcup_{\mathbf{w}} Z'_{\mathbf{w}} = \mathfrak{T}(\mathbf{d})'$ (where $\sqcup$ denotes disjoint union). Also, the connected components of $\mathfrak{T}(\mathbf{d})'$ are given by fixing the dimension of W. Thus, since $|\mathbf{w}| = \dim W$, it suffices to prove that the $Z'_{\mathbf{w}}$ are irreducible and locally closed and that $\dim Z'_{\mathbf{w}}$ is independent of $\mathbf{w}$ for fixed $|\mathbf{w}|$. Consider the maps

$$Z'_{\mathbf{w}} \xrightarrow{p_1} {}^1Z'_{\mathbf{w}} \xrightarrow{p_2} {}^2Z'_{\mathbf{w}}$$

where

$$\begin{aligned} {}^1Z'_{\mathbf{w}} &= \{(\mathbf{D}, W) \,|\, (\mathbf{D}, W, t) \in Z'_{\mathbf{w}} \text{ for some } t\} \\ {}^2Z'_{\mathbf{w}} &= \{\mathbf{D} \,|\, (\mathbf{D}, W) \in {}^1Z'_{\mathbf{w}} \text{ for some } W\} \\ p_1(\mathbf{D}, W, t) &= (\mathbf{D}, W) \\ p_2(\mathbf{D}, W) &= \mathbf{D}. \end{aligned}$$

Then p_1 and p_2 are locally trivial fibrations. Now

$${}^2Z'_{\mathbf{w}} = \{\mathbf{D} = \{\mathbf{D}_i\}_{i=0}^k \,|\, 0 = \mathbf{D}_0 \subset \mathbf{D}_1 \subset \cdots \subset \mathbf{D}_k = D,\ \dim \mathbf{D}_i/\mathbf{D}_{i-1} = \mathbf{d}_i\}$$

is simply a flag manifold. It is a homogeneous space as follows. $GL(D)$ acts transitively on ${}^2Z'_{\mathbf{w}}$ with stabilizer isomorphic to the set of matrices

$$G_0 = \left\{ \begin{pmatrix} M_1 & * & \cdots & * \\ 0 & M_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & M_k \end{pmatrix} \middle| \; M_i \in GL(\mathbf{d}_i) \right\}.$$

Thus,

$$\begin{aligned} \dim {}^2Z'_{\mathbf{w}} &= \dim GL(D) - \dim G_0 \\ &= \sum_{i<j} \mathbf{d}_i \mathbf{d}_j \end{aligned} \tag{1.20}$$

Now, the fiber of p_2 over a point $\mathbf{D} \in {}^2Z'_{\mathbf{w}}$ is

$$F_2 = \{W \subset D \,|\, \alpha(W, \mathbf{D}) = \mathbf{w}\}.$$

The group G_0 acts transitively on this space and the stabilizer is isomorphic to the set of matrices

$$G_1 = \left\{ \begin{pmatrix} M_1 & * & * & * & * & \cdots & \cdots & * \\ 0 & N_1 & 0 & * & 0 & \cdots & \cdots & * \\ 0 & 0 & M_2 & * & * & & & * \\ 0 & 0 & 0 & N_2 & 0 & & & * \\ 0 & 0 & 0 & 0 & \ddots & \ddots & & * \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & M_k & * \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & N_k \end{pmatrix} \middle| \begin{array}{c} M_i \in GL(\mathbf{w}_i), \\ , N_i \in GL(\mathbf{d}_i - \mathbf{w}_i) \end{array} \right\}.$$

So

$$\begin{aligned} \dim F_2 &= \dim G_0 - \dim G_1 \\ &= \sum_{j \leq i} \mathbf{w}_i(\mathbf{d}_j - \mathbf{w}_j) \end{aligned} \tag{1.21}$$

The fiber of p_1 over a point $(\mathbf{D}, W) \in {}^1Z'_{\mathbf{w}}$ is

$$F_1 = \{t \in \operatorname{End} D \,|\, t(\mathbf{D}_i) \subset \mathbf{D}_{i-1},\, \operatorname{im} t \subset W \subset \ker t\}. \tag{1.22}$$

Pick a basis $\{u_i\}_{i=1}^d$ of D such that $\{u_i\}_{i=1}^{\mathbf{d}_1+\cdots+\mathbf{d}_j}$ is a basis for $\mathbf{D}_j$ and

$$\bigcup_{l=0}^{j} \{u_i\}_{i=\mathbf{d}^{(1,l-1)}+1}^{\mathbf{d}^{(1,l-1)}+\mathbf{w}_l}$$

(where $\mathbf{d}^{(1,0)} = 0$) is a basis for $W \cap \mathbf{D}_j$. Then by considering the matrices of t in this basis it is easy to see that F_1 is an affine space of dimension

$$\dim F_1 = \sum_{i<j} \mathbf{w}_i(\mathbf{d}_j - \mathbf{w}_j). \tag{1.23}$$

So from Equations (1.20), (1.21) and (1.23) we see that

$$\begin{aligned} \dim Z'_{\mathbf{w}} &= \sum_{i<j} \mathbf{d}_i \mathbf{d}_j + \sum_{i,j=1}^{k} \mathbf{w}_i(\mathbf{d}_j - \mathbf{w}_j) \\ &= \sum_{i<j} \mathbf{d}_i \mathbf{d}_j + |\mathbf{w}|(d - |\mathbf{w}|) \end{aligned} \tag{1.24}$$

and thus $\dim Z'_{\mathbf{w}}$ is independent of $\mathbf{w}$ (for a fixed value of $|\mathbf{w}|$).

Now, ${}^2Z'_{\mathbf{w}}$, F_1 and F_2 are all smooth and connected, hence irreducible. Also, ${}^2Z'_{\mathbf{w}}$ and F_1 are closed while F_2 is locally closed. The latter statement follows from the fact that F_2 is equal to the closed set $\{W \subset D \,|\, \alpha(W, \mathbf{D}) \geq \mathbf{w}\}$ minus the finite collection of closed sets $\{W \subset D \,|\, \alpha(W, \mathbf{D}) \geq \mathbf{a}\}_{\mathbf{a} > \mathbf{w}}$. Thus each $Z'_{\mathbf{w}}$ is irreducible and locally closed. $\square$

Let

$$A'_{\mathbf{w},\mathbf{r},\mathbf{n}} = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})' \,|\, \alpha(W, \mathbf{D}) = \mathbf{w}, \qquad \alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r},\ \alpha(\ker t, \mathbf{D}) = \mathbf{n}\}. \tag{1.25}$$

We will need the following two propositions in the sequel.

Proposition 1.2.D. *Let $\mathbf{w} \in (\mathbb{Z}_{\geq 0})^k$ with $\mathbf{w}_i \leq \mathbf{d}_i$ for all i and let $M = M(\mathbf{d}, \mathbf{w})$. Then $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ is an open dense subset of $\overline{Z'_{\mathbf{w}}}$. In particular, $\overline{A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}} = \overline{Z'_{\mathbf{w}}}$.*

Proof. It is enough to show that $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ is dense in $Z'_{\mathbf{w}}$ (it is obvious from the definitions that $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M} \subset Z'_{\mathbf{w}}$). Since $\mathbf{r}^M \leq \mathbf{w} \leq \mathbf{n}^M$ by construction of M, we have that the projection of $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ onto ${}^1Z'_{\mathbf{w}}$ is all of ${}^1Z'_{\mathbf{w}}$. Thus it suffices to show that $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ is dense in each fiber. Fix $(\mathbf{D}, W) \in {}^1Z'_{\mathbf{w}}$. The fiber, F_1, of the projection p_1 is given by (1.22). The intersection of F_1 with $(p_1|_{A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}})^{-1}(\mathbf{D}, W)$ is isomorphic to

$$B = \{t \in \operatorname{End} D \,|\, t(\mathbf{D}_i) \subset \mathbf{D}_{i-1},\ \operatorname{im} t \subset W \subset \ker t, \qquad \alpha(\ker t, \mathbf{D}) = \mathbf{n}^M,\ \alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r}^M\}.$$

Choose a basis β of D compatible with the flag $\mathbf{D}$ and subspace W (that is, there exist bases for W and each $\mathbf{D}_i$ which are subsets of β). Now, since $\operatorname{im} t \subset W \subset \ker t$, t can be factored through D/W and considered as a map into W. Each t is uniquely determined by the corresponding $\bar{t} \in \operatorname{End}(D/W, W)$. Consider the matrix of $\bar{t}$ in the basis of D/W given by the projection of the basis β under the natural map $D \to D/W$ and the basis of W which is a subset of β. It must be of the following form:

$$C_t = \begin{pmatrix} 0 & A_{1,2} & A_{1,3} & \cdots & A_{1,k} \\ \vdots & 0 & A_{2,3} & \cdots & A_{2,k} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & A_{k-1,k} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.26}$$

where $A_{i,j}$ is a $(\mathbf{w}_i) \times (\mathbf{d}_j - \mathbf{w}_j)$ matrix. Then $t \in B$ if and only if each submatrix

$$C_t^{i,j} = \begin{pmatrix} A_{i,i+1} & A_{i,i+2} & \cdots & A_{i,j+1} \\ 0 & A_{i+1,i+2} & \cdots & A_{i+1,j+1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{j,j+1} \end{pmatrix}, \quad 1 \le i \le j \le k-1, \tag{1.27}$$

has maximal rank. To see this, consider the diagram M' of non-crossing oriented curves connecting the $V_{\mathbf{d}_i}$ associated to a $t \in F_1$. That is, the number of down-oriented vertices among those associated to $V_{\mathbf{d}_i}$ is given by $\mathbf{w}_i$ and the number of left and right endpoints of curves of M' in $V_{\mathbf{d}_i}$ are given by $\alpha(\operatorname{im} t, \mathbf{D})_i$ and $\mathbf{d}_i - \alpha(\ker t, \mathbf{D})_i$ respectively. A priori, this is not an oriented lower crossingless match (for instance, the unmatched vertices of M' might not be arranged so that those oriented down are to the right of those oriented up). The requirement that $C_t^{i,j}$ has maximal rank is equivalent to the requirement that M' has the maximum possible number of curves connecting $V_{\mathbf{d}_i}$, $V_{\mathbf{d}_{i+1}}$, …, and $V_{\mathbf{d}_{j+1}}$. Thus, referring to the definition of $M(\mathbf{d}, \mathbf{w})$ given in Section 1.1.4, we see that the condition that all the $C_t^{i,j}$ have maximal rank is equivalent to the condition that $M' = M$ (so M' is indeed an oriented crossingless match) and thus equivalent to $\alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r}^{M'} = \mathbf{r}^M$ and $\alpha(\ker t, \mathbf{D}) = \mathbf{n}^{M'} = \mathbf{n}^M$ or $t \in B$. Note that this argument also allows us to see that B is not empty since it contains the element t given by the matrix whose (i,j) entry is 1 if $i < j$ and M contains a curve connecting the i^{th} and j^{th} vertices and zero otherwise. In fact, this is a canonical form of any $t \in B$. That is, by a change of basis (preserving the flag $\mathbf{D}$), we can transform the matrix of any $t \in B$ to this form.

Assume we know that the subset $N_{m,n}$ of the set $M_{m,n}$ of $m \times n$ matrices given by

$$N_{m,n} = \{A \in M_{m,n} \,|\, A \text{ has maximal rank}\}$$

is an open subset of the set $M_{m,n}$. Then $N_{m,n}$ is given by the non-vanishing of a finite collection of polynomials in the matrix elements of $M_{m,n}$ (recall we are working in the Zariski topology). Thus, the requirement that the submatrices $C_t^{i,j}$ have maximal rank is equivalent to the non-vanishing of a finite number of polynomials in the matrix elements of the $C_t^{i,j}$ (and hence of C_t). Therefore, we will have shown that B is the intersection of a finite number of open subsets of F_1 and hence is open (and thus dense since it is not empty) in F_1.

So it remains to show that $N_{m,n}$ is dense in $M_{m,n}$. But if we let $r = \min(m,n)$, then

$$N_{m,n} = \{A \in M_{m,n} \,|\, \text{At least one } r \times r \text{ submatrix of } A \text{ has rank } r\}.$$

which is a union of open subsets of $M_{m,n}$ (since an $r \times r$ matrix has rank r if and only if its determinant is non-zero) and hence open (and dense) in $M_{m,n}$. □

Proposition 1.2.E. *With the notation of Proposition 1.2.D,* $\overline{A'_{\mathbf{a},\mathbf{r}^S,\mathbf{n}^S}} \subset \overline{Z'_{\mathbf{w}}}$ *for all* $S \leq M$, $\mathbf{a} \geq \mathbf{w}$, $|\mathbf{a}| = |\mathbf{w}|$.

Proof. It suffices to show that $A'_{\mathbf{a},\mathbf{r}^S,\mathbf{n}^S}$ is contained in $\overline{Z'_{\mathbf{w}}}$. The image of $A'_{\mathbf{a},\mathbf{r}^S,\mathbf{n}^S}$ under the projection p_1 is $\{(\mathbf{D}, W) \,|\, \alpha(W, \mathbf{D}) = \mathbf{a}\}$ which is contained in $\overline{{}^1Z'_{\mathbf{w}}}$ since $\mathbf{a} \geq \mathbf{w}$ and $|\mathbf{a}| = |\mathbf{w}|$. The fiber of the projection p_1 (restricted to $A'_{\mathbf{a},\mathbf{r}^S,\mathbf{n}^S}$) over a point $(\mathbf{D}, W)$ is

$$\{t \,|\, (\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}),\, \alpha(\ker t, \mathbf{D}) = \mathbf{n}^S,\, \alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r}^S\}$$

and this is in the closure of the set

$$\{t \,|\, (\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}),\, \alpha(\ker t, \mathbf{D}) = \mathbf{n}^M,\, \alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r}^M\}$$

since $S \leq M$. So $A'_{\mathbf{a},\mathbf{r}^S,\mathbf{n}^S} \subset \overline{Z'_{\mathbf{w}}}$. □

We now define the irreducible components of $\mathfrak{T}(\mathbf{d})$ to be the $\mathbb{F}_{q^2}$ points of the irreducible components of $\mathfrak{T}(\mathbf{d})'$. Let $\overline{Z_{\mathbf{w}}}$ denote the set of $\mathbb{F}_{q^2}$ points of the irreducible component $\overline{Z'_{\mathbf{w}}}$ of $\mathfrak{T}(\mathbf{d})'$. We also define the *dense points* of an irreducible component $\overline{Z_{\mathbf{w}}}$ of $\mathfrak{T}(\mathbf{d})$ to be the $\mathbb{F}_{q^2}$ points of the dense subset $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ (where $M = M(\mathbf{d},\mathbf{w})$) of the corresponding irreducible component $\overline{Z'_{\mathbf{w}}}$ of $\mathfrak{T}(\mathbf{d})'$. However, the $\mathbb{F}_{q^2}$ points of $A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$ are exactly the elements of $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$. Thus, the dense points of the irreducible component $\overline{Z_{\mathbf{w}}}$ of $\mathfrak{T}(\mathbf{d})$ are just the points of $A_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$.

1.2.6 Geometric Realization of the Canonical Basis

We are now ready to describe the set of functions mentioned at the end of Section 1.2.4. Define

$$\begin{aligned} h^{\mathbf{d}}_{\mathbf{w}} &= \eta^{-1}_{\mathbf{0},\mathbf{d}}\left(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}\right) \\ g^{\mathbf{d}}_{\mathbf{w}} &= \left(h^{\mathbf{d}}_{\mathbf{w}}\right)^e \\ \mathcal{B}_c &= \{g^{\mathbf{d}}_{\mathbf{w}}\}_{\mathbf{w}}. \end{aligned} \tag{1.28}$$

For a vector $w_1 \otimes \cdots \otimes w_k \in V_{\mathbf{a}_1} \otimes \dots V_{\mathbf{a}_k}$, let $(w_1 \otimes \cdots \otimes w_k)^r = w_k \otimes \cdots \otimes w_1 \in V_{\mathbf{a}_k} \otimes \dots V_{\mathbf{a}_1}$. For an intertwining operator $\gamma : V_{\mathbf{a}_1} \otimes \cdots \otimes V_{\mathbf{a}_k} \to V_{\mathbf{b}_1} \otimes \cdots \otimes V_{\mathbf{b}_l}$ corresponding to a crossingless match S, let $\gamma^\dagger : V_{\mathbf{b}_l} \otimes \cdots \otimes V_{\mathbf{b}_1} \to V_{\mathbf{a}_k} \otimes \cdots \otimes V_{\mathbf{a}_1}$ denote the intertwining operator corresponding to the crossingless match S rotated 180°. It follows easily from the graphical calculus described in [5] that

$$\langle \gamma(v), w \rangle = \left\langle v, (\sigma\gamma^\dagger\sigma)(w) \right\rangle$$

for any $v \in V_{\mathbf{a}_1} \otimes \cdots \otimes V_{\mathbf{a}_k}$ and $w \in V_{\mathbf{b}_l} \otimes \cdots \otimes V_{\mathbf{b}_1}$.

We will need the following results.

Lemma 1.2.F.

$$\gamma_{\mathbf{r}^S,\mathbf{n}^S}\left(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}\right) = \begin{cases} \diamondsuit^{\mathbf{n}^S - \mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S} & \textit{if } S \leq M(\mathbf{d},\mathbf{w}), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. It is apparent from the graphical calculus of [5] that if $S \leq M(\mathbf{d}, \mathbf{w})$, then $\left(\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S}\right)^\dagger \left(\left(\heartsuit^{\mathbf{n}^S-\mathbf{r}^S} v^{\mathbf{w}-\mathbf{r}^S}\right)^r\right) = \left(\heartsuit^{\mathbf{d}} v^{\mathbf{w}}\right)^r$ and that $\left(\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S}\right)^\dagger$ sends other dual canonical basis elements $\left(\heartsuit^{\mathbf{n}^S-\mathbf{r}^S} v^{\mathbf{a}}\right)^r$, $\mathbf{a} \neq \mathbf{w} - \mathbf{r}^S$, to elements of the form $\left(\heartsuit^{\mathbf{d}} v^{\mathbf{a}'}\right)^r$ with $\mathbf{a}' \neq \mathbf{w}$. Therefore

$$\begin{aligned}
\left\langle \gamma_{\mathbf{r}^S,\mathbf{n}^S}\left(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}\right), \left(\heartsuit^{\mathbf{n}^S-\mathbf{r}^S} v^{\mathbf{w}-\mathbf{r}^S}\right)^r \right\rangle &= \left\langle \diamondsuit^{\mathbf{d}} v_{\mathbf{w}}, \left(\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S}\right)^\dagger \left(\left(\heartsuit^{\mathbf{n}^S-\mathbf{r}^S} v^{\mathbf{w}-\mathbf{r}^S}\right)^r\right)\right\rangle \\
&= \left\langle \diamondsuit^{\mathbf{d}} v_{\mathbf{w}}, \left(\heartsuit^{\mathbf{d}} v^{\mathbf{w}}\right)^r \right\rangle \\
&= 1
\end{aligned}$$

and

$$\left\langle \gamma_{\mathbf{r}^S,\mathbf{n}^S}\left(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}\right), \left(\heartsuit^{\mathbf{n}^S-\mathbf{r}^S} v^{\mathbf{a}}\right)^r \right\rangle = 0$$

for all $\mathbf{a} \neq \mathbf{w} - \mathbf{r}^S$. Thus $\gamma_{\mathbf{r}^S,\mathbf{n}^S}\left(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}\right) = \diamondsuit^{\mathbf{n}^S-\mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S}$. A similar argument demonstrates that $\gamma_{\mathbf{r}^S,\mathbf{n}^S}\left(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}\right) = 0$ if $S \not\leq M(\mathbf{d}, \mathbf{w})$ since then the image of $\left(\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S}\right)^\dagger$ is spanned by $\heartsuit^{\mathbf{d}} v^{\mathbf{a}}$ with $\mathbf{a} \neq \mathbf{w}$. □

Proposition 1.2.G.

$$g^{\mathbf{d}}_{\mathbf{w}} = \sum_{S \leq M(\mathbf{d},\mathbf{w})} \left(\eta_{\mathbf{r}^S \mathbf{n}^S}\right)^{-1} \left(\diamondsuit^{\mathbf{n}^S-\mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S}\right).$$

Proof. This follows immediately from Lemma 1.2.F. □

Proposition 1.2.H. *$\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}$ is equal to $\otimes^{\mathbf{d}} v_{\mathbf{w}}$ plus a linear combination of elements $\otimes^{\mathbf{d}} v_{\mathbf{a}}$, $\mathbf{a} > \mathbf{w}$, $|\mathbf{a}| = |\mathbf{w}|$, with coefficients in $q^{-1}\mathbb{N}[q^{-1}]$.*

Proof. This follows from Sections 1.5 and 1.6 of [5]. □

We can now prove one of our main results.

Theorem 1.2.I. *$g^{\mathbf{d}}_{\mathbf{w}}$ is the unique element of $\mathcal{T}_c(\mathbf{d})$, up to a multiplicative constant, satisfying the following conditions.*

1. *$g^{\mathbf{d}}_{\mathbf{w}}$ is equal to a non-zero constant on the set of dense points $A_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ of the irreducible component $\overline{Z_{\mathbf{w}}}$ (where $M = M(\mathbf{d}, \mathbf{w})$).*

2. The support of $g^{\mathbf{d}}_{\mathbf{w}}$ lies in $\overline{Z_{\mathbf{w}}}$.

Furthermore, the set $\{g^{\mathbf{d}}_{\mathbf{w}}\}_{\mathbf{w}}$ is a basis of $\mathcal{T}_c(\mathbf{d})$ and the map

$$\diamondsuit^{\mathbf{d}} v_{\mathbf{w}} \mapsto g^{\mathbf{d}}_{\mathbf{w}}$$

(extended by linearity) is a $\mathbf{U}_q$-module isomorphism $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k} \cong \mathcal{T}_c(\mathbf{d})$.

Proof. In this proof, to simplify notation in calculations, we will suppress the isomorphism $\eta_{\mathbf{r},\mathbf{n}}$ defined by (1.14) and identify the vector $\otimes^{\mathbf{n}-\mathbf{r}} v_{\mathbf{w}-\mathbf{r}}$ with the function $f_{\mathbf{w},\mathbf{r},\mathbf{n}}$. Assume that $g'^{\mathbf{d}}_{\mathbf{w}}$ satisfies the above conditions and let $g'^{\mathbf{d}}_{\mathbf{w}} = \left(h'^{\mathbf{d}}_{\mathbf{w}}\right)^e$. The value of $g'^{\mathbf{d}}_{\mathbf{w}}$ on $A_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ is given by $k_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ times the coefficient of $f_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ when $g'^{\mathbf{d}}_{\mathbf{w}}$ is written as a sum of the basic functions. This coefficient is equal to

$$\left\langle \gamma_{\mathbf{r}^M,\mathbf{n}^M}(h'^{\mathbf{d}}_{\mathbf{w}}), \left(\otimes^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M}\right)^r \right\rangle$$

.

Therefore, since $k_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M} \neq 0$, condition 1 is equivalent to

$$\begin{aligned} &\left\langle \gamma_{\mathbf{r}^M,\mathbf{n}^M}(h'^{\mathbf{d}}_{\mathbf{w}}), \left(\otimes^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M}\right)^r \right\rangle \neq 0 \\ \Leftrightarrow &\left\langle h'^{\mathbf{d}}_{\mathbf{w}}, (\tilde{\gamma}_{\mathbf{r}^M,\mathbf{n}^M})^\dagger \left(\left(\otimes^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M}\right)^r \right) \right\rangle \neq 0. \end{aligned}$$

Now, since $M = M(\mathbf{d},\mathbf{w})$ is the oriented lower crossingless match associated to $\mathbf{w}$, $M(\mathbf{n}^M - \mathbf{r}^M, \mathbf{w} - \mathbf{r}^M)$ has no lower curves and all down arrows are to the right of all up arrows. So after being rotated by 180° (but keeping the original orientation of unmatched vertices – for example, those oriented up remain oriented up), this diagram has all down arrows to the left of all up arrows. Thus, by Section 2.3 of [5], $\left(\otimes^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M}\right)^r = \left(\heartsuit^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M}\right)^r$. It also follows from the graphical calculus of [5] that

$$(\tilde{\gamma}_{\mathbf{r}^M,\mathbf{n}^M})^\dagger \left(\left(\heartsuit^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M}\right)^r \right) = \left(\heartsuit^{\mathbf{d}} v^{\mathbf{w}}\right)^r.$$

Therefore condition 1 is equivalent to

$$\left\langle h'^{\mathbf{d}}_{\mathbf{w}}, \left(\heartsuit^{\mathbf{d}} v^{\mathbf{w}}\right)^r \right\rangle \neq 0. \tag{1.29}$$

Next we consider condition 2. In order for this condition to be satisfied, $g'^{\mathbf{d}}_{\mathbf{w}}$ must be equal to zero on $A_{\mathbf{w}',\mathbf{r}^{M'},\mathbf{n}^{M'}}$ for all $\mathbf{w}' \neq \mathbf{w}$ (where $M' = M(\mathbf{d}, \mathbf{w}')$). By an argument analogous to that given above, this is equivalent to the condition

$$\left\langle h'^{\mathbf{d}}_{\mathbf{w}}, \left(\heartsuit^{\mathbf{d}} v^{\mathbf{w}'}\right)^{r} \right\rangle = 0 \tag{1.30}$$

for all $\mathbf{w}' \neq \mathbf{w}$. Therefore, by (1.29) and (1.30), we must have

$$h'^{\mathbf{d}}_{\mathbf{w}} = c^{\mathbf{d}}_{\mathbf{w}} \cdot \diamondsuit^{\mathbf{d}} v_{\mathbf{w}} = c^{\mathbf{d}}_{\mathbf{w}} \cdot h^{\mathbf{d}}_{\mathbf{w}}$$

for some non-zero constant $c^{\mathbf{d}}_{\mathbf{w}}$ which proves uniqueness up to a multiplicative constant. It still remains to show that $g^{\mathbf{d}}_{\mathbf{w}}$ satisfies the given conditions.

Now, by Proposition 1.2.G, the value of $g^{\mathbf{d}}_{\mathbf{w}}$ on $A_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$, where $M = M(\mathbf{d}, \mathbf{w})$, is equal to $k_{\mathbf{w},\mathbf{r},\mathbf{n}}$ times the coefficient of $\otimes^{\mathbf{n}^M - \mathbf{r}^M} v_{\mathbf{w}-\mathbf{r}^M}$ in the expression of $\diamondsuit^{\mathbf{n}^M - \mathbf{r}^M} v_{\mathbf{w}-\mathbf{r}^M}$ as a linear combination of elementary basis elements. By Proposition 1.2.H, this coefficient is equal to 1. So $g^{\mathbf{d}}_{\mathbf{w}}$ is equal to a non-zero constant on $A_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$. Also, by Propositions 1.2.G and 1.2.H, $g^{\mathbf{d}}_{\mathbf{w}}$ is equal to a linear combination of functions of the form $\left(\eta_{\mathbf{r}^S,\mathbf{n}^S}\right)^{-1}\left(\otimes^{\mathbf{n}^S - \mathbf{r}^S} v_{\mathbf{a}}\right) = f_{\mathbf{a}+\mathbf{r}^S,\mathbf{r}^S,\mathbf{n}^S}$ with $S \leq M$, $|\mathbf{a}| = |\mathbf{w} - \mathbf{r}^S|$ ($\Rightarrow |\mathbf{a} + \mathbf{r}^S| = |\mathbf{w}|$), and $\mathbf{a} \geq \mathbf{w} - \mathbf{r}^S$ ($\Rightarrow \mathbf{a} + \mathbf{r}^s \geq \mathbf{w}$). Thus, by Proposition 1.2.E, the support of $g^{\mathbf{d}}_{\mathbf{w}}$ lies in $\overline{Z_{\mathbf{w}}}$. So we have demonstrated that the functions $g^{\mathbf{d}}_{\mathbf{w}}$ are the unique functions, up to a multiplicative constant, satisfying conditions 1 and 2.

The last two statements of the theorem follow from the fact that the map $\eta_{\mathbf{0},\mathbf{d}}$: $\diamondsuit^{\mathbf{d}} v_{\mathbf{w}} \mapsto h^{\mathbf{d}}_{\mathbf{w}}$ (extended by linearity) is a $\mathbf{U}_q$-module isomorphism $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k} \cong \mathcal{T}_0(\mathbf{d})$ and the fact that ϵ is an isomorphism onto its image. $\square$

1.2.7 A Conjectured Characterization of $\mathcal{T}_c(\mathbf{d})$ and $\mathcal{B}_c$

We present here a conjecture concerning an alternative characterization of the basis $\mathcal{B}_c$. Let $\mathcal{P}$ be the category of semisimple perverse sheaves on $\mathfrak{T}(\mathbf{d})'$ constructible with respect to the stratification given by the $A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$ and let $\mathcal{D} : \mathcal{P} \to \mathcal{P}$ be the operation of Verdier Duality. For $\mathbf{B}^\bullet \in \mathcal{P}$ and $x \in \mathfrak{T}(\mathbf{d})'$, $\mathbf{B}^\bullet_x$ denotes the stalk complex at the point x. We define the action of the involution $\Psi^{(k)}$ on $\mathcal{T}(\mathbf{d})$ by

$$\Psi^{(k)}\left(f_{\mathbf{w},\mathbf{r},\mathbf{n}}\right) = (\eta_{\mathbf{r},\mathbf{n}})^{-1}\Psi^{(k)}\eta_{\mathbf{r},\mathbf{n}}\left(f_{\mathbf{w},\mathbf{r},\mathbf{n}}\right),$$

where on the right hand side $\Psi^{(k)}$ is the involution used to characterize the canonical basis (see [5], Section 1.6). In particular, the canonical basis is invariant under the action of $\Psi^{(k)}$. Now let $\theta : \mathcal{P} \to \mathcal{T}(\mathbf{d})$ be the map such that

$$(\theta(\mathbf{B}^\bullet))(x) = \sum_i (-1)^i q^i \dim(\mathbf{B}^i_x) \text{ for } x \in \mathfrak{T}(\mathbf{d}) \text{ and } \mathbf{B}^\bullet \in \mathcal{P}.$$

For each irreducible component $\overline{Z'_{\mathbf{w}}}$ of $\mathfrak{T}(\mathbf{d})'$ there is an intersection sheaf complex $IC^\bullet_{\mathbf{w}}$ associated to the local system which is the constant sheaf $\mathbb{C}$ (in degree zero) on the dense subset $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ where $M = M(\mathbf{d},\mathbf{w})$ (see [14] for details).

Conjecture 1.2.J. $\theta\left(IC^\bullet_{\mathbf{w}}\right) = k^{-1}_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M} g^{\mathbf{d}}_{\mathbf{w}}$.

The factor of $k^{-1}_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ arises from the fact that $\theta\left(IC^\bullet_{\mathbf{w}}\right)$ is equal to one on the set $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$. The proof of Conjecture 1.2.J would most likely center around the idea that the action of Verdier Duality in $\mathcal{P}$ should correspond to the action of $\Psi^{(k)}$ in $\mathcal{T}(\mathbf{d})$. The precise statement is the following:

Conjecture 1.2.K. $\theta\mathcal{D} = \Psi^{(k)}\theta$.

1.3 Geometric Realization of The Intertwining Operators

1.3.1 Defining the Intertwining Operators

The goal of this section is to decompose $\mathfrak{T}(\mathbf{d})$ into subsets corresponding to a basis for the space of intertwining operators

$$H^\mu_{\mathbf{d}_1,\ldots,\mathbf{d}_k} = \mathrm{Hom}_{\mathbf{U}_q}(V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}, V_\mu). \tag{1.31}$$

Note that $H^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k} = 0$ unless $\mu = d - 2r$ for some $0 \le r \le d/2$ (where $d = |\mathbf{d}|$). Thus, the intertwining operators will be maps from $\mathcal{T}(\mathbf{d})$ to $\mathcal{M}(d)$ since these V_μ are precisely the representations appearing in $\mathcal{M}(d)$ (see Section 1.1.3).

Let $Y \subset \mathfrak{T}(\mathbf{d})$. Define $R_Y : \mathfrak{T}(\mathbf{d}) \to \mathfrak{T}(\mathbf{d})$ to be the map which restricts functions to their values on Y. That is, for $f \in \mathcal{T}(\mathbf{d})$, $R_Y f = \mathbf{1}_Y f$ (where the multiplication of functions is pointwise).

Consider the map $p : \mathfrak{T}(\mathbf{d}) \to \mathfrak{M}(d)$ such that $p(\mathbf{D}, W, t) = (W, t)$. Let $T_Y = p_! R_Y$. Then T_Y is a map from $\mathcal{T}(\mathbf{d})$ to $\mathcal{M}(d)$.

Proposition 1.3.A. *If $Y \subset \mathfrak{T}(\mathbf{d})$ satisfies $\pi_1\pi_2^{-1}(Y) \subset Y$ and $\pi_2\pi_1^{-1}(Y) \subset Y$ where π_1 and π_2 are the maps from* (1.8) *then T_Y is an intertwining operator.*

Proof. It suffices to show that T_Y commutes with the action of E, F and $K^{\pm 1}$ since these elements generate $\mathbf{U}_q$. Note that the condition $\pi_1\pi_2^{-1}(Y) \subset Y$ implies $\pi_2^{-1}(Y) \subset \pi_1^{-1}(Y)$ and the condition $\pi_2\pi_1^{-1}(Y) \subset Y$ implies $\pi_1^{-1}(Y) \subset \pi_2^{-1}(Y)$. Thus $\pi_1^{-1}(Y) = \pi_2^{-1}(Y)$. We first show that $T_Y E = E T_Y$. Now $T_Y E = p_! R_Y E$ and $E T_Y = E p_! R_Y$. Thus it suffices to show that $R_Y E = E R_Y$ and $p_! E = E p_!$. Since $\mathcal{T}(\mathbf{d})$ is spanned by functions of the form $\mathbf{1}_A$ where A is a subvariety of $\mathfrak{T}(\mathbf{d})$, we need only check that actions agree on such functions. For $x = (\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})$

$$\begin{aligned}
R_Y E \mathbf{1}_A(x) &= \mathbf{1}_Y(x)(E\mathbf{1}_A)(x) \\
&= \mathbf{1}_Y(x) q^{-\dim(\pi_1^{-1}(W))}((\pi_1)_! \pi_2^* \mathbf{1}_A)(x) \\
&= q^{-\dim(\pi_1^{-1}(W))} \mathbf{1}_Y(x)((\pi_1)_! \mathbf{1}_{\pi_2^{-1}(A)})(x) \\
&= q^{-\dim(\pi_1^{-1}(W))} \mathbf{1}_Y(x) \chi_q(\pi_1^{-1}(x) \cap \pi_2^{-1}(A)) \\
&= q^{-\dim(\pi_1^{-1}(W))} \chi_q(\pi_1^{-1}(x \cap Y) \cap \pi_2^{-1}(A)) \\
&= q^{-\dim(\pi_1^{-1}(W))} \chi_q(\pi_1^{-1}(x) \cap \pi_1^{-1}(Y) \cap \pi_2^{-1}(A)) \\
&= q^{-\dim(\pi_1^{-1}(W))} \chi_q(\pi_1^{-1}(x) \cap \pi_2^{-1}(Y) \cap \pi_2^{-1}(A)) \\
&= q^{-\dim(\pi_1^{-1}(W))} \chi_q(\pi_1^{-1}(x) \cap \pi_2^{-1}(Y \cap A))
\end{aligned}$$

$$= q^{-\dim(\pi_1^{-1}(W))}(\pi_1)_!\mathbf{1}_{\pi_2^{-1}(Y\cap A)}(x)$$

$$= q^{-\dim(\pi_1^{-1}(W))}(\pi_1)_!\pi_2^*\mathbf{1}_{Y\cap A}(x)$$

$$= q^{-\dim(\pi_1^{-1}(W))}(\pi_1)_!\pi_2^*(\mathbf{1}_Y\mathbf{1}_A)(x)$$

$$= ER_Y\mathbf{1}_A(x),$$

where the fifth equality holds from consideration of the two cases $x \in Y$ and $x \notin Y$.

It remains to show that $p_!E = Ep_!$. For the purposes of this demonstration, we introduce the map

$$p' : \bigcup_w \mathfrak{T}(w, w+1; \mathbf{d}) \to \bigcup_w \mathfrak{M}(w, w+1, d)$$

which acts as $p'(\mathbf{D}, U, W, t) = (U, W, t)$. We have the following commutative diagram.

$$\begin{array}{ccc}
\mathfrak{T}(\mathbf{d}) & \xrightarrow{p} & \mathfrak{M}(d) \\
\uparrow{\scriptstyle \pi_2} & & \uparrow{\scriptstyle \pi_2} \\
\bigcup_v \mathfrak{T}(w, w+1; \mathbf{d}) & \xrightarrow{p'} & \bigcup_w \mathfrak{M}(w, w+1, d) \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\
\mathfrak{T}(\mathbf{d}) & \xrightarrow{p} & \mathfrak{M}(d)
\end{array}$$

As before we use the notation π_1 and π_2 to denote several different, but analogous maps.

Note that $p^{-1}\pi_2 = \pi_2(p')^{-1}$ (both are the map $(U, W, t) \mapsto \{(\mathbf{D}, W', t') \in \mathfrak{T}(\mathbf{d}) \mid W' = W, t' = t\}$). Using this fact we show that $(p')_!\pi_2^* = \pi_2^* p_!$. Let $x \in \mathfrak{T}(\mathbf{d})$. Then

$$\begin{aligned}
(\pi_2^* p_! \mathbf{1}_A)(x) &= (p_!\mathbf{1}_A)(\pi_2(x)) \\
&= \chi_q(p^{-1}(\pi_2(x)) \cap A) \\
&= \chi_q(\pi_2((p')^{-1}(x)) \cap A) \\
&= \chi_q(\pi_2((p')^{-1}(x) \cap \pi_2^{-1}(A))) \\
&= \chi_q((p')^{-1}(x) \cap \pi_2^{-1}(A)) \\
&= (p')_!\mathbf{1}_{\pi_2^{-1}(A)}(x) \\
&= (p')_!\pi_2^*\mathbf{1}_A(x)
\end{aligned}$$

where in the fourth equality we used the general fact that $\pi_2(B) \cap A = \pi_2(B \cap \pi_2^{-1}(A))$ and in the fifth equality we used the fact that if $x = (U', W', t')$ then

$$\begin{aligned}
(p')^{-1}(x) \cap \pi_2^{-1}(A) &= \{(\mathbf{D}, U, W, t) \,|\, p'(\mathbf{D}, U, W, t) = (U', W', t'), \\
&\qquad\qquad\qquad\qquad \pi_2(\mathbf{D}, U, W, t) \in A\} \\
&= \{(\mathbf{D}, U, W, t) \,|\, U = U',\, W = W',\, t = t',\, (\mathbf{D}, W, t) \in A\} \\
&\cong \{(\mathbf{D}, W, t) \,|\, W = W',\, t = t',\, (\mathbf{D}, W, t) \in A\} \\
&= \pi_2((p')^{-1}(x) \cap \pi_2^{-1}(A)).
\end{aligned}$$

We also have that $\pi_1 p' = p\pi_1$ (both are the map $(\mathbf{D}, U, W, t) \mapsto (U, t)$). Thus

$$\begin{aligned}
Ep_! &= q^{-\dim(\pi_1^{-1}(\cdot))}(\pi_1)_!\pi_2^* p_! \\
&= q^{-\dim(\pi_1^{-1}(\cdot))}(\pi_1)_!(p')_!\pi_2^* \\
&= q^{-\dim(\pi_1^{-1}(\cdot))}(\pi_1 p')_!\pi_2^* \\
&= q^{-\dim(\pi_1^{-1}(\cdot))}(p\pi_1)_!\pi_2^* \\
&= q^{-\dim(\pi_1^{-1}(\cdot))}p_!(\pi_1)_!\pi_2^* \\
&= p_! q^{-\dim(\pi_1^{-1}(\cdot))}(\pi_1)_!\pi_2^* \\
&= p_! E
\end{aligned}$$

where we have used the fact that the map $f \mapsto f_!$ is functorial [18].

Thus, we have shown that $T_Y E = E T_Y$. The proof that $T_Y F = F T_Y$ is analogous. Also,

$$\begin{aligned}
K^{\pm 1} T_Y f(\mathbf{D}, W, t) &= q^{\pm(d - 2\dim W)} T_Y f(\mathbf{D}, W, t) \\
&= T_Y q^{\pm(d - 2\dim W)} f(\mathbf{D}, W, t) \\
&= T_Y K^{\pm 1} f(\mathbf{D}, W, t).
\end{aligned}$$

□

1.3.2 A Basis $\mathcal{B}_I$ for the Space of Intertwining Operators

We see from Section 1.1.4 that a basis for the space of intertwining operators $H^{\mu}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ is in one to one correspondence with the set of crossingless matches $CM^{\mu}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$. Note that crossingless matches of the form $CM^{\mu}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ (i.e. with only one box on the top vertical line) are in one to one correspondence with elements of $LCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$. For a given element S of $LCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ simply set μ equal to the number of unmatched vertices of S and join the unmatched vertices to the upper box. Recall that elements $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$ such that $\mathbf{a}_i \leq \mathbf{d}_i$ are in one to one correspondence with the elements of $OLCM_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$. Given such an $\mathbf{a}$, consider its associated oriented lower crossingless match $M(\mathbf{d}, \mathbf{a})$. Note that $|\mathbf{a}|$ is the number of vertices (both matched and unmatched) in $M(\mathbf{d}, \mathbf{a})$ which are oriented down.

For any flag $\mathbf{D}$ and $t \in \operatorname{End} D$ let $\alpha(t, \mathbf{D}) = \alpha(\ker t, \mathbf{D})$. Then let

$$\begin{aligned} Y_{\mathbf{a}} &= \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}) \mid \alpha(t, \mathbf{D}) = \mathbf{n}^{M(\mathbf{d},\mathbf{a})}, \dim W = |\mathbf{a}|\} \\ &= \bigcup_{\mathbf{w}:|\mathbf{w}|=|\mathbf{a}|} \bigcup_{\mathbf{r}} A_{\mathbf{w},\mathbf{r},\mathbf{n}^{M(\mathbf{d},\mathbf{a})}}. \end{aligned} \tag{1.32}$$

Now, note that $\mathbf{n}^{M(\mathbf{d},\mathbf{a})}$ depends only on the lower curves of $\mathbf{a}$ and not on the orientation of the unmatched vertices. Thus, if $\bar{\mathbf{a}}$ denotes the (unoriented) lower crossingless match associated to $\mathbf{a}$, we can unambiguously define $\mathbf{n}^{\bar{\mathbf{a}}} = \mathbf{n}^{M(\mathbf{d},\mathbf{a})}$. Then if b is an unoriented crossingless match, we define

$$\begin{aligned} Y_b &= \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}) \mid \alpha(t, \mathbf{D}) = \mathbf{n}^b\} \\ &= \bigcup_{\mathbf{a}:\bar{\mathbf{a}}=b} Y_{\mathbf{a}}. \end{aligned} \tag{1.33}$$

The last equality arises from the fact that if $(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})$ then $\operatorname{im} t \subset W \subset \ker t$, so $r \leq \dim W \leq d - r$ (where $r = \operatorname{rank} t$). Thus, since $(\mathbf{D}, W, t) \in Y_b$ implies that $r = \operatorname{rank} t$ is the number of lower curves in b, the values $r, r+1, \ldots, d-r$ are precisely the number of down arrows (that is, the $|\mathbf{a}|$) in the various $\mathbf{a}$ such that $\bar{\mathbf{a}} = b$. We also have the following:

Proposition 1.3.B. $\sqcup_b Y_b = \sqcup_{\mathbf{a}} Y_{\mathbf{a}} = \mathfrak{T}(\mathbf{d})$.

Proof. It is obvious that the $Y_{\mathbf{a}}$ are disjoint. Thus, from equation (1.33) we see that it suffices to prove that for every $(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})$, $\alpha(t, \mathbf{D}) = \mathbf{n}^{M(\mathbf{d},\mathbf{a})}$ for some crossingless match $\mathbf{a}$. Fix an $(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})$ and let $\mathbf{a} = \alpha(t, \mathbf{D})$. Now, down arrows of $\mathbf{a}$ represent dimensions of the kernel of t while up arrows of $\mathbf{a}$ represent dimensions of $D/\ker t$. Let c denote the i^{th} up arrow from the left. Since $\operatorname{im} t \subset \ker t$ and $t(\mathbf{D}_j) \subset (\mathbf{D}_{j-1})$, there must be at least i down arrows to the left of c. Since this holds for all i, it follows that each up arrow of $M(\mathbf{d}, \mathbf{a})$ is matched. Thus, since $\mathbf{n}^{M(\mathbf{d},\mathbf{a})}$ is obtained from $\mathbf{a}$ by forcing all unmatched vertices to be oriented down, we have that $\mathbf{n}^{M(\mathbf{d},\mathbf{a})} = \mathbf{a} = \alpha(t, \mathbf{D})$. □

Define

$$\mathcal{B}_I = \left\{ T_{Y_b} \;\middle|\; b \in \bigcup_{\mu} CM^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k} \right\}. \tag{1.34}$$

Proposition 1.3.C. *Each element of $\mathcal{B}_I$ is an intertwining operator and*

$$T_{Y_b}(\mathcal{T}(\mathbf{d})) \subset \mathcal{M}^r(d) \cong V_\mu$$

for $b \in CM^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ and $r = (d-\mu)/2$.

Proof. According to Proposition 1.3.A, to show that T_{Y_b} is an intertwining operator we need only check that $\pi_2\pi_1^{-1}(Y_b) \subset Y_b$ and $\pi_1\pi_2^{-1}(Y_b) \subset Y_b$ for all $b \in CM^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$. If we denote by t^x and $\mathbf{D}^x$ the map t and flag $\mathbf{D}$ of the point $x \in \mathfrak{T}(\mathbf{d})$ (so $x = (\mathbf{D}^x, W, t^x)$ for some W), then $t^y = t^x$ and $\mathbf{D}^y = \mathbf{D}^x$ for all $y \in \pi_2\pi_1^{-1}(x)$. Thus $\alpha(t^x, \mathbf{D}^x) = \alpha(t^y, \mathbf{D}^y)$ for all $y \in \pi_2\pi_1^{-1}(x)$ which implies that $\pi_2\pi_1^{-1}(Y_b) \subset Y_b$ for all b. Similarly $\pi_1\pi_2^{-1}(Y_b) \subset Y_b$ for all b. Now, the image of T_{Y_b} consists of functions on $\mathfrak{M}^r(d)$ where r is the number of lower curves in b. In fact, it is easy to see that for $f \in \mathcal{T}(\mathbf{d})$, $T_{Y_b}(f)(W, t)$ depends only on the dimension of W and the rank of t. So the image of T_{Y_b} is contained in $\mathcal{M}^r(d)$. Recall from Section 1.1.3 that $\mathcal{M}^r(d) \cong V_{d-2r}$. Since r is equal to the number of lower curves in b, $d - 2r$ is equal to the number of middle curves and hence $d - 2r = \mu$. So T_{Y_b} is an intertwining operator into the representation V_μ as it should be. □

1.3.3 The space $\mathcal{T}_s(\mathbf{d})$ and the Basis $\mathcal{B}_s$

For the purposes of this section we will identify the sets $LCM_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ and $\cup_\mu CM^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ as in Section 1.3.2. Also, to simplify notation, we shall identify elements $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^k$ such that $\mathbf{a}_i \leq \mathbf{d}_i$ with their associated oriented lower crossingless matches $M(\mathbf{d}, \mathbf{a})$.

Let $\mathcal{T}_s(\mathbf{d})$ be the space of all functions $f \in \mathcal{T}(\mathbf{d})$ such that

$$\dim W = \dim W',\ \alpha(t, \mathbf{D}) = \alpha(t', \mathbf{D}') \Rightarrow f(\mathbf{D}, W, t) = f(\mathbf{D}', W', t').$$

It is obvious that if we define

$$\mathcal{B}_s = \{\mathbf{1}_{Y_\mathbf{a}} \,|\, \mathbf{a} \in \bigcup_\mu OCM^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}\}, \tag{1.35}$$

then

$$\mathcal{T}_s(\mathbf{d}) = \operatorname{Span} \mathcal{B}_s.$$

Theorem 1.3.D. *$\mathcal{T}_s(\mathbf{d})$ is isomorphic as a $\mathbf{U}_q$-module to $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ and $\mathcal{B}_s$ is a basis for $\mathcal{T}_s(\mathbf{d})$ adapted to its decomposition into a direct sum of irreducible representations. That is, for a given $b \in CM^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, the space $\operatorname{Span}\{\mathbf{1}_{Y_\mathbf{a}} \,|\, \bar{\mathbf{a}} = b\}$ is isomorphic to the irreducible representation V_μ via the map*

$$\mathbf{1}_{Y_\mathbf{a}} \mapsto {}^\mu v_{\mu - 2(\#\text{ of unmatched down arrows in }\mathbf{a})}$$

(extended by linearity).

Proof. For $\mathbf{a} \in LCM_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ such that $\mathbf{a}$ has at least one unmatched up arrow, let $\mathbf{a}^+$ be the element of $LCM_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ obtained from $\mathbf{a}$ by switching the orientation of the rightmost unmatched up arrow. Thus $\overline{\mathbf{a}^+} = \bar{\mathbf{a}}$ and $\mathbf{a}^+$ has one more unmatched down arrow than $\mathbf{a}$. Similarly, if $\mathbf{a} \in LCM_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ has at least one unmatched down arrow, let $\mathbf{a}^-$ be the element of $LCM_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ obtained from $\mathbf{a}$ by switching the orientation of the leftmost unmatched down arrow. Recall from the proofs of Propositions 1.3.A and 1.3.C that $\pi_1^{-1}(Y_b) = \pi_2^{-1}(Y_b)$. It follows from this and the fact that $Y_b = \bigcup_{\mathbf{a}:\bar{\mathbf{a}}=b} Y_\mathbf{a}$

that $\pi_2\pi_1^{-1}(Y_{\mathbf{a}}) = Y_{\mathbf{a}^+}$ if $\mathbf{a}$ has at least one unmatched up arrow and $\pi_2\pi_1^{-1}(Y_{\mathbf{a}}) = \emptyset$ otherwise. Similarly, $\pi_1\pi_2^{-1}(Y_{\mathbf{a}}) = Y_{\mathbf{a}^-}$ if $\mathbf{a}$ has at least one unmatched down arrow and $\pi_1\pi_2^{-1}(Y_{\mathbf{a}}) = \emptyset$ otherwise.

Now, for $x \in \mathfrak{T}(\mathbf{d})$,

$$\begin{aligned} F\mathbf{1}_{Y_{\mathbf{a}}}(x) &= q^{-\dim(\pi_2^{-1}(x))}(\pi_2)_!\pi_1^*\mathbf{1}_{Y_{\mathbf{a}}}(x) \\ &= q^{-\dim(\pi_2^{-1}(x))}(\pi_2)_!\mathbf{1}_{\pi_1^{-1}(Y_{\mathbf{a}})}(x) \\ &= q^{-\dim(\pi_2^{-1}(x))}\chi_q(\pi_2^{-1}(x)\cap\pi_1^{-1}(Y_{\mathbf{a}})). \end{aligned}$$

Now, we already know from the above discussion that $\pi_2^{-1}(x)\cap\pi_1^{-1}(Y_{\mathbf{a}}) = \emptyset$ if $x \notin Y_{\mathbf{a}^+}$. So assuming $x = (\mathbf{D}, W, t) \in Y_{\mathbf{a}^+}$, let $r = \operatorname{rank} t$. Then

$$\begin{aligned} F\mathbf{1}_{Y_{\mathbf{a}}}(\mathbf{D}, W, t) &= q^{-\dim(\pi_2^{-1}(\mathbf{D},W,t))}\chi_q(\pi_2^{-1}(\mathbf{D}, W, t)\cap\pi_1^{-1}(Y_{\mathbf{a}})) \\ &= q^{-\dim \mathbb{P}^{|\mathbf{a}^+|-r-1}}\chi_q\left(\mathbb{P}^{|\mathbf{a}^+|-r-1}\right) \\ &= q^{-(|\mathbf{a}^+|-r-1)}\sum_{i=0}^{|\mathbf{a}^+|-r-1} q^{2i} \\ &= \left[|\mathbf{a}^+| - r\right] \\ &= [(\#\text{ down arrows in }\mathbf{a}^+) - (\#\text{ lower curves in }\mathbf{a}^+)] \\ &= [\#\text{ unmatched down arrows in }\mathbf{a}^+]. \end{aligned}$$

Thus,

$$F\mathbf{1}_{Y_{\mathbf{a}}} = [\#\text{ unmatched down arrows in }\mathbf{a}^+]\mathbf{1}_{Y_{\mathbf{a}^+}}. \tag{1.36}$$

Now,

$$\begin{aligned} E\mathbf{1}_{Y_{\mathbf{a}}}(x) &= q^{-\dim(\pi_1^{-1}(x))}(\pi_1)_!\pi_2^*\mathbf{1}_{Y_{\mathbf{a}}}(x) \\ &= q^{-\dim(\pi_1^{-1}(x))}(\pi_1)_!\mathbf{1}_{\pi_2^{-1}(Y_{\mathbf{a}})}(x) \\ &= q^{-\dim(\pi_1^{-1}(x))}\chi_q(\pi_1^{-1}(x)\cap\pi_2^{-1}(Y_{\mathbf{a}})). \end{aligned}$$

We know that $\pi_1^{-1}(x)\cap\pi_2^{-1}(Y_{\mathbf{a}}) = \emptyset$ if $x \notin Y_{\mathbf{a}^-}$. So assuming $x = (\mathbf{D}, W, t) \in Y_{\mathbf{a}^-}$, let $r = \operatorname{rank} t$. Then

$$E\mathbf{1}_{Y_{\mathbf{a}}}(\mathbf{D}, W, t) = q^{-\dim(\pi_1^{-1}(\mathbf{D},W,t))}\chi_q(\pi_1^{-1}(\mathbf{D}, W, t)\cap\pi_2^{-1}(Y_{\mathbf{a}}))$$

$$= q^{-\dim \mathbb{P}^{d-r-|\mathbf{a}^-|-1}} \chi_q(\mathbb{P}^{d-r-|\mathbf{a}^-|-1})$$

$$= [d - r - |\mathbf{a}^-|]$$

$$= [d - (\#\text{ lower curves in } \mathbf{a}^-) - (\#\text{ down arrows in } \mathbf{a}^-)]$$

$$= [(\#\text{ up arrows in } \mathbf{a}^-) - (\#\text{ lower curves in } \mathbf{a}^-)]$$

$$= [\#\text{ unmatched up arrows in } \mathbf{a}^-].$$

Thus,

$$E\mathbf{1}_{Y_{\mathbf{a}}} = [\#\text{ unmatched up arrows in } \mathbf{a}^-]\mathbf{1}_{Y_{\mathbf{a}^-}}. \tag{1.37}$$

Finally, it is easy to see that

$$\begin{aligned} K\mathbf{1}_{Y_{\mathbf{a}}} &= q^{\pm(d-2|\mathbf{a}|)}\mathbf{1}_{Y_{\mathbf{a}}} \\ &= q^{\pm(\mu - 2(\#\text{ unmatched down arrows in } \mathbf{a}))}\mathbf{1}_{Y_{\mathbf{a}}} \end{aligned} \tag{1.38}$$

where μ is the total number of unmatched arrows in $\mathbf{a}$. Using the fact that μ is the total number of middle curves of b (and hence the total number of unmatched vertices in any $\mathbf{a}$ such that $\bar{\mathbf{a}} = b$), the second statement of the theorem now follows easily from a comparison with (1.1).

Since we know from Section 1.1.4 that the set $CM^{\mu}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ is in one to one correspondence with the set of intertwining operators $H^{\mu}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$, we have that

$$\mathcal{T}_s(\mathbf{d}) \cong \bigoplus_{\mu} H^{\mu}_{\mathbf{d}_1,\ldots,\mathbf{d}_k} \otimes V_{\mu} \cong V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$$

which proves the first statement of the theorem. □

Now, like the canonical basis, the basis $\mathcal{B}_s$ we have constructed here is closely related to the irreducible components of $\mathfrak{T}(\mathbf{d})$. To see this, we first need a proposition. Consider the varieties $Y_{\mathbf{a}}$ and Y_b defined over $\bar{\mathbb{F}}_{q^2}$. To avoid confusion, denote these by $Y'_{\mathbf{a}}$ and Y'_b. Then

Proposition 1.3.E. $\overline{Y'_{\mathbf{a}}} = \overline{Z'_{\mathbf{a}}}$.

Proof. Since the $Y'_{\mathbf{a}}$ are smooth and connected, they are irreducible. Also, from an argument analogous to the one given in the proof of Proposition 1.3.B, we know that $\sqcup_{\mathbf{a}} Y'_{\mathbf{a}} = \mathfrak{T}(\mathbf{d})'$. Thus, since the cardinality of the sets $\{\overline{Y'_{\mathbf{a}}}\}$ and $\{\overline{Z'_{\mathbf{a}}}\}$ are the same, $\{\overline{Y'_{\mathbf{a}}}\}$ must be the set of irreducible components of $\mathfrak{T}(\mathbf{d})'$. Now, $Y'_{\mathbf{a}} \cap Z'_{\mathbf{a}} = \bigcup_{\mathbf{r}} A'_{\mathbf{a},\mathbf{r},\mathbf{n}^{M(\mathbf{d},\mathbf{a})}}$. But, by Proposition 1.2.D, $\overline{A'_{\mathbf{a},\mathbf{r}^{M(\mathbf{d},\mathbf{a})},\mathbf{n}^{M(\mathbf{d},\mathbf{a})}}} = \overline{Z'_{\mathbf{a}}}$. Therefore we must have $\overline{Y'_{\mathbf{a}}} = \overline{Z'_{\mathbf{a}}}$. □

Since $Y_{\mathbf{a}}$ is precisely the set of $\mathbb{F}_{q^2}$ points of $Y'_{\mathbf{a}}$, we have the following characterization of the basis $\mathcal{B}_s$.

Theorem 1.3.F. *The elements $\mathbf{1}_{Y_{\mathbf{a}}}$ of the basis $\mathcal{B}_s$ are the unique elements of $\mathcal{T}_s(\mathbf{d})$ equal to one on the dense points of the irreducible component $\overline{Z_{\mathbf{a}}}$ of $\mathfrak{T}(\mathbf{d})$ with support contained in this irreducible component.*

So, like the elements of $\mathcal{B}_c$, the elements of $\mathcal{B}_s$ are equal to a non-zero constant on the set of dense points of an irreducible component of $\mathfrak{T}(\mathbf{d})$ with supports contained in distinct irreducible components. However, unlike $\mathcal{B}_c$, the elements of $\mathcal{B}_s$ have disjoint supports.

1.3.4 The Multiplicity Variety $\mathfrak{S}(\mathbf{d})$

We briefly describe here the relation between $\mathcal{B}_I$ and $\mathcal{B}_s$ and the multiplicity variety [19]. Let $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^k$ and let D be a $|\mathbf{d}|$-dimensional $\bar{\mathbb{F}}_{q^2}$ vector space. The *multiplicity variety* is the variety (defined over $\bar{\mathbb{F}}_{q^2}$)

$$\mathfrak{S}(\mathbf{d})' = \{(\mathbf{D}, t) \,|\, (\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})' \text{ for some } W \subset D\}.$$

Define the projection $\pi : \mathfrak{T}(\mathbf{d})' \to \mathfrak{S}(\mathbf{d})'$ by $\pi(\mathbf{D}, W, t) = (\mathbf{D}, t)$. It follows easily from the above results that the irreducible components of $\mathfrak{S}(\mathbf{d})'$ are given by the closures of the sets

$$\mathcal{Y}'_b = \{(\mathbf{D}, t) \,|\, \alpha(t, \mathbf{D}) = \mathbf{n}^b\},\ b \in CM^{\mu}_{\mathbf{d}_1,\ldots,\mathbf{d}_k},$$

and that these irreducible components are in one to one correspondence with the irreducible modules in the direct sum decomposition of $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$. Then $Y'_b = \pi_{-1}(\mathcal{Y}'_b)$ and $\{Y_{\mathbf{a}} \,|\, \bar{\mathbf{a}} = b\}$ yields a decomposition of the $\mathbb{F}_{q^2}$ points of the fiber of $\pi|_{Y'_b}$ isomorphic to the decomposition of $\mathfrak{M}^r(d)$ into the subsets $\mathfrak{M}^r(w,d)$ where r is the number of lower curves in b. Thus the bases $\mathcal{B}_I$ and $\mathcal{B}_s$ have natural geometric interpretations in terms of the multiplicity variety and the projection π.

1.3.5 The Action of the Intertwining Operators on $\mathcal{T}_s(\mathbf{d})$

We will now determine how our intertwining operators act on the space $\mathcal{T}_s(\mathbf{d})$. For $\mathbf{a} \in (Z_{\geq 0})^k$, let $\mathbf{a}^j = \mathbf{a}^{(1,j)}$. We will need the following two technical lemmas.

Lemma 1.3.G. *If $\mathbf{D} = (0 = \mathbf{D}_0 \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset \cdots \subset \mathbf{D}_k = D)$ is a flag with $\mathbf{d} = \alpha(D, \mathbf{D})$ and $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^k$ with $\mathbf{a}_i \leq \mathbf{d}_i$, then*

$$\chi_q(\{W \,|\, W \subset D,\, \alpha(W, \mathbf{D}) = \mathbf{a}\}) = c_{\mathbf{d},\mathbf{a}} \stackrel{def}{=} \sum_{\mathbf{b} \in C_{\mathbf{a}}} q^{2\sum_{1 \leq j < i \leq d} \mathbf{b}_i(1-\mathbf{b}_j)}$$

where

$$C_{\mathbf{a}} = \{\mathbf{b} \in (\mathbb{Z}_{\geq 0})^d |\ \mathbf{b}_i \in \{0,1\} \,\forall\, i,\ \mathbf{b}^{(\mathbf{d}_{j-1}+1, \mathbf{d}_j)} = \mathbf{a}_j\}$$

and we set $\mathbf{d}_0 = 0$.

Proof. Complete $\mathbf{D}$ to a flag $\mathbf{F} = (0 \subset \mathbf{F}_1 \subset \mathbf{F}_2 \subset \cdots \subset \mathbf{F}_d = W)$ such that $\dim \mathbf{F}_i = i$ and $\mathbf{F}_{\mathbf{d}^i} = \mathbf{D}_i$ where $d = |\mathbf{d}|$. This gives a decomposition of $Gr^d_{|\mathbf{a}|}$ into cells, each isomorphic to $(\mathbb{F}_{q^2})^j$ for some j. The cells are given by $\{W \,|\, W \subset D,\, \alpha(W, \mathbf{F}) = \mathbf{b}\}$ for a fixed $\mathbf{b}$. The number of points in such a cell is equal to

$$q^{2\sum_{1 \leq j < i \leq d} \mathbf{b}_i(1-\mathbf{b}_j)}.$$

Our variety is the union of those cells such that $\mathbf{b}^{(\mathbf{d}_{j-1}+1, \mathbf{d}_j)} = \mathbf{a}_j$. The result follows. ☐

Specializing to $q = 1$ yields

Lemma 1.3.H.

$$c_{\mathbf{d},\mathbf{a}}|_{q=1} = \prod_{i=1}^{k} \binom{\mathbf{d}_i}{\mathbf{a}_i}.$$

Proof. This follows immediately from Lemma 1.3.G since $\mathbf{b}_i \in \{0, 1\}$ for each cell. □

Theorem 1.3.I. *The set $\mathcal{B}_I$ acting on $\mathcal{T}_s(\mathbf{d})$ spans the space of intertwining operators $\bigoplus_\mu H^\mu_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$. In particular, for $b \in CM^\mu_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$, T_{Y_b} acts on the basis $\mathcal{B}_s$ of $\mathcal{T}_s(\mathbf{d})$ as*

$$T_{Y_b}\mathbf{1}_{Y_\mathbf{a}} = \begin{cases} c_b \mathbf{1}_{\mathfrak{M}^r(|\mathbf{a}|,d)} \in \mathcal{M}^r(d) \cong V_{d-2r} = V_\mu & \text{if } \bar{\mathbf{a}} = b, \\ 0 & \text{if } \bar{\mathbf{a}} \neq b \end{cases}$$

where r is the number of lower curves in b, $d = |\mathbf{d}|$ and c_b is non-zero constant.

Proof. Recall that $T_Y = p_! R_Y$. It is obvious from the fact that $Y_b = \bigcup_{\mathbf{a}:\bar{\mathbf{a}}=b} Y_\mathbf{a}$ that

$$R_{Y_b}\mathbf{1}_{Y_\mathbf{a}} = \mathbf{1}_{Y_b}\mathbf{1}_{Y_\mathbf{a}} = \begin{cases} \mathbf{1}_{Y_\mathbf{a}} & \text{if } \bar{\mathbf{a}} = b \\ 0 & \text{if } \bar{\mathbf{a}} \neq b \end{cases}.$$

So we need only determine $p_!\mathbf{1}_{Y_\mathbf{a}}$ for $\bar{\mathbf{a}} = b$. Now, for $x = (W^x, t^x) \in \mathfrak{M}(d)$,

$$p_!\mathbf{1}_{Y_\mathbf{a}}(x) = \chi_q(p^{-1}(x) \cap Y_\mathbf{a}).$$

Recall that p is the map $(\mathbf{D}, W, t) \mapsto (W, t)$ and

$$Y_\mathbf{a} = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}) \,|\, \alpha(t, \mathbf{D}) = \mathbf{n}^b,\, \dim W = |\mathbf{a}|\}.$$

Thus,

$$p^{-1}(x) \cap Y_\mathbf{a} \cong \{\mathbf{D} \,|\, \dim(\mathbf{D}_i/\mathbf{D}_{i-1}) = \mathbf{d}_i,\, t^x(\mathbf{D}_i) \subset \mathbf{D}_{i-1},\, \alpha(t^x, \mathbf{D}) = \mathbf{n}^b\} \qquad (1.39)$$

if $\dim W^x = |\mathbf{a}|$ and $p^{-1}(x) \cap Y_\mathbf{a} = \emptyset$ otherwise. Note that this variety depends only on the dimension of the kernel of t^x (or equivalently, the rank of t^x) and the dimension of W^x. The variety is empty unless $r = \operatorname{rank} t^x$ is equal to the number of lower curves in $\mathbf{a}$. Thus, $T_{Y_b}\mathbf{1}_{Y_\mathbf{a}}$ is a constant function on $\mathfrak{M}^r(|\mathbf{a}|, d)$. Moreover, this constant c_b, equal to the number of points in the variety in (1.39), depends only on $\bar{\mathbf{a}} = b$ and not on the orientation of $\mathbf{a}$. As long as c_b is non-zero, we know that T_{Y_b} is a non-zero intertwining operator. Moreover, it is obvious that if all the Y_b are non-zero then the intertwining operators T_{Y_b} are linearly independent.

To show that $c_b \neq 0$ it suffices to show that its evaluation at $q = 1$ is non-zero. The variety (1.39) consists of all t_x-stable flags $\mathbf{D} = (0 \subset \mathbf{D}_1 \subset \cdots \subset \mathbf{D}_k = D)$ such that $\dim \mathbf{D}_i = \mathbf{d}^i$ and the intersection of $\mathbf{D}_i$ with $\ker t^x$ is a space of dimension $(\mathbf{n}^b)^j = \sum_{i=1}^{j} \mathbf{n}_i^b$. There is only one choice for $\mathbf{D}_k$, namely D. Assume we have picked $\mathbf{D}_{j+1}$. $\mathbf{D}_j$ can be any subspace of dimension $\mathbf{d}^i$ such that

$$\begin{aligned} &t^x(\mathbf{D}_{j+1}) \subset \mathbf{D}_j \subset \mathbf{D}_{j+1} \\ &\text{and } \dim(\mathbf{D}_j \cap \ker t^x) = (\mathbf{n}^b)^j. \end{aligned}$$

Note that since $\dim(\mathbf{D}_{j+1} \cap \ker t^x) = (\mathbf{n}^b)^{j+1}$ and $\dim \mathbf{D}_{j+1} = \mathbf{d}^{j+1}$, we have that $\dim t^x(\mathbf{D}_{j+1}) = \mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1}$. Also, since $(t^x)^2 = 0$, $t^x(\mathbf{D}_{j+1}) \subset \ker t^x$. Passing to the quotient by $t^x(\mathbf{D}_{j+1})$ and denoting this by a bar, we see that picking a subspace $\mathbf{D}_j$ subject to the above conditions is equivalent to picking a subspace $\overline{\mathbf{D}_j}$ of $\overline{\mathbf{D}_{j+1}}$ of dimension $\mathbf{d}^j - (\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1})$ such that

$$\dim(\overline{\mathbf{D}_j} \cap \overline{\ker t^x}) = (\mathbf{n}^b)^j - (\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1}).$$

Since $\dim \overline{\mathbf{D}_{j+1}} = \mathbf{d}^{j+1} - (\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1}) = (\mathbf{n}^b)^{j+1}$ and $\dim \overline{\mathbf{D}_{j+1}} \cap \overline{\ker t^x} = \dim \overline{\mathbf{D}_{j+1} \cap \ker t^x} = (\mathbf{n}^b)^{j+1} - (\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1}) = 2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1}$ we see by Lemma 1.3.H that the value of χ_q of the variety of such spaces evaluated at $q = 1$ is

$$\begin{aligned} &\binom{2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1}}{(\mathbf{n}^b)^{j+1} + (\mathbf{n}^b)^j - \mathbf{d}^{j+1}} \cdot \binom{(\mathbf{n}^b)^{j+1} - (2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1})}{\mathbf{d}^j - \mathbf{d}^{j+1} + (\mathbf{n}^b)^{j+1} - ((\mathbf{n}^b)^{j+1} + (\mathbf{n}^b)^j - \mathbf{d}^{j+1})} \\ &= \binom{2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1}}{(\mathbf{n}^b)^{j+1} + (\mathbf{n}^b)^j - \mathbf{d}^{j+1}} \cdot \binom{\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1}}{\mathbf{d}^j - (\mathbf{n}^b)^j}. \end{aligned}$$

This is thus strictly positive provided that

$$2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1} \geq 0 \tag{1.40}$$

$$(\mathbf{n}^b)^{j+1} + (\mathbf{n}^b)^j - \mathbf{d}^{j+1} \geq 0 \tag{1.41}$$

$$\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1} \geq 0 \tag{1.42}$$

$$\mathbf{d}^{j} - (\mathbf{n}^b)^{j} \geq 0 \tag{1.43}$$

$$2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1} \geq (\mathbf{n}^b)^{j+1} + (\mathbf{n}^b)^{j} - \mathbf{d}^{j+1} \tag{1.44}$$

$$\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1} \geq \mathbf{d}^{j} - (\mathbf{n}^b)^{j} \tag{1.45}$$

Now, recall that $\mathbf{n}^b$ is obtained from $\mathbf{a}$ by forcing all unmatched arrows to be oriented down. Also, $\mathbf{d}^j$ is the number of vertices associated to $V_{\mathbf{d}_1}$ through $V_{\mathbf{d}_j}$ while $(\mathbf{n}^b)^j$ is number of these vertices with down arrows. Thus $\mathbf{d}^j - (\mathbf{n}^b)^j$ is the number of these vertices with up arrows. So (1.42), (1.43) and (1.45) are obvious. (1.44) follows from the simple fact that $(\mathbf{n}^b)^{j+1} \geq (\mathbf{n}^b)^j$. (1.40) and (1.41) follow from the fact that each up arrow is matched to a down arrow to its left since all unmatched arrows point down and matchings are oriented to the left.

Thus, χ_q of the variety of choices of $\mathbf{D}_j$ given $\mathbf{D}_{j+1}$ is independent of $\mathbf{D}_{j+1}$ (up to isomorphism) and is non-zero. Using the fact that the Euler characteristic of a locally trivial fibered space is equal to the product of the Euler characteristics of the base and the fiber, we see that the evaluation of c_b at 1 is a product of positive numbers and is thus positive. So $c_b \neq 0$. $\square$

1.3.6 The Action of the Intertwining Operators on $\mathcal{T}_c(\mathbf{d})$

We now compute the action of our intertwining operators on the space $\mathcal{T}_c(\mathbf{d})$.

Define the coefficients $\kappa_{\mathbf{a}}^{\mathbf{d},\mathbf{w}}$ by

$$\Diamond^{\mathbf{d}} v_{\mathbf{w}} = \sum_{\mathbf{a}} \kappa_{\mathbf{a}}^{\mathbf{d},\mathbf{w}} \left(\otimes^{\mathbf{d}} v_{\mathbf{a}}\right).$$

For $b \in CM^{\mu}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$, define $\mathbf{l}^b, \mathbf{m}^b \in (\mathbb{Z}_{\geq 0})^k$ such that $\mathbf{l}^b_i$ is equal to the number of left endpoints of lower curves of b in the box corresponding to $V_{\mathbf{d}_i}$ and $\mathbf{m}^b_i$ is equal to the number of endpoints of middle curves of b in the box corresponding to $V_{\mathbf{d}_i}$.

Theorem 1.3.J. *The set $\mathcal{B}_I$ acting on $\mathcal{T}_c(\mathbf{d})$ spans the space of intertwining operators*

$\bigoplus_\mu H^\mu_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$. *In particular, if* $b \in CM^\mu_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ *is such that* $b \leq M(\mathbf{d},\mathbf{w})$*, then*

$$T_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) = \omega \mathbf{1}_{\mathfrak{M}^{|\mathbf{l}^b|}(|\mathbf{w}|,d)}$$

where

$$\begin{aligned}
\omega &= \sum_{\mathbf{a}} \left(\kappa^{\mathbf{m}^b,\mathbf{w}-\mathbf{l}^b}_{\mathbf{a}} k_{\mathbf{a}+\mathbf{l}^b,\mathbf{l}^b,\mathbf{l}^b+\mathbf{m}^b} \prod_{i=1}^{k-1} c_{\mathbf{a}^i_1,\mathbf{a}^i_2} \right), \\
\mathbf{a}^i_1 &= \left((\mathbf{l}^b)^{(i,k)},\, \mathbf{a}^{(i,k)},\, (\mathbf{m}^b - \mathbf{a})^{(i,k)},\, (\mathbf{d} - \mathbf{m}^b - 2\mathbf{l}^b)^{(i,k)} \right), \\
\mathbf{a}^i_2 &= \left(\mathbf{l}^b_i,\, \mathbf{a}_i,\, \mathbf{m}^b_i - \mathbf{a}_i,\, \mathbf{d}_i - \mathbf{m}^b_i - \mathbf{l}^b_i \right)
\end{aligned}$$

Otherwise, $T_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) = 0$.

Proof. For a crossingless match $b \in CM^\mu_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$, $T_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) = p_! R_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}})$ and

$$R_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) = \sum_{S \leq M(\mathbf{d},\mathbf{w})} R_{Y_b}\left(\eta_{\mathbf{r}^S,\mathbf{n}^S}\right)^{-1} \left(\diamondsuit^{\mathbf{n}^S - \mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S} \right).$$

This is equal to zero unless $\bar{S} = b$ for some $S \leq M(\mathbf{d},\mathbf{w})$ (that is, the set of lower curves of b is a subset of the set of lower curves of $M(\mathbf{d},\mathbf{w})$). If this is the case, then

$$R_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) = \left(\eta_{\mathbf{r}^S,\mathbf{n}^S}\right)^{-1} \left(\diamondsuit^{\mathbf{n}^S - \mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S} \right)$$

for the particular $S \leq M(\mathbf{d},\mathbf{w})$ such that $\bar{S} = b$. Then $\mathbf{n}^S = \mathbf{l}^b + \mathbf{m}^b$ and $\mathbf{r}^S = \mathbf{l}^b$. So

$$\begin{aligned}
R_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) &= \left(\eta_{\mathbf{l}^b,\mathbf{l}^b+\mathbf{m}^b}\right)^{-1} \left(\diamondsuit^{\mathbf{m}^b} v_{\mathbf{w}-\mathbf{l}^b} \right) \\
&= \left(\eta_{\mathbf{l}^b,\mathbf{l}^b+\mathbf{m}^b}\right)^{-1} \left(\sum_{\mathbf{a}} \kappa^{\mathbf{m}^b,\mathbf{w}-\mathbf{l}^b}_{\mathbf{a}} \left(\otimes^{\mathbf{m}^b} v_{\mathbf{a}} \right) \right) \\
&= \sum_{\mathbf{a}} \kappa^{\mathbf{m}^b,\mathbf{w}-\mathbf{l}^b}_{\mathbf{a}} f_{\mathbf{a}+\mathbf{l}^b,\mathbf{l}^b,\mathbf{l}^b+\mathbf{m}^b} \\
&= \sum_{\mathbf{a}} \kappa^{\mathbf{m}^b,\mathbf{w}-\mathbf{l}^b}_{\mathbf{a}} k_{\mathbf{a}+\mathbf{l}^b,\mathbf{l}^b,\mathbf{l}^b+\mathbf{m}^b} \mathbf{1}_{A_{\mathbf{a}+\mathbf{l}^b,\mathbf{l}^b,\mathbf{l}^b+\mathbf{m}^b}}.
\end{aligned}$$

Let $(W,t) \in \mathfrak{M}(d)$. Then if the set of lower curves of b is a subset of the set of lower curves of $M(\mathbf{d},\mathbf{w})$,

$$T_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}})(W,t) = \sum_{\mathbf{a}} \kappa^{\mathbf{m}^b,\mathbf{w}-\mathbf{l}^b}_{\mathbf{a}} k_{\mathbf{a}+\mathbf{l}^b,\mathbf{l}^b,\mathbf{l}^b+\mathbf{m}^b} p_! \mathbf{1}_{A_{\mathbf{a}+\mathbf{l}^b,\mathbf{l}^b,\mathbf{l}^b+\mathbf{m}^b}}(W,t)$$

$$= \sum_{\mathbf{a}} \kappa_{\mathbf{a}}^{\mathbf{m}^b,\mathbf{w}-\mathbf{l}^b} k_{\mathbf{a}+\mathbf{l}^b,\mathbf{l}^b,\mathbf{l}^b+\mathbf{m}^b} \chi_q(\{\mathbf{D} \,|\, \alpha(D,\mathbf{D}) = \mathbf{d},\ t(\mathbf{D}_i) \subset \mathbf{D}_{i-1}, \\ \alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{l}^b,\ \alpha(W,\mathbf{D}) = \mathbf{l}^b + \mathbf{a},\ \alpha(\ker t, \mathbf{D}) = \mathbf{l}^b + \mathbf{m}^b\}). \tag{1.46}$$

We see from Proposition 1.2.H that $\kappa_{\mathbf{a}}^{\mathbf{m}^b,\mathbf{w}-\mathbf{l}^b} = 0$ unless $|\mathbf{a}| = |\mathbf{w} - \mathbf{l}^b| = |\mathbf{w}| - |\mathbf{l}^b|$. Therefore, since $|\alpha(W,\mathbf{D})| = \dim W$, (1.46) is zero unless $\dim W = |\mathbf{w}|$. Similarly, it is zero unless $\operatorname{rank} t = \dim(\operatorname{im} t) = |\mathbf{l}^b|$. If these conditions are satisfied, (1.46) is independent of W and t. We can then evaluate ω, the value of the expression in (1.46), using Lemma 1.3.G and the fact that the Euler characteristic of a locally trivial fibered space is the product of the Euler characteristics of the base and the fiber. There is only one possible choice for $\mathbf{D}_0$, namely 0. Assume we have picked $\mathbf{D}_{i-1}$. Then $\mathbf{D}_i$ must satisfy the following conditions:

1. $t(\mathbf{D}_i) \subset \mathbf{D}_{i-1}$ or, equivalently, $\mathbf{D}_i \subset t^{-1}(\mathbf{D}_{i-1})$

2. $\mathbf{D}_i \supset \mathbf{D}_{i-1}$, $\dim \mathbf{D}_i = \mathbf{d}^{(1,i)}$

3. $\dim(\mathbf{D}_i \cap \operatorname{im} t) = (\mathbf{l}^b)^{(1,i)}$

4. $\dim(\mathbf{D}_i \cap W) = (\mathbf{l}^b + \mathbf{a})^{(1,i)}$

5. $\dim(\mathbf{D}_i \cap \ker t) = (\mathbf{l}^b + \mathbf{m}^b)^{(1,i)}$.

Pass to the quotient by $\mathbf{D}_{i-1}$ and denote this by a bar. Let $\mathbf{F}$ be the flag

$$\mathbf{F} = (\mathbf{F}_0 = 0 \subset \mathbf{F}_1 = \overline{\operatorname{im} t} \subset \mathbf{F}_2 = \overline{W} \subset \mathbf{F}_3 = \overline{\operatorname{im} t}, \mathbf{F}_4 = \overline{t^{-1}(\mathbf{D}_{i-1})}).$$

Then the above conditions are equivalent to picking $\overline{\mathbf{D}_i} \subset \overline{t^{-1}(\mathbf{D}_{i-1})}$ such that

$$\alpha(\mathbf{D}_i, \mathbf{F}) = (\mathbf{l}_i^b, \mathbf{a}_i, \mathbf{m}_i^b - \mathbf{a}_i, \mathbf{d}_i - \mathbf{l}_i^b - \mathbf{m}_i^b).$$

Since

$$\dim \overline{\operatorname{im} t} = (\mathbf{l}^b)^{(i,k)},$$

$$\dim \overline{W} = (\mathbf{l}^b + \mathbf{a})^{(i,k)},$$

$$\dim \overline{\ker t} = (\mathbf{l}^b + \mathbf{m}^b)^{(i,k)},$$

and
$$\begin{aligned}\dim \overline{t^{-1}(\mathbf{D}_{i-1})} &= \dim t^{-1}(\mathbf{D}_{i-1}) - \dim \mathbf{D}_{i-1} \\ &= \dim(\operatorname{im} t \cap D_{i-1}) + \dim(\ker t) - \dim \mathbf{D}_{i-1} \\ &= (\mathbf{l}^b)^{(1,i-1)} + (d - |\mathbf{l}^b|) - \mathbf{d}^{(1,i-1)} \\ &= (\mathbf{d} - \mathbf{l}^b)^{(i,k)},\end{aligned}$$

the form of the action of the elements of $\mathcal{B}_I$ follows.

It remains to show that the set $\mathcal{B}_I$ spans the space of intertwining operators $\bigoplus_\mu H^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$. Since it follows from Theorem 1.3.I that the cardinality of $\mathcal{B}_I$ is equal to the dimension of $\bigoplus_\mu H^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, it suffices to show the linear independence of the set $\mathcal{B}_I$. Assume that, acting on the space $\mathcal{T}_s(\mathbf{d})$,

$$\sum_i a_i T_{Y_{b_i}} = 0, \quad a_i \neq 0 \;\forall\; i. \tag{1.47}$$

Since the image of $T_{Y_{b_i}}$ is contained in $\mathcal{M}^{|l_{b_i}|}(d)$ by the above results, we may assume that $|l_{b_i}| = |l_{b_j}|$ for all i and j. Fix an i and consider a $\mathbf{w}$ such that $M(\mathbf{d}, \mathbf{w}) = b_i$. All $T_{Y_{b_j}}$, $j \neq i$, act by zero on $g^{\mathbf{d}}_{\mathbf{w}}$ by the above (since $|l_{b_i}| = |l_{b_j}|$, we cannot have $b_j \leq b_i = M(\mathbf{d}, \mathbf{w})$). Also, $T_{Y_{b_i}} \neq 0$ by the above. Thus $a_i = 0$ which is a contradiction. Thus the theorem is proved. □

1.3.7 An Isomorphism of $\mathcal{T}_c(\mathbf{d})$ with $\mathcal{T}_s(\mathbf{d})$

For $(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})$, let

$$B_{\mathbf{D},W,t} = \{(\mathbf{D}', W', t') \,|\, W' = W,\ t' = t,\ \alpha(t, \mathbf{D}') = \alpha(t, \mathbf{D})\}.$$

For $f \in \mathcal{T}(\mathbf{d})$ let

$$\chi_q(f) = \sum_{x \in \mathfrak{T}(\mathbf{d})} f(x).$$

Let $\xi : \mathcal{T}_c(\mathbf{d}) \to \mathcal{T}_s(\mathbf{d})$ be the map given by

$$\xi(f)(\mathbf{D}, W, t) = \chi_q(R_{B_{\mathbf{D},W,t}} f).$$

The fact that the image of ξ is contained in $\mathcal{T}_s(\mathbf{d})$ follows from the fact that, up to isomorphism, $B_{\mathbf{D},W,t}$ depends only on $\alpha(t,\mathbf{D})$ and $\dim W$.

Proposition 1.3.K. *ξ is an $\mathbf{U}_q$-module isomorphism.*

Proof. This follows easily from Theorems 1.3.I and 1.3.J since

$$\xi = \sum_b \frac{1}{c_b}(T_{Y_b}|_{Y_b})^{-1} \circ T_{Y_b}.$$

□

Chapter 2

Fusion Products

Introduction

In the previous chapter we discussed a geometric realization of integrable highest weight representations V_λ of $\mathfrak{sl}_2$ (or the associated quantum group). One might ask if a similar construction can produce the *fusion* tensor products $V_\lambda \otimes_l V_\mu$, certain truncations of $V_\lambda \otimes V_\mu$. In this chapter, we answer this question affirmatively. We realize $V_\lambda \otimes_l V_\mu$ as the homology of the most natural subvarieties of the tensor product varieties (see Section 3). We also consider the case of a fusion tensor product of arbitrarily many $\mathfrak{sl}_2$-modules $V_{\lambda_1}, \cdots, V_{\lambda_r}$. Finally, we give a combinatorial description of the irreducible components of the fusion tensor product varieties using the notions of graphical calculus and crossingless matches for $\mathfrak{sl}_2$. We do not expect these constructions to generalize to Lie algebras of higher rank.

2.1 Fusion products for $U(\mathfrak{sl}_2)$.

Let $\mathcal{R}$ denote the category of finite-dimensional $\mathfrak{sl}_2$-modules, and for $i \geq 0$ let V_i denote the simple module of highest weight i. Let $\mathbb{C}[\mathcal{R}]$ be the Grothendieck ring of $\mathcal{R}$ and let $[V]$ denote the class of a module V. We have

$$V_i \otimes V_j \simeq \bigoplus_{k=j-i}^{i+j} V_k, \qquad [V_i] \cdot [V_j] = \sum_{k=j-i}^{i+j} [V_k], \qquad \text{for } i \leq j$$

where in the sums k increases by twos.

Now let us fix some positive integer $l \in \mathbb{N}$. Consider the quotient

$$\mathbb{C}_l[\mathcal{R}] = \mathbb{C}[\mathcal{R}]/[V_{l+1}]\mathbb{C}[\mathcal{R}].$$

Denoting by $[V]_l$ the image of $[V]$ in $\mathbb{C}_l[\mathcal{R}]$, we have $\mathbb{C}_l[\mathcal{R}] = \mathbb{C}[V_0]_l \oplus \cdots \oplus \mathbb{C}[V_l]_l$, and

$$[V_i \otimes V_j]_l = \sum_{k=j-i}^{\min(i+j,2l-i-j)} [V_k]_l, \qquad \text{for } i \leq j \leq l.$$

We also set

$$V_i \otimes_l V_j = \bigoplus_{k=j-i}^{\min(i+j,2l-i-j)} V_k, \qquad \text{for } i \leq j \leq l.$$

Again, in the above sums, k increases by twos. The ring $\mathbb{C}_l[\mathcal{R}]$ appears in conformal field theory (as the Grothendieck ring of the modular category of integrable $\widehat{\mathfrak{sl}}_2$-modules of level l) and in quantum group theory (as the Grothendieck ring of a suitable quotient of the category of tilting modules over $U_\epsilon(\mathfrak{sl}_2)$ when ϵ is an $(l+2)$th root of unity).

2.2 Lagrangian construction of $U(\mathfrak{sl}_2)$.

We briefly recall Ginzburg's construction of irreducible representations of $\mathfrak{sl}_2$ in the homology of certain varieties associated to partial flag varieties (cf. [7]). We use the (in this case equivalent) language of quiver varieties (cf. [23]).

Let $v, w \in \mathbb{N}$ and let V and W be $\mathbb{C}$-vector spaces of dimensions v and w respectively. Consider the space

$$M(v,w) = \{(i,j) \mid ij = 0;\ \ker j = \{0\}\} \subset \text{Hom}\,(W,V) \oplus \text{Hom}\,(V,W).$$

We let $GL(V)$ act on $M(v,w)$ via $g \cdot (i,j) = (gi, jg^{-1})$. This action is free and we set $\mathcal{M}(v,w) = M(v,w)/GL(V)$. The assignment $(i,j) \mapsto (ji, \text{Im } j)$ defines an isomorphism between $\mathcal{M}(v,w)$ and the variety

$$\mathcal{F}_{v,w} = \{(t,V_0) \mid V_0 \subset W, \dim V_0 = v, \text{Im } t \subset V_0 \subset \ker t\} \subset \mathcal{N}_W \times \text{Gr}(v,w),$$

where $\mathcal{N}_W$ is the nullcone of $\mathfrak{gl}(W)$ and $\mathrm{Gr}(v,w)$ is the Grassmannian of v-dimensional subspaces in W. We will denote by $\pi : \mathcal{M}(v,w) \to \mathcal{N}_W$, the projection $(i,j) \mapsto ji$. For any $t \in \mathcal{N}_W$ such that $t^2 = 0$ we set $\mathcal{M}(v,w)_t = \pi^{-1}(t)$ and $\mathcal{M}(w)_t = \sqcup_v \mathcal{M}(v,w)_t$. In particular, we set $\mathcal{L}(v,w) = \pi^{-1}(0)$. Observe that $\mathcal{L}(v,w)$ is just $\mathrm{Gr}(v,w)$ and that $\mathcal{M}(v,w)$ is isomorphic to the cotangent bundle of $\mathcal{L}(v,w)$. We have $\dim \mathcal{M}(v,w) = 2\dim \mathcal{L}(v,w) = 2v(w-v)$. For $v_1, v_2, w \in \mathbb{N}$ we also consider the variety of triples

$$Z(v_1,v_2,w) = \{((i_1,j_1),(i_2,j_2)) \mid j_1i_1 = j_2i_2\} \subset \mathcal{M}(v_1,w) \times \mathcal{M}(v_2,w).$$

Then $\dim Z(v_1,v_2,w) = v_1(w-v_1) + v_2(w-v_2)$.

The form $\omega((i,j),(i',j')) = \mathrm{Tr}_V(ij' - i'j)$ defines a symplectic structure on $\mathcal{M}(v,w)$, for which the variety $\mathcal{L}(v,w)$ is Lagrangian. Equip $\mathcal{M}(v_1,w) \times \mathcal{M}(v_2,w)$ with the symplectic form $\omega \times (-\omega)$. Then $Z(v_1,v_2,w)$ is also Lagrangian. Let $Z(w) = \sqcup_{v_1,v_2} Z(v_1,v_2,w)$.

For any complex algebraic variety X we let $H_*(X)$ be the Borel-Moore homology with coefficients in $\mathbb{C}$, and set $H_{top}(X) = H_{2d}(X)$ where $d = \dim X$.

Let $p_{ij} : \mathcal{M}(v_1,w) \times \mathcal{M}(v_2,w) \times \mathcal{M}(v_3,w) \to \mathcal{M}(v_i,w) \times \mathcal{M}(v_j,w)$ be the obvious projections. The map

$$p_{13} : \ p_{12}^{-1}(Z(v_1,v_2,w)) \cap p_{23}^{-1}(Z(v_2,v_3,w)) \to Z(v_1,v_3,w)$$

is proper and we can define the convolution product

$$\begin{aligned} H_i(Z(v_1,v_2,w)) \otimes H_j(Z(v_2,v_3,w)) &\to H_{i+j-d_2}(Z(v_1,v_3,w)) \\ c \otimes c' &\mapsto p_{13*}(p_{12}^*(c) \cap p_{23}^*(c')) \end{aligned}$$

where $d_2 = 4v_2(w-v_2)$. In particular, this gives rise to an algebra structure on $H_{top}(Z(w)) = \bigoplus_{v_1,v_2} H_{top}(Z(v_1,v_2,w))$.

Now let $t \in \mathcal{N}_W$ such that $t^2 = 0$. The projection

$$p_1 : \ Z(v_1,v_2,w) \cap p_2^{-1}(\mathcal{M}(v_2,w)_t) \to \mathcal{M}(v_1,w)_t$$

(where p_1 and p_2 are the obvious projections) is proper and the convolution action

$$H_{top}(Z(v_1, v_2, w)) \otimes H_{top}(\mathcal{M}(v_2, w)_t) \to H_{top}(\mathcal{M}(v_1, w)_t)$$
$$c \otimes c' \mapsto p_{1*}(c \cap p_2^*(c'))$$

makes $H_{top}(\mathcal{M}(w)_t) = \bigoplus_v H_{top}(\mathcal{M}(v, w)_t)$ into a $H_{top}(Z(w))$-module.

Theorem 2.2.A ([7]). *There is a natural surjective homomorphism $\Phi : U(\mathfrak{sl}_2) \to H_{top}(Z(w))$. Under Φ, the module $H_{top}(\mathcal{M}(w)_t)$ is isomorphic to V_{w-2u} where $u = \operatorname{rank} t$.*

We now give the realization of tensor products of $U(\mathfrak{sl}_2)$-modules. Let $w = w_1 + \cdots + w_r$ and fix $W = W_1 \oplus \cdots \oplus W_r$ with dim $W_i = w_i$. Let $W_0 = 0$. The group $GL(W)$ acts on $\mathcal{M}(v, w)$ by $g \cdot (i, j) = (ig^{-1}, gj)$. Consider the embedding

$$\sigma : (\mathbb{C}^*)^{r-1} \to \prod_{i=1}^{r} GL(W_i) \subset GL(W)$$
$$(t_2, t_3, \ldots, t_r) \mapsto (Id, t_2^{-1}, t_2^{-1}t_3^{-1}, \ldots, t_2^{-1} \cdots t_r^{-1})$$

Then, for each v, we have (see e.g [25, Lemma 3.2])

$$\mathcal{M}(v, w)^\sigma = \bigsqcup_{v_1+\cdots+v_r=v} \mathcal{M}(v_1, w_1) \times \cdots \times \mathcal{M}(v_r, w_r).$$

Consider the subvarieties

$$\mathcal{M}(v, w_1, \ldots, w_r) = \{x \in \mathcal{M}(v, w) \mid \lim_{t_i \to 0} \sigma(t_2, \ldots, t_r) \cdot x \text{ exists}\}$$

$$\mathcal{N}_W(w_1, \ldots, w_r) = \{t \in \mathcal{N}_W \mid \lim_{t_i \to 0} \sigma(t_2, \ldots, t_r) \cdot t \text{ exists}\}.$$

For $x \in \mathcal{M}(v, w_1, \ldots, w_r)$, let us set $\tau(x) = \lim_{t_i \to 0} \sigma(t_2, \ldots, t_r) \cdot x$. We define $\tau(t)$ similarly for $t \in \mathcal{N}_W(w_1, \ldots, w_r)$. Now consider

$$\mathcal{L}(v, w_1, \ldots, w_r) = \{x \in \mathcal{M}(v, w_1, \ldots, w_r) \mid \tau(x) \in \prod_i \mathcal{L}(v_i, w_i) \text{ for some } (v_i)\}.$$

Set $\mathcal{L}(w_1, \ldots, w_r) = \sqcup_v \mathcal{L}(v, w_1, \ldots, w_r)$. Note that $\mathcal{L}(w_1, \ldots, w_r) = \pi^{-1}(\tau^{-1}(0))$ so that we have an action of $H_{top}(Z(w))$ on $H_{top}(\mathcal{L}(w_1, \ldots, w_r))$. Moreover, it is easy to check that $\mathcal{L}(w_1,\dots,w_r)$ is Lagrangian. Note that $\mathcal{L}(w_1,\dots,w_r)$ is isomorphic to the variety

$$\{(t,V_0) \mid V_0 \subset W,\ \operatorname{im} t \subset V_0 \subset \ker t,\ t(W_j) \subset W_0 \oplus \cdots \oplus W_{j-1},\ 1 \le j \le r\}.$$

Theorem 2.2.B ([8], [25], [20]). *$H_{top}(\mathcal{L}(w_1,\dots,w_r))$ is isomorphic to $V_{w_1} \otimes \cdots \otimes V_{w_r}$ as a $U(\mathfrak{sl}_2)$-module.*

2.3 Lagrangian construction of the fusion product

Let us fix some positive integer l. We will now describe an open subvariety of $\mathcal{L}(w_1,\dots,w_r)$ whose homology realizes the fusion product $V_{w_1} \otimes_l \cdots \otimes_l V_{w_r}$.

We keep the notation of the previous section For all $k \in \mathbb{N}$ and $t \in \mathcal{N}_{W_1\oplus\cdots\oplus W_k}(w_1,\dots,w_k)$ we set $\tau_k(t) = \lim_{t_k \to 0} \sigma(1,\dots,1,t_k)(t)$. Let us consider the open subvariety $\mathcal{N}^l(w_1,w_2) = \{t \in \mathcal{N}_{W_1\oplus W_2} \mid \dim \ker t \le l\}$ of $\mathcal{N}_{W_1\oplus W_2}$ and define inductively

$$\begin{aligned} \mathcal{N}^l(w_1,\dots,w_k) = \{t \in \mathcal{N}_{W_1\oplus\cdots\oplus W_k} \mid \dim \ker t \le l + \operatorname{rank} \tau_k(t), \\ t|_{W_1\oplus\cdots\oplus W_{k-1}} \in \mathcal{N}^l(w_1,\dots,w_{k-1})\} \end{aligned} \tag{2.1}$$

for $k \ge 3$. Finally, set $\mathcal{L}_l(w_1,\dots,w_r) = \mathcal{L}(w_1,\dots,w_r) \cap \pi^{-1}(\mathcal{N}^l(w_1,\dots,w_r))$. By definition $\mathcal{L}_l(w_1,\dots,w_r)$ is an open subvariety of $\mathcal{L}(w_1,\dots,w_r)$ and therefore $H_{top}(\mathcal{L}_l(w_1,\dots,w_r))$ is a $H_{top}(Z(w))$-module.

Theorem 2.3.A. *$H_{top}(\mathcal{L}_l(w_1,\dots,w_r))$ is isomorphic to $V_{w_1} \otimes_l \cdots \otimes_l V_{w_r}$ as a $U(\mathfrak{sl}_2)$-module.*

Proof. We proceed by induction. Suppose $r = 2$. It is enough to describe the irreducible components of $\mathcal{L}_l(w_1,w_2)$ corresponding to highest weight vectors in the $U(\mathfrak{sl}_2)$-module $H_{top}(\mathcal{L}_l(w_1,w_2))$. The irreducible components of $\mathcal{L}(w_1,w_2)$ corresponding to highest-weight vectors are the

$$I_v = \{(i,j) \mid j(V) \subset W_1,\ i(W_2) = V,\ i(W_1) = 0\}, \qquad \text{for } 0 \le v \le w_1, w_2$$

and the associated highest weight is w_1+w_2-2v. Note that the condition $\dim\ker ji \leq l$ is equivalent to the condition $w_1+w_2-2v \leq 2l-w_1-w_2$. Now suppose that the theorem is proved for tensor products of $r-1$ modules, and let us set $W' = W_1 \oplus \cdots \oplus W_{r-1}$. For each $u \in \mathbb{N}$ let us set $\mathcal{N}_{W'}(u) = \{t \in \mathcal{N}_{W'} \mid \operatorname{rank} t = u\}$. Recall that $\mathcal{L}_l(w_1,\ldots,w_{r-1})$ is Lagrangian and that π is semi-small with all strata being relevant (c.f [23, §10]). Thus $\pi(\mathcal{L}_l(w_1,\ldots,w_{r-1})) \cap \mathcal{N}_{W'}(u)$ is a subvariety of $\mathcal{N}_{W'}(u)$ of dimension $\frac{1}{2}\dim \mathcal{N}_{W'}(u)$. Let $\mathcal{C}_1^u,\ldots,\mathcal{C}_{s(u)}^u$ be its irreducible components. By the induction hypothesis,

$$s(u) = \dim \operatorname{Hom}_{\mathfrak{sl}_2}(V_{w'-2u}, V_{w_1} \otimes_l \cdots \otimes_l V_{w_{r-1}}). \tag{2.2}$$

The irreducible components of $\mathcal{L}_l(v,w_1,\ldots,w_r)$ corresponding to highest weight vectors of $H_{top}(\mathcal{L}_l(w_1,\ldots,w_r))$ are of the form $\overline{I_\chi}$ with

$$I_\chi = \{(i,j) \mid i(W) = V,\ j(V) \subset W',\ (i_{W'}, j) \in \chi\}$$

where χ is an irreducible component of $\mathcal{L}_l(v,w_1,\ldots,w_{r-1})$, and the associated highest weight is $w-2v$ (note that I_χ may be empty). Let us fix $u \in \mathbb{N}$ and $\mathcal{C}_k^u$ for some $k \leq s(u)$. Let $\chi \subset \overline{\pi^{-1}(\mathcal{C}_k^u)} \cap \mathcal{L}_l(v,w_1,\ldots,w_{r-1})$ be an irreducible component. Then $I_\chi \subset \overline{\mathcal{L}_l(w_1,\ldots,w_r)}$ if for all (i,j) in (an open dense subset of) I_χ we have $\dim \operatorname{Im} ji \leq l+u$. This is equivalent to the condition that the corresponding highest weight $w-2v$ satisfies

$$w-2v \leq 2l - w_r - (w'-2u). \tag{2.3}$$

Equations (2.2) and (2.3) together imply that

$$H_{top}(\mathcal{L}_l(w_1,\ldots,w_r)) \simeq (V_{w_1} \otimes_l \cdots \otimes_l V_{w_{r-1}}) \otimes_l V_{w_r}$$

as a $U(\mathfrak{sl}_2)$-module, as desired. □

Remark 2.3.B. i) The above construction is not canonical in the sense that it was made using a choice of a bracketing of the tensor product, namely

$$(\cdots((V_{w_1} \otimes_l V_{w_2}) \otimes_l V_{w_3}) \cdots \otimes_l V_{w_r}).$$

Different bracketings give rise to different (possibly non-isomorphic) open subvarieties of $\mathcal{L}_l(w_1, \ldots, w_r)$ realizing the same fusion tensor product.

ii) One might be tempted to define in an analogous fashion a truncated tensor product for finite-dimensional representations of $U_q(\widehat{\mathfrak{sl}}_2)$ by considering equivariant K-theory of $\mathcal{L}_l(w_1, w_2)$ rather than Borel-Moore homology. However, it is easy to check that (because of Remark i)) the resulting product is not associative.

2.4 A graphical calculus for the fusion product

We first recall some results from the previous chapter on the graphical calculus of tensor products and intertwining operators. In the graphical calculus, V_d is depicted by a box marked d with d vertices. To depict the set $CM^{\mu}_{w_1,\ldots,w_r}$ of crossingless matches, we place the boxes representing the V_{w_i} on a horizontal line and the box representing V_μ on another horizontal line lying above the first one. $CM^{\mu}_{w_1,\ldots,w_r}$ is then the set of non-intersecting curves (up to isotopy) connecting the vertices of the boxes such that the following conditions are satisfied:

1. Each curve connects exactly two vertices.

2. Each vertex is the end point of exactly one curve.

3. No curve joins a box to itself.

4. The curves lie inside the box bounded by the two horizontal lines and the vertical lines through the extreme right and left points.

We call the curves joining two lower boxes *lower curves* and those joining a lower and an upper box *middle curves*. We define the set of oriented crossingless matches $OCM^{\mu}_{w_1,\ldots,w_r}$ to be the set of elements of $CM^{\mu}_{w_1,\ldots,w_r}$ along with an orientation of the curves such that all lower curves are oriented to the left and all middle curves are oriented so that those oriented down are to the right of those oriented up.

As shown in [5], the set of crossingless matches $CM^{\mu}_{w_1,\dots,w_r}$ is in one to one correspondence with a basis of the set of intertwining operators

$$H^{\mu}_{w_1,\dots,w_r} \stackrel{\text{def}}{=} \operatorname{Hom}(V_{w_1} \otimes \cdots \otimes V_{w_r}, V_{\mu}).$$

The matrix coefficients of the intertwining operator associated to a particular crossingless match are given by Theorem 2.1 of [5].

We will also need to define the set of *lower crossingless matches* $LCM^{\mu}_{w_1,\dots,w_r}$ and *oriented lower crossingless matches* $OLCM^{\mu}_{w_1,\dots,w_r}$. Elements of $LCM^{\mu}_{w_1,\dots,w_r}$ and $OLCM^{\mu}_{w_1,\dots,w_r}$ are obtained from elements of $CM^{\mu}_{w_1,\dots,w_r}$ and $OCM^{\mu}_{w_1,\dots,w_r}$ (respectively) by removing the upper box (thus converting lower end points of middle curves to unmatched vertices). For the case of $OLCM^{\mu}_{w_1,\dots,w_r}$, unmatched vertices will still have an orientation (indicated by an arrow attached to the vertex). As for middle curves in the case of $OCM^{\mu}_{w_1,\dots,w_r}$, the unmatched vertices in an element of $OLCM^{\mu}_{w_1,\dots,w_r}$ must be arranged so that those oriented down are to the right of those oriented up.

Note that the set of lower crossingless matches $LCM = LCM_{w_1,\dots,w_r}$ is in one to one correspondence with the set $\bigcup_{\mu} CM^{\mu}_{w_1,\dots,w_r}$. From now on, we will identify these two sets.

Let s be a bracketing of the tensor product $V_{w_1} \otimes \cdots \otimes V_{w_r}$. Pick an ordering of the tensor operations compatible with this bracketing. For each n such that $1 \leq n \leq r-1$, let S_n be the set of the V_{w_i} separated from the n^{th} tensor product operation only by operations ranked lower than or equal to n. Then let ${}^{l}_{s}CM^{\mu}_{w_1,\dots,w_r}$ be the set of elements of $CM^{\mu}_{w_1,\dots,w_r}$ satisfying the following condition: for each n, the number of curves connecting V_{w_i}'s in S_n to either V_{w_i}'s in S_n on the other side of the n^{th} tensor product symbol or V_w's not in S_n is less than or equal to l. Note that this condition does not depend on the particular ordering so long as it is compatible with the bracketing s.

Let ${}^l_sLCM = {}^l_sLCM_{w_1,\ldots,w_r}$ be the set of lower crossingless matches satisfying the same condition (where unmatched vertices are always counted as curves with the other end point outside of any S_n) and identify this set with the set $\bigcup_\mu {}^l_sCM^\mu_{w_1,\ldots,w_r}$. We define ${}^l_sOCM^\mu_{w_1,\ldots,w_r}$ and ${}^l_sOLCM = {}^l_sOLCM_{w_1,\ldots,w_r}$ similarly (and the corresponding identification is made).

Note that in the case $r = 2$ the condition in the definition simplifies to the requirement that the total number of curves (including middle curves) is less than or equal to l. In fact, the given definition simply arises from applying this condition to each tensor product operation (in the given ordering), neglecting curves with both end points in V_{w_i}'s which have already been tensored together.

Proposition 2.4.A. *The set* ${}^l_sCM^\mu_{w_1,\ldots,w_r}$ *is in one to one correspondence with a basis of the space of intertwining operators* ${}^lH^\mu_{w_1,\ldots,w_r} \stackrel{def}{=} \mathrm{Hom}(V_{w_1} \otimes_l \cdots \otimes_l V_{w_r}, V_\mu)$.

Proof. We first consider the case $r = 2$. For any $b \in CM^\mu_{w_1,w_2}$, the total number of curves is equal to $(w_1 + w_2 + \mu)/2$ (since each vertex is an end point of exactly one curve). Thus the condition that the total number of curves is less than or equal to l reduces to $w_1 + w_2 + \mu \leq 2l$ or $\mu \leq 2l - w_1 - w_2$ as desired.

Now assume the result holds for the product of less than r irreducible modules and that for the product of V_{w_1} through V_{w_r}, the r^{th} tensor product operation is the one occurring between V_{w_k} and $V_{w_{k+1}}$ $(k < r)$. Note that

$$\bigoplus_\nu {}^lH^\nu_{w_1,\ldots,w_k} \otimes {}^lH^\mu_{\nu,w_{k+1},\ldots,w_r} \cong {}^lH^\mu_{w_1,\ldots,w_r}$$

via the map $f \otimes g \mapsto g(f \otimes \mathrm{id}_{V_{w_{k+1}} \oplus \cdots \oplus V_{w_r}})$. Now, if s_1 is the bracketing of the first k modules and s_2 is the bracketing of the last $r - k$ modules, it is easy to see that

$$\sum_\nu {}^l_{s_1}CM^\nu_{w_1,\ldots,w_k} \times {}^l_{s_2}CM^\mu_{\nu,w_{k+1},\ldots,w_r} \cong {}^l_sCM^\mu_{w_1,\ldots,w_r} \text{ (as sets).}$$

The result now follows by induction. ∎

From the associativity of the fusion tensor product it follows immediately that the order of the set ${}^{l}_{s}CM^{\mu}_{w_1,\ldots,w_r}$ is independent of the bracketing s. However, we will present here a direct proof.

Proposition 2.4.B. *The order of the set* ${}^{l}_{s}CM^{\mu}_{w_1,\ldots,w_r}$ *is independent of the bracketing* s.

Proof. It suffices to prove the statement for three factors. Let s_1 be the bracketing $(V_{w_1} \otimes V_{w_2}) \otimes V_{w_3}$ and s_2 be the bracketing $V_{w_1} \otimes (V_{w_2} \otimes V_{w_3})$. We will set up a one to one correspondence between ${}^{l}_{s_1}CM^{\mu}_{w_1,\ldots,w_r}$ and ${}^{l}_{s_2}CM^{\mu}_{w_1,\ldots,w_r}$. We will first establish a one to one correspondence between the subsets consisting of those crossingless matches with no curves connecting V_{w_1} and V_{w_3} and a fixed number n of lower curves. Let a (resp. b) denote the number of curves connecting V_{w_1} (resp. V_{w_3}) to V_{w_2}. Thus $a + b = n$. Now, the number of curves with at least one end point in V_{w_1} or V_{w_2} is $w_1 + w_2 - a$ and the total number of curves minus the curves connecting V_{w_1} to V_{w_2} is $w_1 + w_2 + w_3 - n - a$. Thus a crossingless match lies in ${}^{l}_{s_1}CM^{\mu}_{w_1,\ldots,w_r}$ if and only if

$$w_1 + w_2 - a \leq l, \; w_1 + w_2 + w_3 - n - a \leq l.$$

Similarly, a crossingless match lies in ${}^{l}_{s_2}CM^{\mu}_{w_1,\ldots,w_r}$ if and only if

$$w_2 + w_3 - b \leq l, \; w_1 + w_2 + w_3 - n - b \leq l.$$

Now, the largest possible value of a is $\min(w_1, n)$ and the largest possible value of b is $\min(w_3, n)$. Therefore, by counting the possible values of a, the number of crossingless matches in ${}^{l}_{s_1}CM^{\mu}_{w_1,\ldots,w_r}$ with no curves connecting V_{w_1} and V_{w_3} and with n total curves is equal to

$$r_a = \min(w_1, n) - \max(w_1 + w_2 - l, w_1 + w_2 + w_3 - n - l) + 1$$

if this number is positive and zero otherwise. Similarly, the number of crossingless matches in ${}^{l}_{s_2}CM^{\mu}_{w_1,\ldots,w_r}$ with no curves connecting V_{w_1} and V_{w_3} and with n total curves is equal to

$$r_b = \min(w_3, n) - \max(w_2 + w_3 - l, w_1 + w_2 + w_3 - n - l) + 1$$

if this number is positive and zero otherwise. Considering the four cases $n \leq w_1, w_3$; $n \geq w_1, w_3$; $w_1 \leq n \leq w_3$ and $w_3 \leq n \leq w_1$ we easily see that $r_a = r_b$ in all cases.

It remains to establish a one to one correspondence between the elements of ${}^{l}_{s_1}CM^{\mu}_{w_1,\ldots,w_r}$ and ${}^{l}_{s_2}CM^{\mu}_{w_1,\ldots,w_r}$ with $c \geq 1$ curves joining V_{w_1} and V_{w_3}. Fix the number of lower curves with one end point in V_{w_2} to be n. Since V_{w_1} and V_{w_3} are connected, there can be no middles curves with end points in V_{w_2}. Thus $s = w_2$. Define a and b as above. By an argument analogous to that given in the earlier case, the number of crossingless matches in ${}^{l}_{s_1}CM^{\mu}_{w_1,\ldots,w_r}$ with $c \geq 1$ curves connecting V_{w_1} to V_{w_3} and with n lower curves with one end point in V_{w_2} is equal to

$$r_a = \min(w_1 - c, w_2) - \max(w_1 + w_2 - l, w_1 + w_3 - l - c) + 1$$

if this number is positive and zero otherwise. Similarly, the number of crossingless matches in ${}^{l}_{s_2}CM^{\mu}_{w_1,\ldots,w_r}$ with $c \geq 1$ curves connecting V_{w_1} to V_{w_3} and with n lower curves with one end point in V_{w_2} is equal to

$$r_b = \min(w_3 - c, w_2) - \max(w_2 + w_3 - l, w_1 + w_3 - l - c) + 1$$

if this number is positive and zero otherwise. Considering the four cases $w_2 \leq w_1 - c, w_3 - c$; $w_2 \geq w_1 - c, w_3 - c$; $w_1 - c \leq w_2 \leq w_3 - c$ and $w_3 - c \leq w_2 \leq w_1 - c$ we easily see that $r_a = r_b$ in all cases. This concludes the proof. $\square$

From now on, we will use the bracketing $(\cdots((V_{w_1} \otimes V_{w_2}) \otimes V_{w_3})\cdots V_{w_r})$ unless explicitly stated otherwise. Thus, if we omit a subscript s, we take s to be this bracketing.

2.5 The fusion product via constructible functions

Fix a $w = w_1 + \cdots + w_r$ dimensional $\mathbb{C}$-vector space W and recall

$$\mathfrak{T}(w_1, \ldots, w_r) = \{(\mathbf{D} = \{D_i\}_{i=0}^r, V_0, t) \mid 0 = D_0 \subset D_1 \subset \cdots \subset D_r = W,\ V_0 \subset W,$$
$$t \in \operatorname{End} W,\ t(D_i) \in D_{i-1}, \dim(D_i/D_{i-1}) = w_i, \operatorname{im} t \subset V_0 \subset \ker t\}.$$

Consider the projection

$$\mathfrak{T}(w_1, \ldots, w_r) \to \{\mathbf{D} = \{D_i\}_{i=0}^r \mid 0 = D_0 \subset D_1 \subset \cdots \subset D_r = W,$$
$$\dim(D_i/D_{i-1}) = w_i\}$$

given by $(\mathbf{D}, V_0, t) \mapsto \mathbf{D}$. It is easy to see that the fibers of this map are all isomorphic and that one could replace the tensor product variety $\mathfrak{T}(w_1, \ldots, w_r)$ by this fiber, restrict the constructible functions to this fiber and the theory would remain unchanged. Let $\mathfrak{T}_{\mathbf{D}}(w_1, \ldots, w_r)$ denote the fiber over a flag $\mathbf{D}$. If we define

$$D_i = W_0 \oplus \cdots \oplus W_i,\ 0 \le i \le r,$$

then obviously

$$\mathfrak{T}_{\mathbf{D}}(w_1, \ldots, w_r) \cong \mathcal{L}(w_1, \ldots, w_r)$$

and in the sequel we will identify these two varieties.

If $b \in CM^{\mu}_{w_1,\ldots,w_r}$ is an unoriented crossingless match, let

$$Y_b = \{(\mathbf{D}, V_0, t) \in \mathfrak{T}(w_1, \ldots, w_r) \mid \dim(\ker t \cap D_i)/(\ker t \cap D_{i-1}) = b_i\}$$

where b_i is the number of left end points (of lower curves) and lower end points (of middle curves) contained in the box representing V_{w_i}. We know from Proposition 1.3.B that $\sqcup_b Y_b = \mathfrak{T}(w_1, \ldots, w_r)$ and that the closures of the Y_b are precisely the irreducible components of $\mathfrak{T}(w_1, \ldots, w_r)$. Let $X_b = Y_b \cap \mathcal{L}(w_1, \ldots, w_r)$. Then obviously $\mathcal{L}(w_1, \ldots, w_r) = \sqcup_{b \in LCM} X_b$.

Proposition 2.5.A. $\mathcal{L}_l(w_1, \ldots, w_r) = \sqcup_{b \in {}^l LCM} X_b$.

Proof. We see from equation (2.1) that $\mathcal{L}_l(w_1, \dots, w_r)$ is the set of all $(t, V_0) \in \mathcal{L}(w_1, \dots, w_r)$ such that

$$\dim \ker t|_{W_1 \oplus \cdots \oplus W_i} \le l + \operatorname{rank} t|_{W_1 \oplus \cdots \oplus W_{i-1}} \ \forall\, 1 \le i \le r.$$

Now, by the definition of the X_b, if $(t, V_0) \in X_b$ for some $b \in LCM$ then $\dim \ker t|_{W_1 \oplus \cdots \oplus W_i}$ is equal to $\sum_{j=1}^{i} w_j$ minus the number of lower curves with both end points among the lower i boxes. Also, $\operatorname{rank} t|_{W_1 \oplus \cdots \oplus W_{i-1}}$ is equal to the number of lower curves with both end points among the lower $i-1$ boxes. Let c_i denote the number of curves with both end points among the lower i boxes. Then

$$\begin{aligned}
\dim \ker t|_{W_1 \oplus \cdots \oplus W_i} &\le l + \operatorname{rank} t|_{W_1 \oplus \cdots \oplus W_{i-1}} \\
\Leftrightarrow \sum_{j=1}^{i} w_j - c_i &\le l + c_{i-1} \\
\Leftrightarrow \sum_{j=1}^{i} w_j - 2c_{i-1} - \#\{\text{curves with right end point in } i^{th} \text{ box}\} &\le l \\
\Leftrightarrow \sum_{i=1}^{n} w_i - \#\{\text{end points in first } i-1 \text{ boxes of lower curves with both} \\
\text{end points in first } i \text{ boxes}\} &\le l
\end{aligned}$$

and this is easily seen to be equivalent to the condition that $b \in {}^{l}LCM$ (with the default bracketing). ☐

We will now define a $U(\mathfrak{sl}_2)$-module structure on a certain space of constructible functions on $\mathcal{L}_l(w_1, \dots, w_r)$. For $\mathbf{a} \in OLCM_{w_1,\dots,w_r}$, let $\bar{\mathbf{a}}$ be the associated element of $LCM_{w_1,\dots,w_r}$ obtained by forgetting the orientation. Define

$$Y_{\mathbf{a}} = \{(\mathbf{D}, V_0, t) \in Y_{\bar{\mathbf{a}}} \mid \dim W = \#\{\text{up-oriented vertices of } \mathbf{a}\}\}$$

where the right end points of lower curves are oriented up (as well as the up-oriented unmatched vertices). Let $X_{\mathbf{a}} = Y_{\mathbf{a}} \cap \mathcal{L}(w_1, \dots, w_r)$. Then it follows from (1.33) that

$$X_b = \bigcup_{\mathbf{a} : \bar{\mathbf{a}} = b} X_{\mathbf{a}}.$$

Now let

$$\mathcal{B}_s^l = \{\mathbf{1}_{Y_\mathbf{a}} \mid \mathbf{a} \in {}^l OLCM\}$$

where $\mathbf{1}_A$ is the function that is equal to one on the set A and zero elsewhere. Let

$$\mathcal{T}^l = \mathcal{T}_s^l(w_1, \dots, w_r) = \operatorname{Span} \mathcal{B}_s^l.$$

We endow $\mathcal{T}^l$ with the structure of a $U(\mathfrak{sl}_2)$-module in the usual manner.

Theorem 2.5.B. *$\mathcal{T}_s^l(w_1, \dots, w_r)$ is isomorphic as a $U(\mathfrak{sl}_2)$-module to $V_{w_1} \otimes_l \cdots \otimes_l V_{w_r}$ and $\mathcal{B}_s^l$ is a basis for $\mathcal{T}_s^l(w_1, \dots, w_r)$ adapted to its decomposition into a direct sum of irreducible representations. That is, for a given $b \in {}^l CM^\mu_{w_1,\dots,w_r}$, the space* $\operatorname{Span}\{\mathbf{1}_{Y_\mathbf{a}} \mid \bar{\mathbf{a}} = b\}$ *is isomorphic to the irreducible representation V_μ via the map*

$$\mathbf{1}_{Y_\mathbf{a}} \mapsto {}^\mu v_{\mu - 2\#\{\textit{unmatched down-oriented vertices of } \mathbf{a}\}}.$$

Proof. The second part of the theorem follows from Theorem 1.3.D. Then

$$\begin{aligned}
\mathcal{T}^l &= \bigoplus_\mu \bigoplus_{b \in {}^l CM^\mu_{w_1,\dots,w_r}} \operatorname{Span}\{\mathbf{1}_{Y_\mathbf{a}} \mid \bar{\mathbf{a}} = b\} \\
&\cong \bigoplus_\mu \bigoplus_{b \in {}^l CM^\mu_{w_1,\dots,w_r}} V_\mu \\
&\cong \bigoplus_\mu {}^l H^\mu_{w_1,\dots,w_r} \otimes V_\mu \\
&\cong V_{w_1} \otimes_l \cdots \otimes_l V_{w_r}
\end{aligned}$$

where ${}^l H^\mu_{w_1,\dots,w_r}$ is given the trivial module structure. □

Remark 2.5.C. We have used here the standard bracketing $(\cdots(V_{w_1} \otimes_l V_{w_2}) \otimes_l V_{w_3}) \cdots \otimes_l V_{w_r})$. However, one could easily modify the definitions to use any other bracketing. The proofs would need only slight changes. Of course, as noted above, while we would still recover the structure of the fusion product, the varieties involved would be non-isomorphic in general.

Chapter 3

Bases of Type A Affine Lie Algebras

Introduction

A remarkable relation between representation theory of affine Lie algebras and models of statistical mechanics based on the Yang-Baxter equation has been discovered and intensively studied by E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado (see [3, 4] and references therein). One of the important findings of the above authors is that the one-dimensional configuration sums for these models give rise to characters of integrable highest weight representations of affine Lie algebras. This relation yields certain explicit bases in the representations that admit pure combinatorial descriptions and imply various identities for the characters.

Another astonishing relation between representation theory of affine Lie algebras and moduli spaces of solutions of self-dual Yang-Mills equations has been accomplished by H. Nakajima [21, 23], who observed a profound link between his earlier work with P. Kronheimer and the results of G. Lusztig [15, 16]. At the heart of both works that preceded the Nakajima discovery are quiver varieties associated with extended Dynkin diagrams. Nakajima introduced a special class of quiver varieties associated with integrable highest weight representations of affine Lie algebras and obtained a geometric description of the action. He also defined certain Lagrangian subvarieties whose irreducible components yield a geometric basis of the affine Lie algebra representations.

The central goal of the present chapter is to relate the two apparently different bases in the representations of affine Lie algebras of type A: one arising from statistical mechanics, the other from gauge theory. We show that the two are governed by the same combinatorics which also respects the weight space decomposition of the representations. This identification allows one to give a natural conceptual framework to the intricate structure of statistical mechanical models and also to make explicit calculations in a seemingly intractable geometric setting. In particular, we are able to give an alternative and much simpler geometric proof of the main result of [4] on the construction of a basis of affine Lie algebra representations. At the same time, we give a simple parametrization of the irreducible components of Nakajima quiver varieties associated to infinite and cyclic quivers.

The comparison of the two very different theories brings some surprises and suggests interesting new directions. In particular, the Young diagrams that are routinely used in representation theory of type A Lie algebras acquire an explicit geometric meaning: They picture precisely representations of the corresponding quivers satisfying a stability condition for level 1 (see Figure 3.2 in the text). On the other hand, the algebraic constructions of [4] involve substantially the highest weight representations of $\widehat{\mathfrak{gl}}_{n+1}$, which are not directly covered by Nakajima's theory. We define new varieties by relaxing the nilpotency condition in the definition of Nakajima's quiver varieties and show that the irreducible components of these new varieties are in one to one correspondence with bases of highest weight representations of $\widehat{\mathfrak{gl}}_{n+1}$. We also mention some interesting problems that arise as a result of the comparison of geometric and algebraic constructions.

We strongly believe that the main results of the current chapter reflect a very general principle that asserts the profound geometric or gauge theoretic origin of various algebraic and combinatorial structures of integrable models in statistical mechanics. The relation of both subjects to the representation theory of affine Lie algebras is a necessary prerequisite of this principle. However we expect much more; namely that various specific constructions appearing in integrable models of statistical mechanics that include tensor products, fusion products, branching rules, Bethe's ansatz and the Yang-Baxter equation itself reflect certain geometric facts about Nakajima varieties, Malkin-Nakajima tensor product varieties, various Lagrangian subvarieties and corresponding gauge theories on commutative and, possibly, noncommutative spaces. The present chapter is a small but indicative step toward this vast program.

The chapter is organized as follows. In Section 3.1 we recall the definition of Lusztig's quiver varieties and characterizations of the irreducible components in types A_∞ and $A_n^{(1)}$. We also introduce a version of Lusztig's quiver varieties for the Lie algebra $\widehat{\mathfrak{gl}}_{n+1}$. Section 3.2 contains the definition of Nakajima's quiver varieties and the Lie algebra action on a suitable space of constructible functions on these varieties is given in Section 3.3. In Section 3.4 we give an enumeration of the irreducible components of the quiver varieties for level 1 in terms of Young diagrams. We also identify the geometric action of the type A_∞ Lie algebra in the basis enumerated by Young diagrams. In Section 3.5 we extend the enumeration of the irreducible components of the quiver varieties to arbitrary level and we establish a match with the indexing of bases of the corresponding representations coming from statistical mechanics. Finally, in Section 3.6, we compare the weight structure of the bases resulting from quiver varieties and the path realizations of statistical mechanics and make certain their complete coincidence.

3.1 Lusztig's Quiver Varieties

In this section, we will recount the explicit description given in [15] of the irreducible components of Lusztig's quiver variety in the case of types A_∞ and $A_n^{(1)}$. See this reference for details, including proofs.

Let I be a set of vertices of the Dynkin graph of a Kac-Moody Lie algebra $\mathfrak{g}$ and let H be the set of pairs consisting of an edge together with an orientation of it. For $h \in H$, let $\mathrm{in}(h)$ (resp. $\mathrm{out}(h)$) be the incoming (resp. outgoing) vertex of h. We define the involution $^{-} : H \to H$ to be the function which takes $h \in H$ to the element of H consisting of the the same edge with opposite orientation. An *orientation* of our graph is a choice of a subset $\Omega \subset H$ such that $\Omega \cup \bar{\Omega} = H$ and $\Omega \cap \bar{\Omega} = \emptyset$.

Let $\mathcal{V}$ be the category of finite-dimensional I-graded vector spaces $\mathbf{V} = \oplus_{i \in I} \mathbf{V}_i$ over $\mathbb{C}$ with morphisms being linear maps respecting the grading. $\mathbf{V} \in \mathcal{V}$ shall denote that $\mathbf{V}$ is an object of $\mathcal{V}$. The dimension of $\mathbf{V} \in \mathcal{V}$ is given by $\mathbf{v} = \dim \mathbf{V} = (\dim \mathbf{V}_0, \dots, \dim \mathbf{V}_n)$. We identify this dimension with the element $(\dim \mathbf{V}_0)\alpha_0 + \cdots + (\dim \mathbf{V}_n)\alpha_n$ of the root lattice of $\mathfrak{g}$. Here the α_i are the simple roots corresponding to the vertices of our quiver (graph with orientation), whose underlying graph is the Dynkin graph of $\mathfrak{g}$.

Given $\mathbf{V} \in \mathcal{V}$, let

$$\mathbf{E}_{\mathbf{V}} = \bigoplus_{h \in H} \mathrm{Hom}(\mathbf{V}_{\mathrm{out}(h)}, \mathbf{V}_{\mathrm{in}(h)}).$$

For any subset H' of H, let $\mathbf{E}_{\mathbf{V},H'}$ be the subspace of $\mathbf{E}_{\mathbf{V}}$ consisting of all vectors $x = (x_h)$ such that $x_h = 0$ whenever $h \notin H'$. The algebraic group $G_{\mathbf{V}} = \prod_i \mathrm{Aut}(\mathbf{V}_i)$ acts on $\mathbf{E}_{\mathbf{V}}$ and $\mathbf{E}_{\mathbf{V},H'}$ by

$$(g, x) = ((g_i), (x_h)) \mapsto gx = (x'_h) = (g_{\mathrm{in}(h)} x_h g^{-1}_{\mathrm{out}(h)}).$$

Define the function $\varepsilon : H \to \{-1, 1\}$ by $\varepsilon(h) = 1$ for all $h \in \Omega$ and $\varepsilon(h) = -1$ for all $h \in \bar{\Omega}$. Let $\mathbf{V} \in \mathcal{V}$. The Lie algebra of $G_{\mathbf{V}}$ is $\mathbf{gl}_{\mathbf{v}} = \prod_i \mathrm{End}(\mathbf{V}_i)$ and it acts on $\mathbf{E}_{\mathbf{V}}$ by

$$(a, x) = ((a_i), (x_h)) \mapsto [a, x] = (x'_h) = (a_{\mathrm{in}(h)} x_h - x_h a_{\mathrm{out}(h)}).$$

Let $\langle\ ,\ \rangle$ be the nondegenerate, $G_{\mathbf{V}}$-invariant, symplectic form on $\mathbf{E}_{\mathbf{V}}$ with values in

$\mathbb{C}$ defined by

$$\langle x, y\rangle = \sum_{h\in H} \varepsilon(h)\,\mathrm{tr}(x_h y_{\bar{h}}).$$

Note that $\mathbf{E_V}$ can be considered as the cotangent space of $\mathbf{E}_{\mathbf{V},\Omega}$ under this form.

The moment map associated to the $G_\mathbf{v}$-action on the symplectic vector space $\mathbf{E_V}$ is the map $\psi : \mathbf{E_V} \to \mathbf{gl_v}$ with i-component $\psi_i : \mathbf{E_V} \to \mathrm{End}\,\mathbf{V}_i$ given by

$$\psi_i(x) = \sum_{h\in H,\, \mathrm{in}(h)=i} \varepsilon(h) x_h x_{\bar{h}}.$$

Definition 3.1.A ([15]). *An element $x \in \mathbf{E_V}$ is said to be* nilpotent *if there exists an $N \geq 1$ such that for any sequence $h_1', h_2', \ldots, h_N'$ in H satisfying* $\mathrm{out}(h_1') = \mathrm{in}(h_2')$, $\mathrm{out}(h_2') = \mathrm{in}(h_3')$, $\ldots$, $\mathrm{out}(h_{N-1}') = \mathrm{in}(h_N')$, *the composition $x_{h_1'} x_{h_2'} \ldots x_{h_N'} : \mathbf{V}_{\mathrm{out}(h_N')} \to \mathbf{V}_{\mathrm{in}(h_1')}$ is zero.*

Definition 3.1.B ([15]). $\Lambda_\mathbf{V}$ *is the set of all nilpotent elements $x \in \mathbf{E_V}$ such that $\psi_i(x) = 0$ for all $i \in I$.*

3.1.1 Type A_∞

Let $\mathfrak{g}$ be the simple Lie algebra of type A_∞. Let $I = \mathbb{Z}$ be the set of vertices of a graph with the set of oriented edges given by

$$H = \{i \to j \mid i, j \in I,\ i - j = 1\} \cup \{i \leftarrow j \mid i, j \in I, i - j = 1\}.$$

We define the involution $^- : H \to H$ as the function that interchanges $i \to j$ and $i \leftarrow j$. For $h = (i \to j)$, we set $\mathrm{out}(h) = i$ and $\mathrm{in}(h) = j$ and for $h = (i \leftarrow j)$, we set $\mathrm{out}(h) = j$ and $\mathrm{in}(h) = i$. Let Ω be the subset of H consisting of the arrows $i \to j$.

Proposition 3.1.C ([15]). *The irreducible components of $\Lambda_\mathbf{V}$ are the closures of the conormal bundles of the various $G_\mathbf{V}$-orbits in $\mathbf{E}_{\mathbf{V},\Omega}$.*

Proof. The case where $\mathfrak{g}$ is of type A_n is proven in [15]. The A_∞ case follows by passing to the direct limit. □

For two integers $k' \leq k$, define $\mathbf{V}_\infty(k',k) \in \mathcal{V}$ to be the vector space with basis $\{e_r \mid k' \leq r \leq k\}$. We require that e_r has degree $r \in I$. Let $x_\infty(k',k) \in \mathbf{E}_{\mathbf{V}_\infty(k',k),\Omega}$ be defined by $x_\infty(k',k) : e_r \mapsto e_{r-1}$ for $k' \leq r \leq k$, where $e_{k'-1} = 0$. It is clear that $(\mathbf{V}_\infty(k',k), x_\infty(k',k))$ is an indecomposable representation of our quiver. Conversely, any indecomposable finite-dimensional representation $(\mathbf{V}, x)$ of our quiver is isomorphic to some $(\mathbf{V}_\infty(k',k), x_\infty(k',k))$.

Let Z^∞ be the set of all pairs $(k' \leq k)$ of integers and let $\tilde{Z}^\infty$ be the set of all functions $Z^\infty \to \mathbb{N}$ with finite support.

It is easy to see that for $\mathbf{V} \in \mathcal{V}$, the set of $G_\mathbf{V}$-orbits in $\mathbf{E}_{\mathbf{V},\Omega}$ is naturally indexed by the subset $\tilde{Z}^\infty_\mathbf{V}$ of $\tilde{Z}^\infty$ consisting of those $f \in \tilde{Z}^\infty$ such that

$$\sum_{k' \leq i \leq k} f(k',k) = \dim \mathbf{V}_i$$

for all $i \in I$. Here the sum is over all $k' \leq k$ such that $k' \leq i \leq k$. Corresponding to a given f is the orbit consisting of all representations isomorphic to a sum of the indecomposable representations $x_\infty(k',k)$, each occuring with multiplicity $f(k',k)$. Denote by $\mathcal{O}_f$ the $G_\mathbf{V}$-orbit corresponding to $f \in \tilde{Z}^\infty_\mathbf{V}$. Let $\mathcal{C}_f$ be the conormal bundle to $\mathcal{O}_f$ and let $\bar{\mathcal{C}}_f$ be its closure. We then have the following

Proposition 3.1.D. *$f \to \bar{\mathcal{C}}_f$ is a 1-1 correspondence between the set $\tilde{Z}^\infty_\mathbf{V}$ and the set of irreducible components of $\Lambda_\mathbf{V}$.*

Proof. This follows immediately from Proposition 3.1.C. □

3.1.2 Type $A_n^{(1)}$

Let $\mathfrak{g}$ be the affine Lie algebra of type $A_n^{(1)}$. That is, it is the Lie algebra generated by the set of elements E_k, F_k, H_k $(k = 0, 1, \ldots, n)$ and d satisfying the following relations:

$$[E_k, F_l] = \delta_{kl} H_k, \qquad [H_k, E_l] = a_{kl} E_l, \qquad [H_k, F_l] = -a_{kl} F_l,$$

$$[d, E_k] = \delta_{k0} E_k, \qquad [d, F_k] = -\delta_{k0} F_k,$$

$$(\operatorname{ad} E_k)^{1-a_{kl}} E_l = 0, \qquad (\operatorname{ad} F_k)^{1-a_{kl}} F_l = 0 \quad \text{for } k \neq l.$$

Here

$$a_{kl} = 2\delta(k,l) - \delta(k,l+1) - \delta(k,l-1),$$

where $\delta(k,l) = 1$ if $k \equiv l \bmod (n+1)$ and is equal to zero otherwise.

Let $I = \mathbb{Z}/(n+1)$ be the set of vertices of a graph with the set of oriented edges given by

$$H = \{i \to j \mid i,j \in I,\ i - j = 1\} \cup \{i \leftarrow j \mid i,j \in I, i - j = 1\}.$$

For two integers $k' \leq k$, define $\mathbf{V}(k',k) \in \mathcal{V}$ to be the vector space with basis $\{e_r \mid k' \leq r \leq k\}$. We require that e_r has degree $i \in I$ where $r \equiv i \pmod{n+1}$. Let $x(k',k) \in \mathbf{E}_{\mathbf{V}(k',k),\Omega}$ be defined by $x(k',k) : e_r \mapsto e_{r-1}$ for $k' \leq r \leq k$, where $e_{k'-1} = 0$. It is clear that $(\mathbf{V}(k',k), x(k',k))$ is an indecomposable representation of our quiver and that $x(k',k)$ is nilpotent. Also, the isomorphism class of this representation does not change when k' and k are simultaneously translated by a multiple of $n+1$. Conversely, any indecomposable finite-dimensional representation $(\mathbf{V}, x)$ of our quiver, with x nilpotent, is isomorphic to some $(\mathbf{V}(k',k), x(k',k))$ where k' and k are uniquely defined up to a simultaneous translation by a multiple of $n+1$.

Let Z be the set of all pairs $(k' \leq k)$ of integers defined up to simultaneous translation by a multiple of $n+1$ and let $\tilde{Z}$ be the set of all functions $Z \to \mathbb{N}$ with finite support.

It is easy to see that for $\mathbf{V} \in \mathcal{V}$, the set of $G_{\mathbf{V}}$-orbits on the set of nilpotent elements in $\mathbf{E}_{\mathbf{V},\Omega}$ is naturally indexed by the subset $\tilde{Z}_{\mathbf{V}}$ of $\tilde{Z}$ consisting of those $f \in \tilde{Z}$ such that

$$\sum_{k' \leq k} f(k',k) \#\{r \mid k' \leq r \leq k,\ r \equiv i \pmod{n+1}\} = \dim \mathbf{V}_i$$

for all $i \in I$. Here the sum is taken over all $k' \leq k$ up to simultaneous translation by a multiple of $n+1$. Corresponding to a given f is the orbit consisting of all representations isomorphic to a sum of the indecomposable representations $x(k',k)$, each occuring with multiplicity $f(k',k)$. Denote by $\mathcal{O}_f$ the $G_{\mathbf{V}}$-orbit corresponding to $f \in \tilde{Z}_{\mathbf{V}}$.

We say that $f \in \tilde{Z}_{\mathbf{V}}$ is *aperiodic* if for any $k' \leq k$, not all $f(k',k)$, $f(k'+1,k+1)$, $\ldots$, $f(k'+n,k+n)$ are > 0. For any $f \in \tilde{Z}_{\mathbf{V}}$, let $\mathcal{C}_f$ be the conormal bundle of $\mathcal{O}_f$ and let $\bar{\mathcal{C}}_f$ be its closure.

Proposition 3.1.E ([15, Prop 15.5]). *Let $f \in \tilde{Z}_{\mathbf{V}}$. The following two conditions are equivalent.*

1. *$\mathcal{C}_f$ consists entirely of nilpotent elements.*

2. *f is aperiodic.*

Proposition 3.1.F ([15, 15.6]). *$f \to \bar{\mathcal{C}}_f$ is a 1-1 correspondence between the set of aperiodic elements in $\tilde{Z}_{\mathbf{V}}$ and the set of irreducible components of $\Lambda_{\mathbf{V}}$.*

Proposition 3.1.G ([15, 12.8]). *Let $x' \in \mathbf{E}_{\mathbf{V},\Omega}$ and $x'' \in \mathbf{E}_{\mathbf{V},\bar{\Omega}}$. Then $\psi_i(x'+x'')=0$ for all $i \in I$ if and only if x'' is orthogonal with respect to $\langle\ ,\ \rangle$ to the tangent space to the $G_{\mathbf{V}}$-orbit of x', regarded as a vector subspace of $\mathbf{E}_{\mathbf{V},\Omega}$.*

3.1.3 $\widehat{\mathfrak{gl}}_{n+1}$ Case

Since $\widehat{\mathfrak{gl}}_{n+1}$ is not a Kac-Moody algebra in a strict sense, this case is not covered by Lusztig's theory and requires certain modifications. We preserve the notation of the previous subsection.

Definition 3.1.H. *$\tilde{\Lambda}_{\mathbf{V}}$ is the set of all elements $x = x' + x''$, where $x' \in E_{\mathbf{V},\Omega}$ and $x'' \in E_{\mathbf{V},\bar{\Omega}}$, such that x' is nilpotent and $\psi_i(x) = 0$ for all $i \in I$.*

For any $f \in \tilde{Z}_{\mathbf{V}}$, we denote by $\mathcal{O}_f$ the corresponding $G_{\mathbf{V}}$-orbit and by $\mathcal{C}_f$ its conormal bundle.

Proposition 3.1.I. *Let $f \in \tilde{Z}_{\mathbf{V}}$. Then*

1. *$\mathcal{C}_f$ consists entirely of elements of $\tilde{\Lambda}_{\mathbf{V}}$, and*

2. *$\tilde{\Lambda}_{\mathbf{V}}$ is the union of $\bar{\mathcal{C}}_f$ for all $f \in \tilde{Z}_{\mathbf{V}}$.*

Proof. This follows from Proposition 3.1.G. □

Proposition 3.1.J. *$f \to \bar{\mathcal{C}}_f$ is a 1-1 correspondence between the set $\tilde{Z}_{\mathbf{V}}$ and the set of irreducible components of $\Lambda_{\mathbf{V}}$.*

Proof. This follows easily since the conormal bundles $\mathcal{C}_f$ are irreducible of the same dimension. □

3.2 Nakajima's Quiver Varieties

We introduce here a description of the quiver varieties first presented in [21] in the case of types A_∞ and $A_n^{(1)}$.

Definition 3.2.A ([21]). *For $\mathbf{v}, \mathbf{w} \in \mathbb{Z}_{\geq 0}^I$, choose I-graded vector spaces $\mathbf{V}$ and $\mathbf{W}$ of graded dimension $\mathbf{v}$ and $\mathbf{w}$ respectively. Then define*

$$\Lambda \equiv \Lambda(\mathbf{v}, \mathbf{w}) = \Lambda_{\mathbf{V}} \times \sum_{i \in I} \operatorname{Hom}(\mathbf{V}_i, \mathbf{W}_i).$$

Now, suppose $\mathbf{S}$ is an I-graded subspace of $\mathbf{V}$. For $x \in \Lambda_{\mathbf{V}}$ we say that $\mathbf{S}$ is *x-stable* if $x(\mathbf{S}) \subset \mathbf{S}$.

Definition 3.2.B ([21]). *$\Lambda^{st} = \Lambda(\mathbf{v}, \mathbf{w})^{st}$ is the set of all $(x, j) \in \Lambda(\mathbf{v}, \mathbf{w})$ satisfying the following condition: If $\mathbf{S} = (\mathbf{S}_i)$ with $\mathbf{S}_i \subset \mathbf{V}_i$ is x-stable and $j_i(\mathbf{S}_i) = 0$ for $i \in I$, then $\mathbf{S}_i = 0$ for $i \in I$.*

$G_{\mathbf{V}}$ acts on $\Lambda(\mathbf{v}, \mathbf{w})$ via

$$(g, (x, j)) = ((g_i), ((x_h), (j_i))) \mapsto ((g_{\mathrm{in}(h)} x_h g_{\mathrm{out}(h)}^{-1}), j_i g_i^{-1}).$$

and the stabilizer of any point of $\Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}}$ in $G_{\mathbf{V}}$ is trivial (see [23, Lemma 3.10]). We then make the following

Definition 3.2.C ([21]). $\mathcal{L} \equiv \mathcal{L}(\mathbf{v}, \mathbf{w}) = \Lambda(\mathbf{v}, \mathbf{w})^{st}/G_{\mathbf{V}}$.

Let $\operatorname{Irr}\mathcal{L}(\mathbf{v}, \mathbf{w})$ (resp. $\operatorname{Irr}\Lambda(\mathbf{v}, \mathbf{w})$) be the set of irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{w})$ (resp. $\Lambda(\mathbf{v}, \mathbf{w})$). Then $\operatorname{Irr}\mathcal{L}(\mathbf{v}, \mathbf{w})$ can be identified with

$$\{Y \in \operatorname{Irr}\Lambda(\mathbf{v}, \mathbf{w}) \,|\, Y \cap \Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}} \neq \emptyset\}.$$

Specifically, the irreducible components of $\operatorname{Irr}\mathcal{L}(\mathbf{v}, \mathbf{w})$ are precisely those

$$X_f \overset{\text{def}}{=} \left(\left(\bar{\mathcal{C}}_f \times \sum_{i \in I} \operatorname{Hom}(\mathbf{V}_i, \mathbf{W}_i) \right) \cap \Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}} \right) /G_{\mathbf{V}}$$

which are nonempty.

The following will be used in the sequel.

Lemma 3.2.D.

$$\Lambda^{st} = \{x \in \Lambda \,|\, \ker x_{i \to i-1} \cap \ker x_{i+1 \leftarrow i} \cap \ker j_i = 0 \ \forall i\}$$

Proof. Since each $\ker x_{i \to i-1} \cap \ker x_{i+1 \leftarrow i}$ is x-stable, the left hand side is obviously contained in the right hand side. Now suppose x is an element of the right hand side. Let $\mathbf{S} = (\mathbf{S}_i)$ with $\mathbf{S}_i \subset \mathbf{V}_i$ be x-stable and $j_i(\mathbf{S}_i) = 0$ for $i \in I$. Assume that $\mathbf{S} \neq 0$. Since all elements of Λ are nilpotent, we can find a minimal value of N such that the condition in Definition 3.1.A is satisfied. Then we can find a $v \in \mathbf{S}_i$ for some i and a sequence $h'_1, h'_2, \ldots, h'_{N-1}$ (empty if $N = 1$) in H such that $\operatorname{out}(h'_1) = \operatorname{in}(h'_2)$, $\operatorname{out}(h'_2) = \operatorname{in}(h'_3)$, $\ldots$, $\operatorname{out}(h'_{N-2}) = \operatorname{in}(h'_{N-1})$ and $v' = x_{h'_1} x_{h'_2} \ldots x_{h'_{N-1}}(v) \neq 0$. Now, $v' \in \mathbf{S}_{i'}$ for some $i' \in I$ by the stability of $\mathbf{S}$ (hence $j_{i'}(v') = 0$) and $v' \in \ker x_{i' \to i'-1} \cap \ker x_{i'+1 \to i'}$ by our choice of N. This contradicts the fact that x is an element of the right hand side. $\square$

In the case of $\widehat{\mathfrak{gl}}_{n+1}$, we define the varieties $\tilde{\Lambda}(\mathbf{v}, \mathbf{w})$, $\tilde{\Lambda}(\mathbf{v}, \mathbf{w})^{st}$ and $\tilde{\mathcal{L}}(\mathbf{v}, \mathbf{w})$ by replacing $\Lambda_{\mathbf{V}}$ by $\tilde{\Lambda}_{\mathbf{V}}$ in the above.

3.3 The Lie Algebra Action

We summarize here some results from [21] that will be needed in the sequel. See this reference for more details, including proofs. We keep the notation of Sections 3.1 and 3.2 (with $\mathfrak{g}$ arbitrary).

Let $\mathbf{w}, \mathbf{v}, \mathbf{v}', \mathbf{v}'' \in \mathbb{Z}^I_{\geq 0}$ be such that $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$. Consider the maps

$$\Lambda(\mathbf{v}'', \mathbf{0}) \times \Lambda(\mathbf{v}', \mathbf{w}) \overset{p_1}{\leftarrow} \tilde{\mathbf{F}}(\mathbf{v}, \mathbf{w}; \mathbf{v}'') \overset{p_2}{\rightarrow} \mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'') \overset{p_3}{\rightarrow} \Lambda(\mathbf{v}, \mathbf{w}), \tag{3.1}$$

where the notation is as follows. A point of $\mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ is a point $(x, j) \in \Lambda(\mathbf{v}, \mathbf{w})$ together with an I-graded, x-stable subspace $\mathbf{S}$ of $\mathbf{V}$ such that $\dim \mathbf{S} = \mathbf{v}' = \mathbf{v} - \mathbf{v}''$. A point of $\tilde{\mathbf{F}}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ is a point $(x, j, \mathbf{S})$ of $\mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ together with a collection of isomorphisms $R'_i : \mathbf{V}'_i \cong \mathbf{S}_i$, $R''_i : \mathbf{V}''_i \cong \mathbf{V}_i/\mathbf{S}_i$ for each $i \in I$. Then we define $p_2(x, j, \mathbf{S}, R', R'') = (x, j, \mathbf{S})$, $p_3(x, j, \mathbf{S}) = (x, j)$ and $p_1(x, j, \mathbf{S}, R', R'') = (x'', x', j')$ where x'', x', j' are determined by

$$R'_{\mathrm{in}(h)} x'_h = x_h R'_{\mathrm{out}(h)} : \mathbf{V}'_{\mathrm{out}(h)} \to \mathbf{S}_{\mathrm{in}(h)},$$
$$j'_i = j_i R'_i : \mathbf{V}'_i \to \mathbf{W}_i$$
$$R''_{\mathrm{in}(h)} x''_h = x_h R''_{\mathrm{out}(h)} : \mathbf{V}''_{\mathrm{out}(h)} \to \mathbf{V}_{\mathrm{in}(h)}/\mathbf{S}_{\mathrm{in}(h)}.$$

It follows that x' and x'' are nilpotent.

Lemma 3.3.A ([21, Lemma 10.3]).

$$(p_3 \circ p_2)^{-1}(\Lambda(\mathbf{v}, \mathbf{w})^{st}) \subset p_1^{-1}(\Lambda(\mathbf{v}'', \mathbf{0}) \times \Lambda(\mathbf{v}', \mathbf{w})^{st}).$$

Thus, we can restrict (3.1) to Λ^{st}, forget the $\Lambda(\mathbf{v}'', \mathbf{0})$-factor and consider the quotient by $G_{\mathbf{V}}$, $G_{\mathbf{V}'}$. This yields the diagram

$$\mathcal{L}(\mathbf{v}', \mathbf{w}) \overset{\pi_1}{\leftarrow} \mathcal{F}(\mathbf{v}, \mathbf{w}; \mathbf{v} - \mathbf{v}') \overset{\pi_2}{\rightarrow} \mathcal{L}(\mathbf{v}, \mathbf{w}), \tag{3.2}$$

where

$$\mathcal{F}(\mathbf{v}, \mathbf{w}, \mathbf{v} - \mathbf{v}') \overset{\mathrm{def}}{=} \{(x, j, \mathbf{S}) \in \mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v} - \mathbf{v}') \,|\, (x, j) \in \Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}}\}/G_{\mathbf{V}}.$$

Let $M(\mathcal{L}(\mathbf{v},\mathbf{w}))$ be the vector space of all constructible functions on $\mathcal{L}(\mathbf{v},\mathbf{w})$. For a subvariety Y of a variety A, let $\mathbf{1}_Y$ denote the function on A which takes the value 1 on Y and 0 elsewhere. Let $\chi(Y)$ denote the Euler characteristic of the algebraic variety Y. Then for a map π between algebraic varieties A and B, let $\pi_!$ denote the map between the abelian groups of constructible functions on A and B given by

$$\pi_!(\mathbf{1}_Y)(y) = \chi(\pi^{-1}(y) \cap Y),\ Y \subset A$$

and let π^* be the pullback map from functions on B to functions on A acting as $\pi^* f(y) = f(\pi(y))$. Then define

$$H_i : M(\mathcal{L}(\mathbf{v},\mathbf{w})) \to M(\mathcal{L}(\mathbf{v},\mathbf{w})); \quad H_i f = u_i f,$$

$$E_i : M(\mathcal{L}(\mathbf{v},\mathbf{w})) \to M(\mathcal{L}(\mathbf{v}-\mathbf{e}^i,\mathbf{w})); \quad E_i f = (\pi_1)_!(\pi_2^* f),$$

$$F_i : M(\mathcal{L}(\mathbf{v}-\mathbf{e}^i,\mathbf{w})) \to M(\mathcal{L}(\mathbf{v},\mathbf{w})); \quad F_i g = (\pi_2)_!(\pi_1^* g).$$

Here

$$\mathbf{u} = {}^t(u_0,\ldots,u_n) = \mathbf{w} - C\mathbf{v}$$

where C is the Cartan matrix of $\mathfrak{g}$ and we are using diagram (3.2) with $\mathbf{v}' = \mathbf{v} - \mathbf{e}^i$ where $\mathbf{e}^i$ is the vector whose components are given by $\mathbf{e}^i_{i'} = \delta_{ii'}$.

Now let φ be the constant function on $\mathcal{L}(\mathbf{0},\mathbf{w})$ with value 1. Let $L(\mathbf{w})$ be the vector space of functions generated by acting on φ with all possible combinations of the operators F_i. Then let $L(\mathbf{v},\mathbf{w}) = M(\mathcal{L}(\mathbf{v},\mathbf{w})) \cap L(\mathbf{w})$.

Proposition 3.3.B ([21, Thm 10.14]). *The operators E_i, F_i, H_i on $L(\mathbf{w})$ provide the structure of the irreducible highest weight integrable representation of $\mathfrak{g}$ with highest weight $\mathbf{w}$. Each summand of the decomposition $L(\mathbf{w}) = \bigoplus_{\mathbf{v}} L(\mathbf{v},\mathbf{w})$ is a weight space with weight $\mathbf{w} - C\mathbf{v}$.*

Let $X \in \operatorname{Irr} \mathcal{L}(\mathbf{v},\mathbf{w})$ and define a linear function $T_X : L(\mathbf{v},\mathbf{w}) \to \mathbb{C}$ as in [16, 3.8]. T_X associates to a constructible function $f \in L(\mathbf{v},\mathbf{w})$ the (constant) value of f on a suitable open dense subset of X. The fact that $L(\mathbf{v}, \mathbf{w})$ is finite-dimensional allows us to take such an open set on which *any* $f \in L(\mathbf{v}, \mathbf{w})$ is constant. So we have a linear map

$$\Phi : L(\mathbf{v}, \mathbf{w}) \to \mathbb{C}^{\operatorname{Irr} \mathcal{L}(\mathbf{v},\mathbf{w})}.$$

The following is proved in [16, 4.16] (slightly generalized in [21]).

Proposition 3.3.C ([21, Prop 10.15]). Φ *is an isomorphism; for any* $X \in \operatorname{Irr} \mathcal{L}(\mathbf{v}, \mathbf{w})$, *there is a unique function* $g_X \in L(\mathbf{v}, \mathbf{w})$ *such that for some open dense subset* O *of* X *we have* $g_X|_O = 1$ *and such that for some closed* $G_{\mathbf{V}}$*-invariant subset* $K \subset \mathcal{L}(\mathbf{v}, \mathbf{w})$ *of dimension* $< \dim \mathcal{L}(\mathbf{v}, \mathbf{w})$ *we have* $g_X = 0$ *outside* $X \cup K$. *The functions* g_X *for* $X \in \operatorname{Irr} \Lambda(\mathbf{v}, \mathbf{w})$ *form a basis of* $L(\mathbf{v}, \mathbf{w})$.

3.4 Level One Representations

We now seek to describe the irreducible components of Nakajima's quiver variety. By the comment made in Section 3.2, it suffices to determine which irreducible components of $\Lambda(\mathbf{v}, \mathbf{w})$ are not killed by the stability condition. By Definition 3.2.A and Lemma 3.2.D, these are precisely those irreducible components which contain points x such that

$$\dim\left(\ker x_{i\to i-1} \cap \ker x_{i+1\leftarrow i}\right) \le \mathbf{w}_i \ \forall i. \tag{3.3}$$

We first consider the basic representation of highest weight Λ_0 where $\Lambda_0(\alpha_i) = \delta_{0i}$. This corresponds to $\mathbf{w} = \mathbf{w}^0$, the vector with zero component 1 and all other components equal to zero.

3.4.1 Type A_∞

Consider the case where $\mathfrak{g}$ is of type A_∞. Let $\mathcal{Y}$ be the set of all Young diagrams. That is, it is the set of all weakly decreasing sequences $[l_1, \ldots, l_s]$ of non-negative integers ($l_j = 0$ for $j > s$). For $Y = [l_1, \ldots, l_s] \in \mathcal{Y}$, let A_Y be the set $\{(1-i, l_i-i) \mid 1 \le i \le s\}$.

Theorem 3.4.A. *The irreducible components of* $\mathcal{L}(\mathbf{v}, \mathbf{w}^0)$ *are precisely those* X_f *where* $f \in \tilde{Z}^{\infty}_{\mathbf{V}}$ *such that*

$$\{(k', k) \mid f(k', k) = 1\} = A_Y$$

for some $Y \in \mathcal{Y}$ *and* $f(k', k) = 0$ *for* $(k', k) \notin A_Y$. *Denote the component corresponding to such an* f *by* X_Y. *Thus,* $Y \leftrightarrow X_Y$ *is a natural 1-1 correspondence between the set* $\mathcal{Y}$ *and the irreducible components of* $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w}^0)$.

Proof. Consider the two representations $(\mathbf{V}_\infty(k_1', k_1), x_\infty(k_1', k_1))$ and $(\mathbf{V}_\infty(k_2', k_2), x_\infty(k_2', k_2))$ of our oriented graph as described in Section 3.1 where the basis of $\mathbf{V}_\infty(k_i', k_i)$ is $\{e_r^i \mid k_i' \le r \le k_i\}$. Let W be the conormal bundle to the $G_{\mathbf{V}}$-orbit through the point

$$x_\Omega = (x_h)_{h\in\Omega} = x_\infty(k_1', k_1) \oplus x_\infty(k_2', k_2) \in \mathbf{E}_{\mathbf{V}_\infty(k_1',k_1)\oplus\mathbf{V}_\infty(k_2',k_2),\Omega}.$$

By Proposition 3.1.G, $x = (x_\Omega, x_{\bar{\Omega}}) = (x_h)_{h\in H}$ is in W if and only if

$$x_{i+1\to i} x_{i+1\leftarrow i} = x_{i\leftarrow i-1} x_{i\to i-1}$$

for all i.

Let $e_r^i = 0$ for $r < k_i'$ or $r > k_i$. Now, $x_{r+1\leftarrow r}(e_r^2) = c_r e_{r+1}^1$ for some $c_r \in \mathbb{C}$ since $x_{r+1\leftarrow r}(e_r^2)$ can have no e_{r+1}^2 component by nilpotency. Suppose that $k_1' \le r+1 \le k_1$ and $c_r \neq 0$ (that is, $x_{r+1\leftarrow r}(e_r^2) \neq 0$) . Then if $r+1 > k_1'$,

$$x_{r\leftarrow r-1}(e_{r-1}^2) = x_{r\leftarrow r-1} x_{r\to r-1}(e_r^2) = x_{r+1\to r} x_{r+1\leftarrow r}(e_r^2) = c_r e_r^1 \neq 0.$$

In particular, $e_{r-1}^2 \neq 0$ and so $r - 1 \ge k_2'$. Continuing in this manner, we see that $x_{k_1'\leftarrow k_1'-1}(e_{k_1'-1}^2) \neq 0$ and thus $k_2' < k_1'$.

Now, if $r+1 \le k_2$ then

$$x_{r+2\to r+1} x_{r+2\leftarrow r+1}(e_{r+1}^2) = x_{r+1\leftarrow r} x_{r+1\to r}(e_{r+1}^2) = x_{r+1\leftarrow r}(e_r^2) \neq 0.$$

Therefore, $x_{r+2\leftarrow r+1}(e_{r+1}^2) \neq 0$. But $x_{r+2\leftarrow r+1}(e_{r+1}^2)$ must be a multiple of e_{r+2}^1 as above. Thus we must have $r+2 \le k_1$ and $x_{r+2\leftarrow r+1}(e_{r+1}^2) \neq 0$. Continuing in this manner we see that $k_2 < k_1$. Refer to Figure 3.1 for illustration.

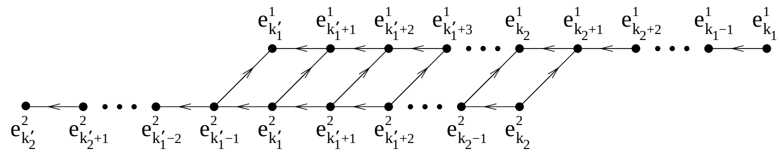

Figure 3.1: If $x_{r+1\leftarrow r}(e^2_r) \neq 0$ for some r, the commutativity of the above diagram forces $k'_2 < k'_1$ and $k_2 < k_1$. Vertices represent the spans of the indicated vectors. Those aligned vertically lie in the same $\mathbf{V}_i$. The arrows indicate the action of the obvious component of x.

Now, let x lie in the conormal bundle to the point

$$\bigoplus_{i=1}^{s} x(k'_i, k'_i + l_i - 1) \in \mathbf{E}_{\oplus_{i=1}^{s}\mathbf{v}_\infty(k'_i,k'_i+l_i-1),\Omega}. \tag{3.4}$$

We can assume (by reordering the indices if necessary) that $k'_1 \geq k'_2 \geq \cdots \geq k'_s$. Now, by the above arguments, $x_{r+1\leftarrow r}(e^i_r)$ must be a linear combination of $\{e^j_{r+1}\}_{j<i}$. Thus

$$e^1_{k'_1} \in \ker x_{k'_1\to k'_1-1} \cap \ker x_{k'_1+1\leftarrow k'_1}.$$

By the stability condition, we must then have $k'_1 = 0$ and there can be no other e^i_r in $\ker x_{r\to r-1} \cap \ker x_{r+1\leftarrow r}$ for any r. Now, by the above considerations, $e^2_{k'_2}$ is in $\ker x_{k'_2\to k'_2-1} \cap \ker x_{k'_2+1\leftarrow k'_2}$ unless $k'_2 + 1 = k'_1$ and $x_{k'_1\leftarrow k'_2}(e^2_{k'_2})$ is a non-zero multiple of $e^1_{k'_1}$. Continuing in this manner, we see that we must have $k'_{i+1} + 1 = k'_i$ and $x_{k'_i\leftarrow k'_{i+1}}(e^{i+1}_{k'_{i+1}}) = c_i e^i_{k'_i} \neq 0$ for $1 \leq i \leq s-1$. Then by the above we must have $k_{i+1} < k_i$ for $1 \leq i \leq s-1$. Setting $l_i = k_i - k'_i + 1$ the theorem follows. □

The Young diagrams enumerating the irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{w}^0)$ can be visualized as in Figure 3.2. Note that the vertices in our diagram correspond to the boxes in the classical Young diagram, and our arrows intersect the classical diagram edges.

For the level one A_∞ case, it is relatively easy to compute the geometric action of the generators E_k and F_k of $\mathfrak{g}$. We first note that for every $\mathbf{v}$, $\mathcal{L}(\mathbf{v}, \mathbf{w}^0)$ is either empty or is a point. It follows that each X_Y is equal to $\mathcal{L}(\mathbf{v}, \mathbf{w}^0)$ for some unique $\mathbf{v}$ which we will denote $\mathbf{v}_Y$.

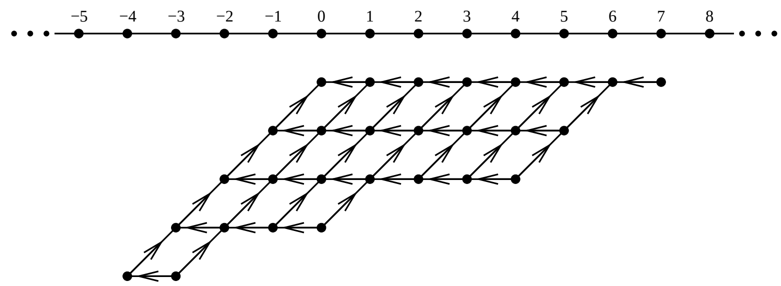

Figure 3.2: The irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{w}^0)$ are enumerated by Young diagrams. The top line is the Dynkin graph of type A_∞. The other horizontal lines represent $x_\infty(k', k)$ where k' and k are the positions of the leftmost and rightmost vertices.

Lemma 3.4.B. *The function g_{X_Y} corresponding to the irreducible component X_Y where $Y \in \mathcal{Y}$ is simply $\mathbf{1}_{X_Y}$, the function on X_Y with constant value one.*

Proof. This is obvious since X_Y is a point. $\square$

Proposition 3.4.C. *$F_k \mathbf{1}_{X_Y} = \mathbf{1}_{X_{Y'}}$ where $\mathbf{v}_{Y'} = \mathbf{v}_Y + \mathbf{e}^k$ if such a Y' exists and $F_k \mathbf{1}_{X_Y} = 0$ otherwise. $E_k \mathbf{1}_{X_Y} = \mathbf{1}_{X_{Y''}}$ where $\mathbf{v}_{Y''} = \mathbf{v}_Y - \mathbf{e}^k$ if such a Y'' exists and $E_k \mathbf{1}_{X_Y} = 0$ otherwise.*

Proof. It is clear from the definitions that $F_k \mathbf{1}_{X_Y} = c_1 \mathbf{1}_{X_{Y'}}$ and $E_k \mathbf{1}_{X_Y} = c_2 \mathbf{1}_{X_{Y''}}$ for some constants c_1 and c_2 if Y' and Y'' exist as described above and that these actions are zero otherwise. We simply have to compute the constants c_1 and c_2.

$$\begin{aligned} F_k \mathbf{1}_{X_Y}(x) &= (\pi_2)_! \pi_1^* \mathbf{1}_{X_Y}(x) \\ &= \chi(\{S \mid S \text{ is } x\text{-stable, } x|_S \in X_Y\}) \\ &= \chi(\mathrm{pt}) \\ &= 1 \end{aligned}$$

if $x \in X_{Y'}$ where $\mathbf{v}_{Y'} = \mathbf{v}_Y + \mathbf{e}^k$ and zero otherwise. The fact that the above set is simply a point follows from the fact that S_k must be the sum of the images of x_h such that $\mathrm{in}(h) = k$. Thus $c_1 = 1$ as desired.

Note that if there exists a Y' such that $\mathbf{v}_{Y'} = \mathbf{v}_Y + \mathbf{e}^k$ then there cannot exist a Y'' such that $\mathbf{v}_{Y''} = \mathbf{v}_Y - \mathbf{e}^k$ and vice versa. Therefore if such a Y'' exists, $F_k \mathbf{1}_{X_Y} = 0$ and so

$$H_k \mathbf{1}_{X_Y} = [E_k, F_k] \mathbf{1}_{X_Y} = -F_k E_k \mathbf{1}_{X_Y}.$$

One can easily check that $H_k \mathbf{1}_{X_Y} = -\mathbf{1}_{X_Y}$ if a Y'' exists as described above and thus $F_k E_k \mathbf{1}_{X_Y} = \mathbf{1}_{X_Y}$. It then follows from the above that we must have $c_2 = 1$. □

The above action of the type A_∞ Lie algebra in the space spanned by a basis indexed by Young diagrams is well known in a purely algebraic context (see e.g. [10]).

Remark 3.4.D. All the results of this section can be repeated with minor modifications for fundamental representations of finite-dimensional Lie algebras of type A_n. In this case, the bases of fundamental representations will be enumerated by Young diagrams of size bounded by an $m \times (n + 1 - m)$ rectangle, where $m = 1, 2, \ldots, n$ is the number of the fundamental representation. Note that the same Young diagrams also naturally enumerate the Schubert cells of the Grassmannians $Gr(m, n + 1)$ for type A_n or the semi-infinite Grassmannian for type A_∞.

3.4.2 Type $A_n^{(1)}$

Let $\mathcal{Y}_n$ be the set of all Young diagrams $[l_1, \ldots, l_s]$ satisfying $l_i > l_{i+n}$ for all $i = 1, \ldots, s$ ($l_j = 0$ for $j > s$). For $Y = [l_1, \ldots, l_s] \in \mathcal{Y}_n$, let A_Y be the set $\{(1 - i, l_i - i) \mid 1 \le i \le s\}$.

Theorem 3.4.E. *The irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{w}^0)$ are precisely those X_f where $f \in \tilde{Z}_{\mathbf{V}}$ such that*

$$\{(k', k) \mid f(k', k) = 1\} = A_Y$$

for some $Y \in \mathcal{Y}_n$ and $f(k', k) = 0$ for $(k', k) \notin A_Y$ (up to simultaneous translation of k' and k by $n + 1$). Denote the component corresponding to such an f by X_Y.

Thus, $Y \leftrightarrow X_Y$ is a natural 1-1 correspondence between the set $\mathcal{Y}_n$ and the irreducible components of $\cup_{\mathbf{v}}\mathcal{L}(\mathbf{v}, \mathbf{w}^0)$.

Proof. The argument is exactly analogous to that used in the proof of Theorem 3.4.A. We need only note that a point in the conormal bundle to the orbit through the point

$$\sum_{i=1}^{s} x(k_i', k_i' + l_i - 1) \in \mathbf{E}_{\oplus_{i=1}^{s} \mathbf{V}(k_i', k_i'+l_i-1), \Omega} \tag{3.5}$$

lies in $\Lambda_{\mathbf{V}}(\mathbf{v}, \mathbf{w}^0)$ if and only if $l_i > l_{i+n}$ for all $i = 1, \ldots, s$ ($l_i = 0$ for $i > s$) by the aperiodicity condition. □

Note that Nakajima's construction yields an action of the Lie algebra on the basis $\{g_{X_Y}\}_{Y \in \mathcal{Y}_n}$ of the basic representation. However, this action is not as straightforward to compute as in the A_∞ case and will be considered in a future work.

3.4.3 $\widehat{\mathfrak{gl}}_{n+1}$ Case

We define A_Y for $Y \in \mathcal{Y}$ as in Section 3.4.1.

Theorem 3.4.F. *The irreducible components of $\tilde{\mathcal{L}}(\mathbf{v}, \mathbf{w}^0)$ are precisely those X_f where $f \in \tilde{Z}_{\mathbf{V}}$ such that*

$$\{(k', k) \mid f(k', k) = 1\} = A_Y$$

for some $Y \in \mathcal{Y}$ and $f(k', k) = 0$ for $(k', k) \notin A_Y$ (up to simultaneous translation of k' and k by $n + 1$). Denote the component corresponding to such an f by X_Y. Thus, $Y \leftrightarrow X_Y$ is a natural 1-1 correspondence between the set $\mathcal{Y}$ and the irreducible components of $\cup_{\mathbf{v}}\tilde{\mathcal{L}}(\mathbf{v}, \mathbf{w}^0)$.

Proof. The argument is exactly analogous to that used in the proof of Theorem 3.4.A. □

As noted in Section 3.1.3, since $\widehat{\mathfrak{gl}}_{n+1}$ is not a Kac-Moody algebra we need to modify Nakajima's construction of highest weight representations. Note that for any n, the difference between $\widehat{\mathfrak{gl}}_{n+1}$ and $\widehat{\mathfrak{sl}}_{n+1}$ is the same Heisenberg algebra $\widehat{\mathfrak{gl}}_1$. The representations of Heisenberg algebras in the context of geometric representation theory were first constructed by Grojnowski [9] and Nakajima [22] (see [24] for a review). However, it is not obvious how to adapt this representation theory to the new quiver varieties $\tilde{\mathcal{L}}(\mathbf{v}, \mathbf{w}^0)$, obtaining the desired commutation relations with the generators of $\widehat{\mathfrak{sl}}_{n+1}$. This problem will be considered in a future work.

3.5 Arbitrary Level Representations

3.5.1 Type A_∞

We first recall some definitions of [4]. A *Maya diagram* is a bijection $m : \mathbb{Z} \to \mathbb{Z}$ such that $(m(j))_{j<0}$ and $(m(j))_{j\geq 0}$ are both increasing. For each Maya diagram there exists a unique $\gamma \in \mathbb{Z}$ such that $m(j) - j = \gamma$ for $|j| \gg 0$. This γ is called the *charge* of m. We denote the set of Maya diagrams of charge γ by $\mathcal{M}[\gamma]$. For $m \in \mathcal{M}[\gamma]$ we let

$$m[r] = (m(j) + r)_{j\in\mathbb{Z}} \in \mathcal{M}[\gamma + r].$$

We can visualize a Maya diagram by a Young diagram. Consider a lattice on the right half plane with lattice points $\{(i, j) \in \mathbb{Z}^2 \mid i \geq 0\}$. Each edge on the lattice is oriented, starting at (i, j) and ending at $(i + 1, j)$ or $(i, j + 1)$ and is numbered by the integer $i + j$. A *path* on the lattice is a map e from $\mathbb{Z}$ to the set of edges on the lattice such that $e(j)$ has number j and the ending site of $e(j)$ is the starting site of $e(j + 1)$. To each Maya diagram of charge γ, we associate the unique path satisfying the following conditions.

1. For $j \ll 0$, $e(j)$ is the edge from $(0, j)$ to $(0, j + 1)$,
2. The edge $e(m(j))$ is vertical (resp. horizontal) if $j < 0$ (resp. $j \geq 0$).

Note that these conditions imply that for $j \gg 0$, $e(j)$ is the edge from $(j - \gamma, \gamma)$ to $(j - \gamma + 1, \gamma)$. See Figure 3.3.

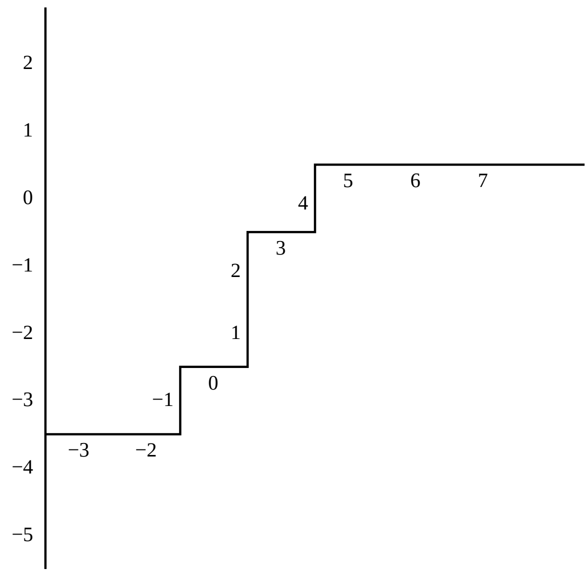

Figure 3.3: The Maya diagram corresponding to $(m(j))_{j\geq 0} = (-3, -2, 0, 3, 5, 6, 7, \dots)$, $(m(j))_{j<0} = (\dots, -6, -5, -4, , -1, 1, 2, 4)$.

Such a path divides the right half plane into two components. The upper half is an *infinite Young diagram* $\mathfrak{Y}$ which consists of a quadrant and a (finite) Young diagram Y attached along a horizontal line at height γ. Thus the set of Maya diagrams are in one to one correspondence with the set of pairs (Y, γ) where $Y \in \mathcal{Y}$ and $\gamma \in \mathbb{Z}$.

Lemma 3.5.A ([4]). *Let $m \in \mathcal{M}[\gamma]$, $m' \in \mathcal{M}[\gamma']$, and let $\mathfrak{Y}$, $\mathfrak{Y}'$ be the corresponding infinite Young diagrams. Then the following are equivalent.*

1. $m(j) \leq m'(j)$ *for* $j \geq 0$,

2. $\gamma \leq \gamma'$ *and* $m(j - \gamma) \geq m'(j - \gamma')$ *for* $j < \gamma$,

3. $\mathfrak{Y} \supset \mathfrak{Y}'$.

We put a partial ordering on the set of Maya diagrams by letting $m \leq m'$ if the conditions in Lemma 3.5.A hold.

Let $\mathbf{w} = \Lambda = \Lambda_{\gamma_1} + \dots \Lambda_{\gamma_l}$ where $\gamma_1 \leq \dots \leq \gamma_l$ and the Λ_i are fundamental weights of $\mathfrak{g}$. Let $\mathbf{w} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$ (that is, $\mathbf{w}$ is function from $\mathbb{Z}$ to $\mathbb{Z}_{\geq 0}$) be the vector with i^{th}

component equal to the number of γ_j equal to i. Let

$$\mathcal{M}[\Lambda] = \mathcal{M}[\gamma_1] \times \cdots \times \mathcal{M}[\gamma_l].$$

For $Y = [l_1, \ldots, l_s] \in \mathcal{Y}$, let A_Y^γ be the set $\{(\gamma + 1 - i, \gamma + l_i - i) \,|\, 1 \leq i \leq s\}$. For $M = ((Y_1, \gamma_1), \ldots, (Y_l, \gamma_l)) \in \mathcal{M}[\Lambda]$, let $A_M = \cup_{i=1}^l A_{Y_i}^{\gamma_i}$ and let $f_M \in \tilde{Z}^\infty$ be the function such that $f(k', k)$ is equal to the number of times (k', k) appears in the set A_M.

Theorem 3.5.B. *The irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{w})$ are precisely those X_f where $f = f_M$ for some $M \in \mathcal{M}[\Lambda]$. Denote the component X_{f_M} by X_M. $M \leftrightarrow X_M$ is a natural 1-1 correspondence between the set*

$$\{(m_1, \ldots, m_l) \in \mathcal{M}[\Lambda] \,|\, m_1 \leq \cdots \leq m_l\}$$

and the irreducible components of $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w})$.

Proof. Recall that irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{w})$ are the closures of the $G_{\mathbf{V}}$-orbits (or isomorphism classes) in $\mathbf{E}_{\mathbf{V},\Omega}$ and that there is a representative of each orbit of the form

$$x = \bigoplus_{(k' \leq k) \in K} x_\infty(k', k) \tag{3.6}$$

for some finite set of pairs K. By picturing $x_\infty(k', k)$ as the string of vertices $k', k' + 1, \ldots, k$, we can represent such an x by a set of finite strings of vertices corresponding to the various $x_\infty(k', k)$ appearing in (3.6). We call the number of vertices in a string its *length.* Each vertex of a string represents a basis vector of $\mathbf{V}$ with degree given by the location of the vertex. The action of x maps each of these basis vectors to the basis vector corresponding to the next (one lower) vertex in the string (or to zero if no such vertex exists). See Figure 3.4.

It is then a straightforward extension of the proof of Theorem 3.4.A that the allowable sets of strings are precisely those that can be grouped into subsets, one

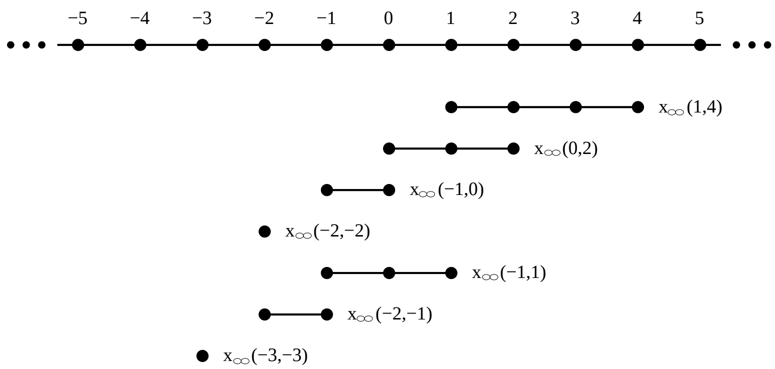

Figure 3.4: The strings associated to some $x \in \mathbf{E}_{\mathbf{V},\Omega}$. The top line is the Dynkin diagram of type A_∞.

for each γ_i, such that the subset corresponding to γ_i, when ordered by decreasing leftmost vertex, has weakly decreasing lengths, the first leftmost vertex is γ_i and the leftmost vertices decrease by one as we move through the subset in order (by leftmost, we mean the vertex with the smallest index). This is precisely the first claim of the Theorem.

It is easy to see that many different $M \in \mathcal{M}[\Lambda]$ may correspond to the same irreducible component. For example, for $\Lambda = \Lambda_{-1} + \Lambda_1$, both

$$M = (([3,2,1],-1),([4,3,2,1],1)), \text{ and}$$
$$M' = (([2,1],-1),([4,3,3,2,1],1))$$

belong to $\mathcal{M}[\Lambda]$ and correspond to the set of strings shown in Figure 3.4 (and hence to the same irreducible component). However, we can associate a unique $M \in \mathcal{M}[\Lambda]$ to each set of strings described above as follows. Associate to γ_1 the longest string with leftmost vertex γ_1 and remove this string from the set. Now do the same for γ_2, etc. When we have associated a string to γ_l, we start again at γ_1 but this time select the longest string with leftmost vertex $\gamma_1 - 1$ and so forth. If at any point, there is no string to associate with a given γ_i, we remove this γ_i from further steps. In this way we associate to each γ_i a sequence of strings of weakly decreasing length (by our condition on the possible sets of strings) with leftmost vertices decreasing by one. The lengths of the strings associated to γ_i give a Young diagram Y_i and we set $m_i = (Y_i, \gamma_i)$. By construction, the length of any string associated to γ_i is greater than the length of a string with the same left end point associated to γ_j for $j > i$. This immediately yields the condition $m_1 \leq \cdots \leq m_l$. Our construction thus gives us the one to one correspondence asserted in the Theorem. □

Note that the enumeration of the irreducible components given in Theorem 3.5.B matches that of Proposition 4.6 of [4].

3.5.2 Type $A_n^{(1)}$

We now consider the case where $\mathfrak{g}$ is of type $A_n^{(1)}$. For an element $M = (m_1, \ldots, m_l) \in \mathcal{M}[\Lambda]$, let R_M be the set (with multiplicity) of pairs (i, l_i) where l_i is the length of a row with top edge having y-coordinate i belonging to one of the m_j. We say that M is *n-reduced* if

$$\{(k+i, l) \,|\, 0 \leq i \leq n\} \not\subset R_M$$

for all k and l.

Define f_M for $M \in \mathcal{M}[\Lambda]$ as in the previous section (except that now our pairs are defined only up to simultaneous translation by $n+1$).

Theorem 3.5.C. *The irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{w})$ are precisely those X_f where $f = f_M$ for some n-reduced $M \in \mathcal{M}[\Lambda]$. Denote the component X_{f_M} by X_M. Then $M \leftrightarrow X_M$ is a natural 1-1 correspondence between the set*

$$\{(m_1, \ldots, m_l) \in \mathcal{M}[\Lambda] \,|\, m_1 \leq \cdots \leq m_l \leq m_1[n+1],\ M \text{ is } n\text{-reduced}\}$$

and the irreducible components of $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w})$.

Proof. Each irreducible component corresponds to a set of strings as in the proof of Theorem 3.5.B with the added condition that we cannot have $n+1$ strings, each of the same length, with left end points the $n+1$ vertices of our quiver. That is, we must have that M is n-reduced. Note that the process described in the proof of Theorem 3.5.B yields $m_i = (Y_i, \gamma_i)$ satisfying $m_1 \leq \cdots \leq m_l \leq m_1[n+1]$ as desired. The Theorem follows. □

Again, as noted in Section 3.4.2, Nakajima's construction yields an action of the Lie algebra on the bases $\{g_{X_M}\}$ of the irreducible representations in both the A_∞ and $A_n^{(1)}$ cases which is more difficult to directly compute than in the level 1 A_∞ case.

3.5.3 $\widehat{\mathfrak{gl}}_{n+1}$ Case

Theorem 3.5.D. *The irreducible components of* $\tilde{\mathcal{L}}(\mathbf{v}, \mathbf{w})$ *are precisely those* X_f *where* $f = f_M$ *for some* $M \in \mathcal{M}[\Lambda]$*. Denote the component* X_{f_M} *by* X_M*. Then* $M \leftrightarrow X_M$ *is a natural 1-1 correspondence between the set*

$$\{(m_1, \ldots, m_l) \in \mathcal{M}[\Lambda] \,|\, m_1 \leq \cdots \leq m_l \leq m_1[n+1]\}$$

and the irreducible components of $\cup_{\mathbf{v}} \tilde{\mathcal{L}}(\mathbf{v}, \mathbf{w})$*.*

Proof. The argument is the same as the proof of Theorem 3.5.C except that we do not have the aperiodicity condition and thus do not require that M is n-reduced. □

Note that the enumeration of the irreducible components of $\cup_{\mathbf{v}} \tilde{\mathcal{L}}(\mathbf{v}, \mathbf{w})$ given by Theorem 3.5.D is the same as that given by Proposition 4.7 of [4] for a spanning set of the dual to the irreducible highest weight representation of $\widehat{\mathfrak{gl}}_{n+1}$. In order to extend the geometric construction of highest weight representations of $\widehat{\mathfrak{sl}}_{n+1}$ to $\widehat{\mathfrak{gl}}_{n+1}$ for an arbitrary level, one would need a representation of the Heisenberg algebra as discussed in Section 3.4.3. Here one might use the construction of the Heisenberg algebra by Baranovsky [1] that generalizes the Grojnowski/Nakajima construction to higher levels.

Remark 3.5.E. One can also give a geometric interpretation of the full Fock space of [4] with basis indexed by $\mathcal{M}[\Lambda]$ via the "smooth" U_l-instanton moduli space $\sqcup_r \mathcal{M}(r, l)$

which has the same generating function for cohomology (see e.g. [24], Chapter 5) as the full Fock space with the basis $\mathcal{M}[\Lambda]$. The types $A_n^{(1)}$ or A_∞ are reflected in the respective action of the groups $\mathbb{Z}/(n+1)\mathbb{Z}$ or $\mathbb{C}^*$ on the moduli space, and $\gamma_1, \ldots, \gamma_l$ is the set of one-dimensional representations of these groups that determine this action.

3.6 A Comparison With The Path Space Representation

The authors of [3] constructed the basic representation of $A_n^{(1)}$ on the space of paths. In [4], this path realization is generalized to arbitrary level. We now compare the geometric presentation $L(\mathbf{v}, \mathbf{w})$ with theirs. We will slightly modify the definitions of [3] to agree with the more general definitions of [4].

3.6.1 The Level One Case

A *basic path* is a sequence $p = (\lambda_0, \lambda_1, \ldots)$ of integers $\lambda_i \in \{0, 1, \ldots, n\}$. The basic path

$$(\overline{j})_{j\geq 0} = (0, 1, \ldots, n, 0, 1, \ldots, n, \ldots)$$

is called the *ground state*. Here $\overline{k}$ for $k \in \mathbb{Z}$ signifies the unique integer such that $0 \leq \overline{k} \leq n$ and $\overline{k} = k \bmod n+1$. Let

$$\mathcal{P}_b = \{p = (\lambda_0, \lambda_1, \ldots) \mid \lambda_j = \overline{j} \text{ for all but a finite number of } j\}.$$

For a basic path $p = (\lambda_0, \lambda_1, \ldots) \in \mathcal{P}_b$, let

$$\omega(p) = \sum_{i=1}^{\infty} i(H(\lambda_i, \lambda_{i+1}) - H(\overline{i}, \overline{i+1})),$$

where

$$H(\lambda, \mu) = \begin{cases} 0 & \text{if } \lambda < \mu \\ 1 & \text{if } \lambda \geq \mu \end{cases}.$$

Basic paths in $\mathcal{P}_b$ can be labeled by Young diagrams as we now describe. The set $\mathcal{M}[0]$ is in one to one correspondence with the set of strictly increasing sequences of integers $m = (m(0), m(1), \dots)$ such that $m(j) = j$ for j large and

$$\#\{j \mid m(j) < 0\} = \#(\{0, 1, 2, \dots\} - \{m(j) \mid m(j) \geq 0\}).$$

Such a sequence represents the Young diagram of signature $[\dots 3^{r_3} 2^{r_2} 1^{r_1}]$ where $r_j = m(j) - m(j-1) - 1$ and vice versa. To a Maya diagram $m = (m(0), m(1), \dots) \in \mathcal{Y}_n$ we associate the basic path $p = (\overline{m(0)}, \overline{m(1)}, \dots) \in \mathcal{P}_b$. Then the ground state corresponds to the empty Young diagram ϕ. In the sequel, we will identify n-reduced Young diagrams (that is, elements of $\mathcal{Y}_n$) and basic paths via the above correspondence.

For $Y = [l_1, \dots, l_s] \in \mathcal{Y}_n$, let

$$\Delta_k(Y) = \delta(k, -s) + \sum_{i=1}^{s} \left(\delta(k, l_i - i + 1) - \delta(k, l_i - i)\right).$$

Proposition 3.6.A. $H_k g_{X_Y} = \Delta_k(Y) g_{X_Y}$.

Proof. Let $Y = [l_1, \dots, l_s]$. Then $g_{X_Y} \in L(\mathbf{v}, \mathbf{w}^0)$ where

$$\begin{aligned} \mathbf{v} &= \dim \bigoplus_{i=1}^{s} \mathbf{V}(1-i, l_i - i) \\ &= \sum_{i=1}^{s} \sum_{l=1-i}^{l_i - i} \alpha_{\bar{l}}. \end{aligned}$$

Recall that the weight of the space $L(\mathbf{v}, \mathbf{w}^0)$ is

$$(u_0, \dots, u_n) = \mathbf{w}^0 - C\mathbf{v}$$

and thus $H_k g_{X_Y} = u_k g_{X_Y}$ with

$$\begin{aligned} u_k &= \Lambda_0(\alpha_k) - \sum_{i=1}^{s} \sum_{l=1-i}^{l_i - i} \langle \alpha_k, \alpha_{\bar{l}} \rangle \\ &= \delta(k, 0) - \sum_{i=1}^{s} \sum_{l=1-i}^{l_i - i} \left(2\delta(k, l) - \delta(k, l-1) - \delta(k, l+1)\right) \\ &= \delta(k, 0) - \sum_{i=1}^{s} \left(\delta(k, 1-i) - \delta(k, -i) + \delta(k, l_i - i) - \delta(k, l_i - i + 1)\right) \end{aligned}$$

$$= \delta(k,-s) + \sum_{i=1}^{s} (\delta(k, l_i - i + 1) - \delta(k, l_i - i))$$
$$= \Delta_k(Y).$$

□

Proposition 3.6.B. $d(g_{X_Y}) = -\omega(Y) g_{X_Y}$.

Proof. We first compute the left hand side. It is obvious that

$$d(g_{X_Y}) = -\mathbf{v}_0 g_{X_Y}$$

where $X_Y \subset \mathcal{L}(\mathbf{v}, \mathbf{w}^0)$. Consider the representation $(\mathbf{V}(k', k'+l-1), x(k', k'+l-1))$ where $l = (n+1)a + b$ with $0 \le b \le n$. Then

$$\begin{aligned}\mathbf{v}_0 &= \dim \mathbf{V}(k', k'+l-1)_0 \\ &= a + \begin{cases} 1 & \text{if } \overline{k'-1} + b > n \\ 0 & \text{if } \overline{k'-1} + b \le n \end{cases}.\end{aligned}$$

Thus, for $Y = [l_1, \dots, l_s] \in \mathcal{Y}_n$ where $l_i = (n+1)a_i + b_i$ with $0 \le b_i \le n$,

$$\mathbf{v}_0 = \sum_{i=1}^{s} \left(a_i + \begin{cases} 1 & \text{if } \overline{-i} + b_i > n \\ 0 & \text{otherwise} \end{cases} \right).$$

We now show that this is equal to $\omega(Y)$. Let $Y_i = [l_1, \dots, l_i]$ for $0 \le i \le s$ where $Y_0 = \phi$ and let $(\lambda_0^i, \lambda_1^i, \dots)$ be the corresponding basic path. Then the first l_i positions of the basic path corresponding to Y_{i-1} are

$$(\overline{1-i}, \overline{1-i}+1, \dots, n, 0, 1, \dots, n, 0, 1, \dots, n, \dots, 0, 1, \dots, n, 0, 1, \dots, \overline{b_i - i}).$$

Here there are a_i repetitions of $0, 1, \dots, n$ if $\overline{i-1} < b_i$ and $a_i - 1$ repetitions if $\overline{i-1} \ge b_i$.

The first l_i positions of the basic path corresponding to Y_i are simply obtained from the above by lowering all the entries by 1 (interpreting -1 as n). The entries of Y_i and Y_{i-1} numbered $l_i + 1$ and above are equal. Then by considering the cases $\overline{i-1} < b_i$ and $\overline{i-1} \ge b_i$, we see that

$$\sum_{j=1}^{\infty} j(H(\lambda_j^i, \lambda_{j+1}^i) - H(\lambda_j^{i-1}, \lambda_{j+1}^{i-1})) = a_i + \begin{cases} 1 & \text{if } \overline{-i} + b_i > n \\ 0 & \text{otherwise} \end{cases}.$$

and the result follows. □

Theorem 3.6.C. *$g_{X_Y} \mapsto Y$ is a weight-preserving vector space isomorphism between the geometric presentation $L(\mathbf{w}^0)$ of $L(\Lambda_0)$ and the basic path space representation given in [3].*

Proof. This follows directly from the previous two propositions and the action of the H_i and d given in [3]. □

3.6.2 Arbitrary Level

We first recall some definitions of [4]. Let $\epsilon_\mu = (0, \ldots, \overset{\mu-\text{th}}{1}, \ldots, 0)$ for $0 \leq \mu \leq n$ denote the standard basis vectors of $\mathbb{Z}^{n+1}$. We extend the indices to $\mathbb{Z}$ by setting $\epsilon_{\mu+n+1} = \epsilon_\mu$. Fix a positive integer l (the level of our representation). A *path* is a sequence $\eta = (\eta(k))_{k\geq 0}$ consisting of elements $\eta(k) \in \mathbb{Z}^{n+1}$ of the form $\epsilon_{\mu_1(k)} + \cdots + \epsilon_{\mu_l(k)}$ with $\mu_1(k), \ldots, \mu_l(k) \in \mathbb{Z}$. To a level l dominant integral weight $\Lambda = \Lambda_{\gamma_1} + \ldots \Lambda_{\gamma_l}$ is associated the path

$$\eta_\Lambda = (\eta_\Lambda(k))_{k\geq 0},\ \eta_\Lambda(k) = \epsilon_{\gamma_1+k} + \cdots + \epsilon_{\gamma_l+k}.$$

η is called a Λ-path if $\eta(k) = \eta_\Lambda(k)$ for $k \gg 0$. The set of Λ-paths is denoted by $\mathcal{P}(\Lambda)$. Define the weight λ_η of η by

$$\lambda_\eta = \Lambda - \sum_{k\geq 0} \pi(\eta(k) - \eta_\Lambda(k)) - \omega(\eta)\delta$$

where

$$\omega(\eta) = \sum_{k\geq 1} k\left(H(\eta(k-1), \eta(k)) - H(\eta_\Lambda(k-1), \eta_\Lambda(k))\right).$$

Here δ is the null root and π is the $\mathbb{Z}$-linear map from $\mathbb{Z}^{n+1}$ to the weight lattice of the Lie algebra of type $A_n^{(1)}$ such that $\pi(\epsilon_\mu) = \Lambda_{\mu+1} - \Lambda_\mu$ (here $\Lambda_{n+1} = \Lambda_0$). The function H is defined as follows: if $\alpha = \epsilon_{\mu_1} + \cdots + \epsilon_{\mu_l}$ and $\beta = \epsilon_{\nu_1} + \cdots + \epsilon_{\nu_l}$ $(0 \leq \mu_i, \nu_i \leq n)$, then

$$H(\alpha, \beta) = \min_{\sigma\in S_l} \sum_{i=1}^{l} \theta(\mu_i - \nu_{\sigma(i)})$$

where S_l is the permutation group on l letters, and

$$\theta(\mu) = \begin{cases} 1 & \text{if } \mu \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Note that we have redefined the notation ω and H of subsection 3.6.1. However, our new definitions reduce to the old ones in the case $\Lambda = \Lambda_0$ and so, to avoid a proliferation of notation, we denote the new functions by the same symbols.

Let η be a Λ-path. An element $M = (m_1, \ldots, m_l) \in \mathcal{M}[\Lambda]$ is called a *lift* of η if and only if

$$m_1 \leq \cdots \leq m_l \leq m_1[r] \tag{3.7}$$

and

$$\eta(k) = \epsilon_{m_1(k)} + \cdots + \epsilon_{m_l(k)}. \tag{3.8}$$

If $M = (m_1, \ldots, m_l)$ and $M' = (m'_1, \ldots, m'_l)$ are lifts of a Λ-path η then we say $M \geq M'$ if and only if $m_j \geq m'_j$ for $1 \leq j \leq l$.

Recall the definition of R_M given in Section 3.5.2. For $M, M' \in \mathcal{M}[\Lambda]$, we say that M is an *n-reduction* of M' if R_M is obtained from $R_{M'}$ by the removal of sets of the form $\{(k+i, l) \,|\, 0 \leq i \leq n\}$ for some k and l.

Proposition 3.6.D. *Suppose $M = (m_1, \ldots, m_l)$ is an n-reduction of $M' = (m'_1, \ldots, m'_l)$ and $m_1 \leq \ldots, \leq m_l \leq m_1[n+1]$, $m'_1 \leq \cdots \leq m'_l \leq m'_1[n+1]$. Then M and M' are lifts of the same path and $M \geq M'$.*

Proof. Recall the construction in the proof of Theorem 3.5.B. Note that choosing arbitrary strings instead of the longest string at each step will not change the values of the right hand side of (3.8) (for any k). Thus, let us form $M'' = (m''_1, \ldots, m''_l) \in \mathcal{M}[\Lambda]$ from the same strings comprising M' but where one of the m''_i contains the entire set of strings of the form $\{(k+i, l) \,|\, 0 \leq i \leq n\}$ which is removed from $R_{M'}$ to obtain R_M. Now, removing this set of strings from M'' simply amounts to removing this set from m''_i. But this just cuts an $(n+1) \times l$ square out of the Maya diagram m''_i and shifts the part of the diagram below the cut up $n+1$ units. See Figure 3.5.

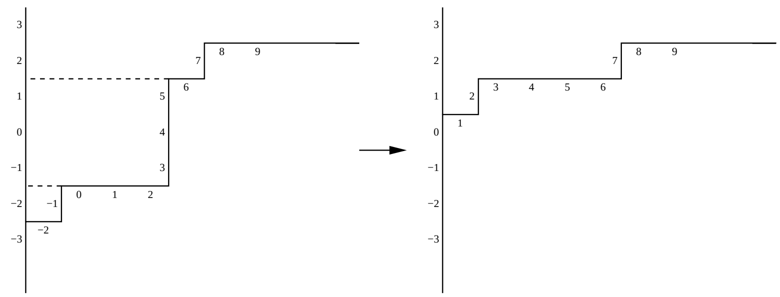

Figure 3.5: Removing an $(n+1) \times l$ square from a Maya diagram (here $n = 2$ and $l = 4$). Notice that the enumeration of the horizontal edges does not change mod $(n+1)$.

Since $\epsilon_{\mu+n+1} = \epsilon_\mu$, the right hand sides of (3.8) for M' and M'' are the same. However, M is simply obtained from M'' by applying the procedure of Theorem 3.5.B to the strings of M'' and as mentioned above, this does not change the right hand sides of (3.8). Thus M and M' are lifts of the same path.

To show that $M \geq M'$, note that by the construction in the proof of Theorem 3.5.B, M is uniquely determined by R_M. Now, we obtain R_M from $R_{M'}$ by removing a set of the form $\{(k+i, l) \,|\, 0 \leq i \leq n\}$ for some k and l. Thus, at each stage in our construction of M, we chose a string of length less than or equal to the string chosen in the construction of M'. Thus we have that $M \geq M'$. $\square$

Proposition 3.6.E ([4]). *For each Λ-path η there exists a unique highest lift M of η such that $M \geq M'$ for any lift M' of η.*

Corollary 3.6.F. *The set*

$$\{M = (m_1, \dots, m_l) \in \mathcal{M}[\Lambda] \,|\, M \text{ is } n\text{-reduced}, m_1 \leq \cdots \leq m_1 \leq m_1[n+1]\}$$

is precisely the set of highest lifts of paths in $\mathcal{P}(\Lambda)$.

Let M_η be the n-reduced element of $\mathcal{M}[\Lambda]$ corresponding to $\eta \in \mathcal{P}(\Lambda)$ and let $\mathfrak{g}$ be the affine Lie algebra of type $A_n^{(1)}$. Define

$$\mathcal{P}(\Lambda)_\mu = \{\eta \in \mathcal{M}[\Lambda] \,|\, \lambda_\eta = \mu\}.$$

In [4], the authors introduced a basis $\{\xi_\eta \,|\, \eta \in \mathcal{P}(\Lambda)_\mu\}$ of the μ weight space of the restricted dual of the highest weight representation of $\mathfrak{g}$ of highest weight Λ [4, Thm 5.4]. The weight of ξ_η is λ_η [4, Thm 5.7].

Theorem 3.6.G. *$g_{X_{M_\eta}} \mapsto \xi_\eta$ is a weight-preserving vector space isomorphism between the geometric presentation $L(\mathbf{w})$ of $L(\Lambda)$ and the path space representation of [4].*

Proof. The fact that we have a vector space isomorphism follows from Proposition 3.6.E and Corollary 3.6.F. It remains to show that the map is weight-preserving. The definition of a path agrees with the definition of a basic path when $\Lambda = \Lambda_0$ and the weights are the same in this case. Thus we have the result for $\Lambda = \Lambda_0$ from the previous section. Then the result for arbitrary level one representations follows easily.

Now, if

$$M_\eta = ((Y_1, \gamma_1), \ldots, (Y_l, \gamma_l))$$

and $\mathbf{V}^i$ is the space corresponding to the strings (see the proof of Proposition 3.5.B) of (Y_i, γ_i) (that is, its dimension in degree j is equal to the number of vertices of these strings that are numbered j) then the weight of g_{M_η} is

$$\sum_{i=1}^{l} (\Lambda_{\gamma_i} - \dim \mathbf{V}^i)$$

where $\dim \mathbf{V}^i$ is identified with an element of the root lattice as in Section 3.1. But this is equal to $\sum_{i=1}^{l} \lambda_{\eta_i}$ where (Y_i, γ_i) is a lift of η_i by the level 1 result. By Proposition 5.6 of [4], this is λ_η as desired. $\square$

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