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Bases of Representations of Type A Affine Lie Algebras via Quiver Varieties and Statistical Mechanics

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1 Introduction

A remarkable relation between representation theory of affine Lie algebras and models of statistical mechanics based on the Yang-Baxter equation has been discovered and intensively studied by Date et al. (see [2, 3] and the references therein). One of the important findings of the above authors is that the one-dimensional configuration sums for these models give rise to characters of integrable highest weight representations of affine Lie algebras. This relation yields certain explicit bases in the representations that admit pure combinatorial descriptions and imply various identities for the characters.

Another astonishing relation between representation theory of affine Lie algebras and moduli spaces of solutions of self-dual Yang-Mills equations has been accomplished by Nakajima [8, 10], who observed a profound link between his earlier work with P. Kronheimer and the results of Lusztig [6, 7]. At the heart of both works that preceded the Nakajima discovery are quiver varieties associated with extended Dynkin diagrams. Nakajima introduced a special class of quiver varieties associated with integrable highest weight representations of affine Lie algebras and obtained a geometric description of the action. He also defined certain Lagrangian subvarieties whose irreducible components yield a geometric basis of the affine Lie algebra representations.

The central goal of the present paper is to relate the two apparently different bases in the representations of affine Lie algebras of type A: one arising from statistical mechanics and the other from gauge theory. We show that the two are governed by

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the same combinatorics that also respects the weight space decomposition of the representations. This identification allows one to give a natural conceptual framework to the intricate structure of statistical mechanical models and also to make explicit calculations in a seemingly intractable geometric setting. In particular, we are able to give an alternative and much simpler geometric proof of the main result of [3] on the construction of a basis of affine Lie algebra representations. At the same time, we give a simple parametrization of the irreducible components of Nakajima quiver varieties associated to infinite and cyclic quivers.

The comparison of the two very different theories brings some surprises and suggests interesting new directions. In particular, the Young diagrams that are routinely used in representation theory of type A Lie algebras acquire an explicit geometric meaning: they picture precisely representations of the corresponding quivers satisfying a stability condition for level one (see Figure 5.2 in the text). On the other hand, the algebraic constructions of [3] involve substantially the highest weight representations of $\widehat{\mathfrak{gl}}_{n+1}$, which are not directly covered by Nakajima's theory. We define new varieties by relaxing the nilpotency condition in the definition of Nakajima's quiver varieties and show that the irreducible components of these new varieties are in one-to-one correspondence with bases of the highest weight representations of $\widehat{\mathfrak{gl}}_{n+1}$. We also mention some interesting problems that arise as a result of the comparison of geometric and algebraic constructions.

We strongly believe that the main results of the current paper reflect a very general principle that asserts the profound geometric or gauge-theoretic origin of various algebraic and combinatorial structures of integrable models in statistical mechanics. The relation of both subjects to the representation theory of affine Lie algebras is a necessary prerequisite of this principle. However, we expect much more; namely, that various specific constructions appearing in integrable models of statistical mechanics that include tensor products, fusion products, branching rules, Bethe ansatz, and the Yang-Baxter equation itself reflect certain geometric facts about Nakajima varieties, Malkin-Nakajima tensor product varieties, various Lagrangian subvarieties, and corresponding gauge theories on commutative and, possibly, noncommutative spaces. The present paper is a small but indicative step toward this vast program.

The paper is organized as follows. In Section 2, we recall the definition of Lusztig's quiver varieties and characterizations of the irreducible components in types A_∞ and $A_n^{(1)}$. We also introduce a version of Lusztig's quiver varieties for the Lie algebra $\widehat{\mathfrak{gl}}_{n+1}$. Section 3 contains the definition of Nakajima's quiver varieties, and the Lie algebra action on a suitable space of constructible functions on these varieties is given in Section 4. In Section 5, we give an enumeration of the irreducible components of the quiver varieties for level one in terms of Young diagrams. We also identify the geometric action of the type A_∞ Lie algebra in the basis enumerated by Young diagrams. In Section 6, we extend the enumeration of the irreducible components of the quiver varieties to arbitrary level, and we establish a match with the indexing of bases of the corresponding representations coming from statistical mechanics. Finally, in Section 7, we compare the weight structure of the bases resulting from quiver varieties and the path realizations of statistical mechanics and make certain of their complete coincidence.

2 Lusztig's quiver varieties

In this section, we recount the explicit description, given in [6], of the irreducible components of Lusztig's quiver variety in the case of types A_∞ and $A_n^{(1)}$; see this reference for details including proofs.

Let I be a set of vertices of the Dynkin graph of a Kac-Moody Lie algebra $\mathfrak{g}$, and let H be the set of pairs consisting of an edge, together with an orientation of it. For $h \in H$, let $\mathrm{in}(h)$ (resp., $\mathrm{out}(h)$) be the incoming (resp., outgoing) vertex of h. We define the involution $^-: H \to H$ to be the function which takes $h \in H$ to the element of H consisting of the same edge with opposite orientation. An *orientation* of our graph is a choice of a subset $\Omega \subset H$ such that $\Omega \cup \bar{\Omega} = H$ and $\Omega \cap \bar{\Omega} = \varnothing$.

Let $\mathcal{V}$ be the category of finite-dimensional I-graded vector spaces $\mathbf{V} = \oplus_{i \in I} \mathbf{V}_i$ over $\mathbb{C}$ with morphisms being linear maps respecting the grading. Then $\mathbf{V} \in \mathcal{V}$ shall denote that $\mathbf{V}$ is an object of $\mathcal{V}$. The dimension of $\mathbf{V} \in \mathcal{V}$ is given by $\boldsymbol{v} = \dim \mathbf{V} = (\dim \mathbf{V}_0, \dots, \dim \mathbf{V}_n)$. We identify this dimension with the element $(\dim \mathbf{V}_0)\alpha_0 + \cdots + (\dim \mathbf{V}_n)\alpha_n$ of the root lattice of $\mathfrak{g}$. Here, the α_i are the simple roots corresponding to the vertices of our quiver (graph with orientation) whose underlying graph is the Dynkin graph of $\mathfrak{g}$.

Given $\mathbf{V} \in \mathcal{V}$, let

$$E_{\mathbf{V}} = \bigoplus_{h \in H} \mathrm{Hom}\left(\mathbf{V}_{\mathrm{out}(h)}, \mathbf{V}_{\mathrm{in}(h)}\right). \tag{2.1}$$

For any subset H' of H, let $E_{\mathbf{V},H'}$ be the subspace of $E_{\mathbf{V}}$ consisting of all vectors $x = (x_h)$ such that $x_h = 0$, whenever $h \notin H'$. The algebraic group $G_{\mathbf{V}} = \prod_i \mathrm{Aut}(\mathbf{V}_i)$ acts on $E_{\mathbf{V}}$ and $E_{\mathbf{V},H'}$ by

$$(g, x) = \left((g_i), (x_h)\right) \longmapsto gx = \left(x'_h\right) = \left(g_{\mathrm{in}(h)} x_h g^{-1}_{\mathrm{out}(h)}\right). \tag{2.2}$$

Define the function $\varepsilon : H \to \{-1, 1\}$ by $\varepsilon(h) = 1$ for all $h \in \Omega$ and $\varepsilon(h) = -1$ for all $h \in \bar{\Omega}$. Let $\mathbf{V} \in \mathcal{V}$. The Lie algebra of $G_{\mathbf{V}}$ is $\mathbf{gl}_{\mathbf{V}} = \prod_i \mathrm{End}(\mathbf{V}_i)$ and it acts on $E_{\mathbf{V}}$ by

$$(a, x) = \big((a_i), (x_h)\big) \longmapsto [a, x] = (x'_h) = \big(a_{\mathrm{in}(h)} x_h - x_h a_{\mathrm{out}(h)}\big). \tag{2.3}$$

Let $\langle \cdot, \cdot \rangle$ be the nondegenerate, G_V-invariant, symplectic form on $\mathbf{E}_V$ with values in $\mathbb{C}$ defined by

$$\langle x, y \rangle = \sum_{h \in H} \varepsilon(h) \operatorname{tr}\big(x_h y_{\bar{h}}\big). \tag{2.4}$$

Note that $\mathbf{E}_V$ can be considered as the cotangent space of $\mathbf{E}_{V,\Omega}$ under this form.

The moment map associated to the G_v-action on the symplectic vector space $\mathbf{E}_V$ is the map $\psi : \mathbf{E}_V \to \mathbf{gl}_V$ with i-component $\psi_i : \mathbf{E}_V \to \operatorname{End} \mathbf{V}_i$ given by

$$\psi_i(x) = \sum_{h \in H,\, \mathrm{in}(h) = i} \varepsilon(h) x_h x_{\bar{h}}. \tag{2.5}$$

Definition 2.1 (see [6]). An element $x \in \mathbf{E}_V$ is said to be *nilpotent* if there exists an $N \geq 1$ such that for any sequence $h'_1, h'_2, \ldots, h'_N$ in H satisfying $\mathrm{out}(h'_1) = \mathrm{in}(h'_2)$ and $\mathrm{out}(h'_2) = \mathrm{in}(h'_3), \ldots, \mathrm{out}(h'_{N-1}) = \mathrm{in}(h'_N)$, the composition $x_{h'_1} x_{h'_2} \cdots x_{h'_N} : \mathbf{V}_{\mathrm{out}(h'_N)} \to \mathbf{V}_{\mathrm{in}(h'_1)}$ is zero.

Definition 2.2 (see [6]). Let Λ_V be the set of all nilpotent elements $x \in \mathbf{E}_V$ such that $\psi_i(x) = 0$ for all $i \in I$.

2.1 Type A_∞

Let $\mathfrak{g}$ be the simple Lie algebra of type A_∞. Let $I = \mathbb{Z}$ be the set of vertices of a graph with the set of oriented edges given by

$$H = \{i \longrightarrow j \mid i, j \in I,\ i - j = 1\} \cup \{i \longleftarrow j \mid i, j \in I,\ i - j = 1\}. \tag{2.6}$$

We define the involution $^{-} : H \to H$ as the function that interchanges $i \to j$ and $i \leftarrow j$. For $h = (i \to j)$, we set $\mathrm{out}(h) = i$ and $\mathrm{in}(h) = j$, and for $h = (i \leftarrow j)$, we set $\mathrm{out}(h) = j$ and $\mathrm{in}(h) = i$. Let Ω be the subset of H consisting of the arrows $i \to j$.

Proposition 2.3 (see[6])**.** The irreducible components of Λ_V are the closures of the conormal bundles of the various G_V-orbits in $\mathbf{E}_{V,\Omega}$. □

Proof. The case where $\mathfrak{g}$ is of type A_n is proven in [6]. The A_∞ case follows by passing to the direct limit. ■

For two integers $k' \leq k$, define $\mathbf{V}_\infty(k', k) \in \mathcal{V}$ to be the vector space with basis $\{e_r \mid k' \leq r \leq k\}$. We require that e_r has degree $r \in I$. Let $x_\infty(k', k) \in \mathbf{E}_{\mathbf{V}_\infty(k',k),\Omega}$

be defined by $x_\infty(k', k) : e_r \mapsto e_{r-1}$ for $k' \le r \le k$, where $e_{k'-1} = 0$. It is clear that $(\mathbf{V}_\infty(k', k), x_\infty(k', k))$ is an indecomposable representation of our quiver. Conversely, any indecomposable finite-dimensional representation $(\mathbf{V}, x)$ of our quiver is isomorphic to some $(\mathbf{V}_\infty(k', k), x_\infty(k', k))$.

Let Z^∞ be the set of all pairs $(k' \le k)$ of integers, and let $\tilde{Z}^\infty$ be the set of all functions $Z^\infty \to \mathbb{N}$ with finite support.

It is easy to see that for $\mathbf{V} \in \mathcal{V}$, the set of $G_\mathbf{V}$-orbits in $\mathbf{E}_{\mathbf{V},\Omega}$ is naturally indexed by the subset $\tilde{Z}^\infty_\mathbf{V}$ of $\tilde{Z}^\infty$ consisting of those $f \in \tilde{Z}^\infty$ such that

$$\sum_{k' \le i \le k} f(k', k) = \dim \mathbf{V}_i, \tag{2.7}$$

for all $i \in I$. Here, the sum is over all $k' \le k$ such that $k' \le i \le k$. Corresponding to a given f is the orbit consisting of all representations isomorphic to a sum of the indecomposable representations $x_\infty(k', k)$, each occuring with multiplicity $f(k', k)$. Denote by $\mathcal{O}_f$ the $G_\mathbf{V}$-orbit corresponding to $f \in \tilde{Z}^\infty_\mathbf{V}$. Let $\mathcal{C}_f$ be the conormal bundle to $\mathcal{O}_f$ and let $\bar{\mathcal{C}}_f$ be its closure. We then have the following proposition.

Proposition 2.4. The map $f \to \bar{\mathcal{C}}_f$ is a one-to-one correspondence between the set $\tilde{Z}^\infty_\mathbf{V}$ and the set of irreducible components of $\Lambda_\mathbf{V}$. □

Proof. This follows immediately from Proposition 2.3. ■

2.2 Type $A_n^{(1)}$

Let $\mathfrak{g}$ be the affine Lie algebra of type $A_n^{(1)}$, that is, the Lie algebra generated by the set of elements E_k, F_k, H_k $(k = 0, 1, \ldots, n)$, and d satisfying the following relations:

$$\begin{aligned} &[E_k, F_l] = \delta_{kl} H_k, \qquad [H_k, E_l] = a_{kl} E_l, \qquad [H_k, F_l] = -a_{kl} F_l, \\ &[d, E_k] = \delta_{k0} E_k, \qquad [d, F_k] = -\delta_{k0} F_k, \\ &\left(\operatorname{ad} E_k\right)^{1-a_{kl}} E_l = 0, \quad \left(\operatorname{ad} F_k\right)^{1-a_{kl}} F_l = 0 \quad \text{for } k \ne l. \end{aligned} \tag{2.8}$$

Here

$$a_{kl} = 2\delta(k, l) - \delta(k, l+1) - \delta(k, l-1), \tag{2.9}$$

where $\delta(k, l) = 1$ if $k \equiv l \bmod (n+1)$ and is equal to zero otherwise.

Let $I = \mathbb{Z}/(n+1)\mathbb{Z}$ be the set of vertices of a graph with the set of oriented edges given by

$$H = \{i \longrightarrow j \mid i, j \in I,\ i - j = 1\} \cup \{i \longleftarrow j \mid i, j \in I,\ i - j = 1\}. \tag{2.10}$$

For two integers $k' \le k$, define $\mathbf{V}(k', k) \in \mathcal{V}$ to be the vector space with basis $\{e_r \mid k' \le r \le k\}$. We require that e_r has degree $i \in I$, where $r \equiv i (\bmod n+1)$. Let $x(k', k) \in \mathbf{E}_{\mathbf{V}(k',k),\Omega}$ be defined by $x(k', k) : e_r \mapsto e_{r-1}$ for $k' \le r \le k$, where $e_{k'-1} = 0$. It is clear that $(\mathbf{V}(k', k), x(k', k))$ is an indecomposable representation of our quiver and that $x(k', k)$ is nilpotent. Also, the isomorphism class of this representation does not change when k' and k are simultaneously translated by a multiple of $n+1$. Conversely, any indecomposable finite-dimensional representation $(\mathbf{V}, x)$ of our quiver, with x nilpotent, is isomorphic to some $(\mathbf{V}(k', k), x(k', k))$, where k' and k are uniquely defined up to a simultaneous translation by a multiple of $n+1$.

Let Z be the set of all pairs $(k' \le k)$ of integers defined up to simultaneous translation by a multiple of $n+1$, and let $\tilde{Z}$ be the set of all functions $Z \to \mathbb{N}$ with finite support.

It is easy to see that for $\mathbf{V} \in \mathcal{V}$, the set of $G_\mathbf{V}$-orbits on the set of nilpotent elements in $\mathbf{E}_{\mathbf{V},\Omega}$ is naturally indexed by the subset $\tilde{Z}_\mathbf{V}$ of $\tilde{Z}$ consisting of those $f \in \tilde{Z}$ such that

$$\sum_{k' \le k} f(k', k) \# \{r \mid k' \le r \le k,\ r \equiv i (\bmod n+1)\} = \dim \mathbf{V}_i, \tag{2.11}$$

for all $i \in I$. Here, the sum is taken over all $k' \le k$, up to simultaneous translation by a multiple of $n+1$. Corresponding to a given f is the orbit consisting of all representations isomorphic to a sum of the indecomposable representations $x(k', k)$, each occuring with multiplicity $f(k', k)$. Denote by $\mathcal{O}_f$ the $G_\mathbf{V}$-orbit corresponding to $f \in \tilde{Z}_\mathbf{V}$.

We say that $f \in \tilde{Z}_\mathbf{V}$ is *aperiodic* if for any $k' \le k$, not all $f(k', k)$ and $f(k'+1, k+1), \dots, f(k'+n, k+n)$ are greater than zero. For any $f \in \tilde{Z}_\mathbf{V}$, let $\mathcal{C}_f$ be the conormal bundle of $\mathcal{O}_f$ and let $\bar{\mathcal{C}}_f$ be its closure.

Proposition 2.5 (see [6]). Let $f \in \tilde{Z}_\mathbf{V}$. The following two conditions are equivalent:

(1) $\mathcal{C}_f$ consists entirely of nilpotent elements;

(2) f is aperiodic. □

Proposition 2.6 (see [6]). The map $f \to \bar{\mathcal{C}}_f$ is a one-to-one correspondence between the set of aperiodic elements in $\tilde{Z}_\mathbf{V}$ and the set of irreducible components of $\Lambda_\mathbf{V}$. □

Proposition 2.7 (see[6]). Let $x' \in \mathbf{E}_{\mathbf{V},\Omega}$ and $x'' \in \mathbf{E}_{\mathbf{V},\bar{\Omega}}$. Then, $\psi_i(x' + x'') = 0$ for all $i \in I$ if and only if x'' is orthogonal with respect to $\langle \cdot, \cdot \rangle$ to the tangent space to the $G_\mathbf{V}$-orbit of x', regarded as a vector subspace of $\mathbf{E}_{\mathbf{V},\Omega}$. □

2.3 $\widehat{\mathfrak{gl}}_{n+1}$ case

Since $\widehat{\mathfrak{gl}}_{n+1}$ is not a Kac-Moody algebra in a strict sense, this case is not covered by Lusztig's theory and requires certain modifications. We preserve the notation of the previous subsection.

Definition 2.8. Let $\tilde{\Lambda}_V$ be the set of all elements $x = x' + x''$, where $x' \in E_{V,\Omega}$ and $x'' \in E_{V,\bar{\Omega}}$, such that x' is nilpotent and $\psi_i(x) = 0$ for all $i \in I$.

For any $f \in \tilde{Z}_V$, we denote by $\mathcal{O}_f$ the corresponding G_V-orbit and by $\mathcal{C}_f$ its conormal bundle.

Proposition 2.9. Let $f \in \tilde{Z}_V$. Then,

(1) $\mathcal{C}_f$ consists entirely of elements of $\tilde{\Lambda}_V$;

(2) $\tilde{\Lambda}_V$ is the union of $\bar{\mathcal{C}}_f$ for all $f \in \tilde{Z}_V$. □

Proof. This follows from Proposition 2.7. ■

Proposition 2.10. The map $f \to \bar{\mathcal{C}}_f$ is a one-to-one correspondence between the set $\tilde{Z}_V$ and the set of irreducible components of Λ_V. □

Proof. This follows easily since the conormal bundles $\mathcal{C}_f$ are irreducible of the same dimension. ■

3 Nakajima's quiver varieties

We introduce here a description of the quiver varieties first presented in [8] in the case of types A_∞ and $A_n^{(1)}$.

Definition 3.1 (see [8]). For $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{Z}^I_{\geq 0}$, choose I-graded vector spaces $\boldsymbol{V}$ and $\boldsymbol{W}$ of graded dimensions $\boldsymbol{v}$ and $\boldsymbol{w}$, respectively. Then, define

$$\Lambda \equiv \Lambda(\boldsymbol{v}, \boldsymbol{w}) = \Lambda_V \times \sum_{i \in I} \mathrm{Hom}\left(\boldsymbol{V}_i, \boldsymbol{W}_i\right). \tag{3.1}$$

Now, suppose that $\boldsymbol{S}$ is an I-graded subspace of $\boldsymbol{V}$. For $x \in \Lambda_V$, we say that $\boldsymbol{S}$ is x-*stable* if $x(\boldsymbol{S}) \subset \boldsymbol{S}$.

Definition 3.2 (see [8]). Let $\Lambda^{st} = \Lambda(\boldsymbol{v}, \boldsymbol{w})^{st}$ be the set of all $(x, j) \in \Lambda(\boldsymbol{v}, \boldsymbol{w})$ satisfying the following condition: if $\boldsymbol{S} = (\boldsymbol{S}_i)$ with $\boldsymbol{S}_i \subset \boldsymbol{V}_i$ is x-stable and $j_i(\boldsymbol{S}_i) = 0$ for $i \in I$, then $\boldsymbol{S}_i = 0$ for $i \in I$.

The group G_V acts on $\Lambda(\boldsymbol{v}, \boldsymbol{w})$ via

$$(g,(x,j)) = \left((g_i),\left((x_h),(j_i)\right)\right) \longmapsto \left(\left(g_{\mathrm{in}(h)} x_h g_{\mathrm{out}(h)}^{-1}\right), j_i g_i^{-1}\right), \tag{3.2}$$

and the stabilizer of any point of $\Lambda(\boldsymbol{v}, \boldsymbol{w})^{\mathrm{st}}$ in G_V is trivial (see [10, Lemma 3.10]). We then make the following definition.

Definition 3.3 (see [8]). Let $\mathcal{L} \equiv \mathcal{L}(\boldsymbol{v}, \boldsymbol{w}) = \Lambda(\boldsymbol{v}, \boldsymbol{w})^{\mathrm{st}}/G_V$.

Let $\operatorname{Irr} \mathcal{L}(\boldsymbol{v}, \boldsymbol{w})$ (resp., $\operatorname{Irr} \Lambda(\boldsymbol{v}, \boldsymbol{w})$) be the set of irreducible components of $\mathcal{L}(\boldsymbol{v}, \boldsymbol{w})$ (resp., $\Lambda(\boldsymbol{v}, \boldsymbol{w})$). Then, $\operatorname{Irr} \mathcal{L}(\boldsymbol{v}, \boldsymbol{w})$ can be identified with

$$\left\{Y \in \operatorname{Irr} \Lambda(\boldsymbol{v}, \boldsymbol{w}) \mid Y \cap \Lambda(\boldsymbol{v}, \boldsymbol{w})^{\mathrm{st}} \neq \varnothing\right\}. \tag{3.3}$$

Specifically, the irreducible components of $\operatorname{Irr} \mathcal{L}(\boldsymbol{v}, \boldsymbol{w})$ are precisely those

$$X_f \stackrel{\mathrm{def}}{=} \left(\left(\bar{\mathcal{C}}_f \times \sum_{i \in I} \operatorname{Hom}\left(\boldsymbol{V}_i, \boldsymbol{W}_i\right)\right) \cap \Lambda(\boldsymbol{v}, \boldsymbol{w})^{\mathrm{st}}\right)/G_V \tag{3.4}$$

which are nonempty.

The following lemma will be used in the sequel.

Lemma 3.4. One has

$$\Lambda^{\mathrm{st}} = \left\{x \in \Lambda \mid \ker x_{i \to i-1} \cap \ker x_{i+1 \leftarrow i} \cap \ker j_i = 0 \ \forall i\right\}. \tag{3.5}$$

□

Proof. Since each $\ker x_{i \to i-1} \cap \ker x_{i+1 \leftarrow i}$ is x-stable, the left-hand side is obviously contained in the right-hand side. Now, suppose that x is an element of the right-hand side. Let $\boldsymbol{S} = (\boldsymbol{S}_i)$ with $\boldsymbol{S}_i \subset \boldsymbol{V}_i$ be x-stable and $j_i(\boldsymbol{S}_i) = 0$ for $i \in I$. Assume that $\boldsymbol{S} \neq 0$. Since all elements of Λ are nilpotent, we can find a minimal value of N such that the condition in Definition 2.1 is satisfied. Then, we can find a $v \in \boldsymbol{S}_i$ for some i and a sequence $h_1', h_2', \ldots, h_{N-1}'$ (empty if $N = 1$) in H such that $\mathrm{out}(h_1') = \mathrm{in}(h_2')$, $\mathrm{out}(h_2') = \mathrm{in}(h_3'), \ldots,$ $\mathrm{out}(h_{N-2}') = \mathrm{in}(h_{N-1}')$, and $v' = x_{h_1'} x_{h_2'} \cdots x_{h_{N-1}'}(v) \neq 0$. Now, $v' \in \boldsymbol{S}_{i'}$ for some $i' \in I$ by the stability of $\boldsymbol{S}$ (hence, $j_{i'}(v') = 0$) and $v' \in \ker x_{i' \to i'-1} \cap \ker x_{i'+1 \to i'}$ by our choice of N. This contradicts the fact that x is an element of the right-hand side. ■

In the case of $\widehat{\mathfrak{gl}}_{n+1}$, we define the varieties $\tilde{\Lambda}(\boldsymbol{v}, \boldsymbol{w})$, $\tilde{\Lambda}(\boldsymbol{v}, \boldsymbol{w})^{\mathrm{st}}$, and $\tilde{\mathcal{L}}(\boldsymbol{v}, \boldsymbol{w})$ by replacing Λ_V by $\tilde{\Lambda}_V$ in the above.

4 The Lie algebra action

We summarize here some results from [8] that will be needed in the sequel; see this reference for more details including proofs. We keep the notation of Sections 2 and 3 (with $\mathfrak{g}$ arbitrary).

Let $w, v, v', v'' \in \mathbb{Z}^I_{\geq 0}$ be such that $v = v' + v''$. Consider the maps

$$\Lambda(v'', 0) \times \Lambda(v', w) \xleftarrow{p_1} \tilde{F}(v, w; v'') \xrightarrow{p_2} F(v, w; v'') \xrightarrow{p_3} \Lambda(v, w), \tag{4.1}$$

where the notation is as follows. A point of $F(v, w; v'')$ is a point $(x, j) \in \Lambda(v, w)$ together with an I-graded, x-stable subspace S of V such that $\dim S = v' = v - v''$. A point of $\tilde{F}(v, w; v'')$ is a point (x, j, S) of $F(v, w; v'')$ together with a collection of isomorphisms $R'_i : V'_i \cong S_i$ and $R''_i : V''_i \cong V_i/S_i$ for each $i \in I$. Then, we define $p_2(x, j, S, R', R'') = (x, j, S)$, $p_3(x, j, S) = (x, j)$, and $p_1(x, j, S, R', R'') = (x'', x', j')$, where x'', x', and j' are determined by

$$\begin{aligned} &R'_{\mathrm{in}(h)} x'_h = x_h R'_{\mathrm{out}(h)} : V'_{\mathrm{out}(h)} \longrightarrow S_{\mathrm{in}(h)}, \\ &j'_i = j_i R'_i : V'_i \longrightarrow W_i, \\ &R''_{\mathrm{in}(h)} x''_h = x_h R''_{\mathrm{out}(h)} : V''_{\mathrm{out}(h)} \longrightarrow V_{\mathrm{in}(h)}/S_{\mathrm{in}(h)}. \end{aligned} \tag{4.2}$$

It follows that x' and x'' are nilpotent.

Lemma 4.1 (see [8, Lemma 10.3]). One has

$$\left(p_3 \circ p_2\right)^{-1}\left(\Lambda(v, w)^{\mathrm{st}}\right) \subset p_1^{-1}\left(\Lambda(v'', 0) \times \Lambda(v', w)^{\mathrm{st}}\right). \tag{4.3}$$

□

Thus, we can restrict (4.1) to Λ^{st}, forget the $\Lambda(v'', 0)$-factor, and consider the quotient by G_V and $G_{V'}$. This yields the diagram

$$\mathcal{L}(v', w) \xleftarrow{\pi_1} \mathcal{F}(v, w; v - v') \xrightarrow{\pi_2} \mathcal{L}(v, w), \tag{4.4}$$

where

$$\mathcal{F}(v, w, v - v') \stackrel{\mathrm{def}}{=} \left\{(x, j, S) \in F(v, w; v - v') \mid (x, j) \in \Lambda(v, w)^{\mathrm{st}}\right\}/G_V. \tag{4.5}$$

Let $M(\mathcal{L}(v, w))$ be the vector space of all constructible functions on $\mathcal{L}(v, w)$. For a subvariety Y of a variety A, let 1_Y denote the function on A which takes the value 1 on Y and 0 elsewhere. Let $\chi(Y)$ denote the Euler characteristic of the algebraic variety Y. Then, for a map π between algebraic varieties A and B, let $\pi_!$ denote the map between the abelian groups of constructible functions on A and B given by

$$\pi_!(\mathbf{1}_Y)(y) = \chi(\pi^{-1}(y) \cap Y), \quad Y \subset A, \tag{4.6}$$

and let π^* be the pullback map from functions on B to functions on A acting as $\pi^* f(y) = f(\pi(y))$. Then, define

$$\begin{aligned} &H_i : M(\mathcal{L}(\boldsymbol{v}, \boldsymbol{w})) \longrightarrow M(\mathcal{L}(\boldsymbol{v}, \boldsymbol{w})); && H_i f = u_i f, \\ &E_i : M(\mathcal{L}(\boldsymbol{v}, \boldsymbol{w})) \longrightarrow M(\mathcal{L}(\boldsymbol{v} - \boldsymbol{e}^i, \boldsymbol{w})); && E_i f = (\pi_1)_!(\pi_2^* f), \\ &F_i : M(\mathcal{L}(\boldsymbol{v} - \boldsymbol{e}^i, \boldsymbol{w})) \longrightarrow M(\mathcal{L}(\boldsymbol{v}, \boldsymbol{w})); && F_i g = (\pi_2)_!(\pi_1^* g). \end{aligned} \tag{4.7}$$

Here

$$\boldsymbol{u} = {}^t(u_0, \ldots, u_n) = \boldsymbol{w} - C\boldsymbol{v}, \tag{4.8}$$

where C is the Cartan matrix of $\mathfrak{g}$ and we are using diagram (4.4) with $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{e}^i$, where $\boldsymbol{e}^i$ is the vector whose components are given by $\boldsymbol{e}^i_{i'} = \delta_{ii'}$.

Now, let φ be the constant function on $\mathcal{L}(0, \boldsymbol{w})$ with value 1. Let $L(\boldsymbol{w})$ be the vector space of functions generated by acting on φ with all possible combinations of the operators F_i. Then, let $L(\boldsymbol{v}, \boldsymbol{w}) = M(\mathcal{L}(\boldsymbol{v}, \boldsymbol{w})) \cap L(\boldsymbol{w})$.

Proposition 4.2 (see [8, Theorem 10.14])**.** The operators E_i, F_i, and H_i on $L(\boldsymbol{w})$ provide the structure of the irreducible highest weight integrable representation of $\mathfrak{g}$ with highest weight $\boldsymbol{w}$. Each summand of the decomposition $L(\boldsymbol{w}) = \bigoplus_{\boldsymbol{v}} L(\boldsymbol{v}, \boldsymbol{w})$ is a weight space with weight $\boldsymbol{w} - C\boldsymbol{v}$. □

Let $X \in \operatorname{Irr} \mathcal{L}(\boldsymbol{v}, \boldsymbol{w})$, and define a linear map $T_X : L(\boldsymbol{v}, \boldsymbol{w}) \to \mathbb{C}$ as in [7]. The map T_X associates to a constructible function $f \in L(\boldsymbol{v}, \boldsymbol{w})$ the (constant) value of f on a suitable open dense subset of X. The fact that $L(\boldsymbol{v}, \boldsymbol{w})$ is finite-dimensional allows us to take such an open set on which *any* $f \in L(\boldsymbol{v}, \boldsymbol{w})$ is constant. So, we have a linear map

$$\Phi : L(\boldsymbol{v}, \boldsymbol{w}) \longrightarrow \mathbb{C}^{\operatorname{Irr} \mathcal{L}(\boldsymbol{v}, \boldsymbol{w})}. \tag{4.9}$$

The following proposition is proved in [7] (slightly generalized in [8, Proposition 10.15]).

Proposition 4.3. The map Φ is an isomorphism; for any $X \in \operatorname{Irr} \mathcal{L}(\boldsymbol{v}, \boldsymbol{w})$, there is a unique function $g_X \in L(\boldsymbol{v}, \boldsymbol{w})$ such that for some open dense subset O of X, $g_X|_O = 1$ and for some closed G_V-invariant subset $K \subset \mathcal{L}(\boldsymbol{v}, \boldsymbol{w})$ of dimension $< \dim \mathcal{L}(\boldsymbol{v}, \boldsymbol{w})$, $g_X = 0$ outside $X \cup K$. The functions g_X for $X \in \operatorname{Irr} \Lambda(\boldsymbol{v}, \boldsymbol{w})$ form a basis of $L(\boldsymbol{v}, \boldsymbol{w})$. □

5 Level-one representations

We now seek to describe the irreducible components of Nakajima's quiver variety. By the comment made in Section 3, it suffices to determine which irreducible components of $\Lambda(\boldsymbol{v}, \boldsymbol{w})$ are not killed by the stability condition. By Definition 3.1 and Lemma 3.4, these are precisely those irreducible components which contain points x such that

$$\dim\left(\ker x_{i\to i-1} \cap \ker x_{i+1\leftarrow i}\right) \le \boldsymbol{w}_i, \quad \forall i. \tag{5.1}$$

We first consider the basic representation of highest weight Λ_0, where $\Lambda_0(\alpha_i) = \delta_{0i}$. This corresponds to $\boldsymbol{w} = \boldsymbol{w}^0$, the vector with zero-component 1 and all other components equal to zero.

5.1 Type A_∞

Consider the case where $\mathfrak{g}$ is of type A_∞. Let $\mathcal{Y}$ be the set of all Young diagrams, that is, the set of all weakly decreasing sequences $[l_1, \dots, l_s]$ of nonnegative integers ($l_j = 0$ for $j > s$). For $Y = [l_1, \dots, l_s] \in \mathcal{Y}$, let A_Y be the set $\{(1-i, l_i - i) \mid 1 \le i \le s\}$.

Theorem 5.1. The irreducible components of $\mathcal{L}(\boldsymbol{v}, \boldsymbol{w}^0)$ are precisely those X_f, where $f \in \tilde{Z}_V^\infty$ such that

$$\left\{(k', k) \mid f(k', k) = 1\right\} = A_Y \tag{5.2}$$

for some $Y \in \mathcal{Y}$ and $f(k', k) = 0$ for $(k', k) \notin A_Y$. Denote the component corresponding to such an f by X_Y. Thus, $Y \leftrightarrow X_Y$ is a natural one-to-one correspondence between the set $\mathcal{Y}$ and the irreducible components of $\cup_{\boldsymbol{v}} \mathcal{L}(\boldsymbol{v}, \boldsymbol{w}^0)$. $\square$

Proof. Consider the two representations $(\mathbf{V}_\infty(k_1', k_1), x_\infty(k_1', k_1))$ and $(\mathbf{V}_\infty(k_2', k_2), x_\infty(k_2', k_2))$ of our oriented graph as described in Section 2, where the basis of $\mathbf{V}_\infty(k_i', k_i)$ is $\{e_r^i \mid k_i' \le r \le k_i\}$. Let W be the conormal bundle to the G_V-orbit through the point

$$x_\Omega = \left(x_h\right)_{h\in\Omega} = x_\infty\left(k_1', k_1\right) \oplus x_\infty\left(k_2', k_2\right) \in \mathbf{E}_{\mathbf{V}_\infty(k_1', k_1)\oplus \mathbf{V}_\infty(k_2', k_2), \Omega}. \tag{5.3}$$

By Proposition 2.7, $x = (x_\Omega, x_{\bar{\Omega}}) = (x_h)_{h\in H}$ is in W if and only if

$$x_{i+1\to i} x_{i+1\leftarrow i} = x_{i\leftarrow i-1} x_{i\to i-1}, \tag{5.4}$$

for all i.

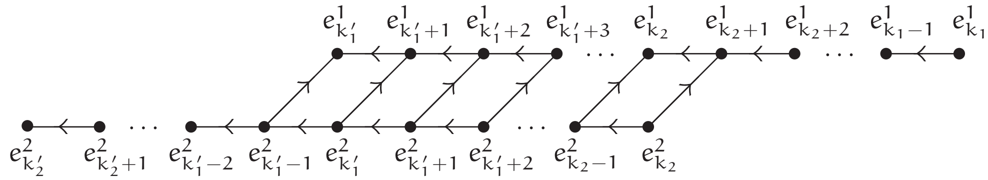

Figure 5.1 If $x_{r+1\leftarrow r}(e^2_r) \neq 0$ for some r, the commutativity of this diagram forces $k'_2 < k'_1$ and $k_2 < k_1$. Vertices represent the spans of the indicated vectors. Those aligned vertically lie in the same $\mathbf{V}_i$. The arrows indicate the action of the obvious component of x.

Let $e^i_r = 0$ for $r < k'_i$ or $r > k_i$. Now, $x_{r+1\leftarrow r}(e^2_r) = c_r e^1_{r+1}$ for some $c_r \in \mathbb{C}$ since $x_{r+1\leftarrow r}(e^2_r)$ can have no e^2_{r+1}-component by nilpotency. Suppose that $k'_1 \leq r+1 \leq k_1$ and $c_r \neq 0$ (i.e., $x_{r+1\leftarrow r}(e^2_r) \neq 0$). Then, if $r+1 > k'_1$,

$$x_{r\leftarrow r-1}\left(e^2_{r-1}\right) = x_{r\leftarrow r-1}x_{r\to r-1}\left(e^2_r\right) = x_{r+1\to r}x_{r+1\leftarrow r}\left(e^2_r\right) = c_r e^1_r \neq 0. \tag{5.5}$$

In particular, $e^2_{r-1} \neq 0$, and so $r-1 \geq k'_2$. Continuing in this manner, we see that $x_{k'_1\leftarrow k'_1-1}(e^2_{k'_1-1}) \neq 0$ and thus $k'_2 < k'_1$.

Now, if $r+1 \leq k_2$, then

$$x_{r+2\to r+1}x_{r+2\leftarrow r+1}\left(e^2_{r+1}\right) = x_{r+1\leftarrow r}x_{r+1\to r}\left(e^2_{r+1}\right) = x_{r+1\leftarrow r}\left(e^2_r\right) \neq 0. \tag{5.6}$$

Therefore, $x_{r+2\leftarrow r+1}(e^2_{r+1}) \neq 0$. But $x_{r+2\leftarrow r+1}(e^2_{r+1})$ must be a multiple of e^1_{r+2} as above. Thus, we must have $r+2 \leq k_1$ and $x_{r+2\leftarrow r+1}(e^2_{r+1}) \neq 0$. Continuing in this manner, we see that $k_2 < k_1$. Refer to Figure 5.1 for illustration.

Now, let x lie in the conormal bundle to the point

$$\bigoplus_{i=1}^{s} x\left(k'_i, k'_i + l_i - 1\right) \in \mathbf{E}_{\oplus_{i=1}^{s} V_\infty(k'_i, k'_i+l_i-1), \Omega}. \tag{5.7}$$

We can assume (by reordering the indices if necessary) that $k'_1 \geq k'_2 \geq \cdots \geq k'_s$. Now, by the above arguments, $x_{r+1\leftarrow r}(e^i_r)$ must be a linear combination of $\{e^j_{r+1}\}_{j<i}$. Thus,

$$e^1_{k'_1} \in \ker x_{k'_1\to k'_1-1} \cap \ker x_{k'_1+1\leftarrow k'_1}. \tag{5.8}$$

By the stability condition, we must then have $k'_1 = 0$, and there can be no other e^i_r in $\ker x_{r\to r-1} \cap \ker x_{r+1\leftarrow r}$ for any r. Now, by the above considerations, $e^2_{k'_2}$ is in $\ker x_{k'_2\to k'_2-1}$ $\cap \ker x_{k'_2+1\leftarrow k'_2}$ unless $k'_2+1 = k'_1$ and $x_{k'_1\leftarrow k'_2}(e^2_{k'_2})$ is a nonzero multiple of $e^1_{k'_1}$. Continuing in this manner, we see that we must have $k'_{i+1}+1 = k'_i$ and $x_{k'_i\leftarrow k'_{i+1}}(e^{i+1}_{k'_{i+1}}) = c_i e^i_{k'_i} \neq 0$

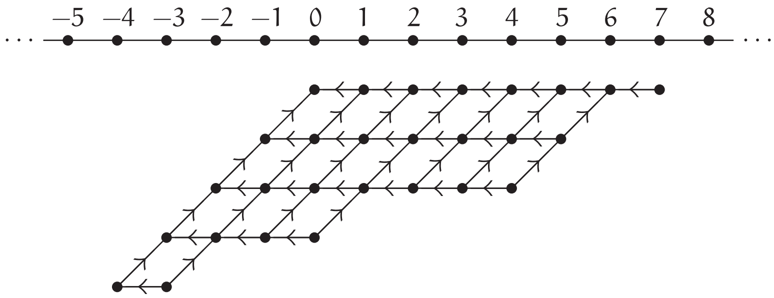

Figure 5.2 The irreducible components of $\mathcal{L}(\boldsymbol{v}, \boldsymbol{w}^0)$ are enumerated by Young diagrams. The top line is the Dynkin graph of type A_∞. The other horizontal lines represent $x_\infty(k', k)$, where k' and k are the positions of the leftmost and rightmost vertices.

for $1 \leq i \leq s-1$. Then, by the above, we must have $k_{i+1} < k_i$ for $1 \leq i \leq s-1$. Setting $l_i = k_i - k'_i + 1$, the theorem follows. ∎

The Young diagrams enumerating the irreducible components of $\mathcal{L}(\boldsymbol{v}, \boldsymbol{w}^0)$ can be visualized as in Figure 5.2. Note that the vertices in our diagram correspond to the boxes in the classical Young diagram, and our arrows intersect the classical diagram edges.

For the level-one A_∞ case, it is relatively easy to compute the geometric action of the generators E_k and F_k of $\mathfrak{g}$. We, first, note that for every $\boldsymbol{v}$, $\mathcal{L}(\boldsymbol{v}, \boldsymbol{w}^0)$ is either empty or is a point. It follows that each X_Y is equal to $\mathcal{L}(\boldsymbol{v}, \boldsymbol{w}^0)$ for some unique $\boldsymbol{v}$, which we will denote $\boldsymbol{v}_Y$.

Lemma 5.2. The function g_{X_Y} corresponding to the irreducible component X_Y, where $Y \in \mathcal{Y}$, is simply $\mathbf{1}_{X_Y}$, the function on X_Y with constant value one. □

Proof. This is obvious since X_Y is a point. ∎

Proposition 5.3. One has $F_k \mathbf{1}_{X_Y} = \mathbf{1}_{X_{Y'}}$, where $\boldsymbol{v}_{Y'} = \boldsymbol{v}_Y + \boldsymbol{e}^k$ if such a Y' exists and $F_k \mathbf{1}_{X_Y} = 0$ otherwise. Also $E_k \mathbf{1}_{X_Y} = \mathbf{1}_{X_{Y''}}$, where $\boldsymbol{v}_{Y''} = \boldsymbol{v}_Y - \boldsymbol{e}^k$ if such a Y'' exists and $E_k \mathbf{1}_{X_Y} = 0$ otherwise. □

Proof. It is clear from the definitions that $F_k \mathbf{1}_{X_Y} = c_1 \mathbf{1}_{X_{Y'}}$ and $E_k \mathbf{1}_{X_Y} = c_2 \mathbf{1}_{X_{Y''}}$ for some constants c_1 and c_2 if Y' and Y'' exist as described above and that these actions are zero otherwise. We simply have to compute the constants c_1 and c_2. Now,

$$F_k \mathbf{1}_{X_Y}(x) = (\pi_2)_! \pi_1^* \mathbf{1}_{X_Y}(x) = \chi\big(\{S \mid S \text{ is } x\text{-stable}, \ x|_S \in X_Y\}\big) = \chi(pt) = 1 \qquad (5.9)$$

if $x \in X_{Y'}$, where $\boldsymbol{v}_{Y'} = \boldsymbol{v}_Y + \boldsymbol{e}^k$ and zero otherwise. The fact that the above set is simply a point follows from the fact that S_k must be the sum of the images of x_h such that $\mathrm{in}(h) = k$. Thus, $c_1 = 1$ as desired.

Note that if there exists a Y' such that $\boldsymbol{v}_{Y'} = \boldsymbol{v}_Y + \boldsymbol{e}^k$, then there cannot exist a Y'' such that $\boldsymbol{v}_{Y''} = \boldsymbol{v}_Y - \boldsymbol{e}^k$ and vice versa. Therefore, if such a Y'' exists, $F_k \mathbf{1}_{X_Y} = 0$, and so

$$H_k \mathbf{1}_{X_Y} = \left[E_k, F_k\right] \mathbf{1}_{X_Y} = -F_k E_k \mathbf{1}_{X_Y}. \tag{5.10}$$

One can easily check that $H_k \mathbf{1}_{X_Y} = -\mathbf{1}_{X_Y}$ if a Y'' exists as described above, and thus $F_k E_k \mathbf{1}_{X_Y} = \mathbf{1}_{X_Y}$. It then follows from the above that we must have $c_2 = 1$. ■

The above action of the type A_∞ Lie algebra in the space spanned by a basis indexed by Young diagrams, is well known in a purely algebraic context (see, e.g., [5]).

Remark 5.4. All the results of this section can be repeated with minor modifications for fundamental representations of finite-dimensional Lie algebras of type A_n. In this case, the bases of fundamental representations will be enumerated by Young diagrams of size bounded by an $m \times (n+1-m)$ rectangle, where $m = 1, 2, \ldots, n$ is the index of the fundamental representation. Note that the same Young diagrams also naturally enumerate the Schubert cells of the Grassmannians $\mathrm{Gr}(m, n+1)$ for type A_n or the semi-infinite Grassmannian for type A_∞.

5.2 Type $A_n^{(1)}$

Let $\mathcal{Y}_n$ be the set of all Young diagrams $[l_1, \ldots, l_s]$ satisfying $l_i > l_{i+n}$ for all $i = 1, \ldots, s$ ($l_j = 0$ for $j > s$). For $Y = [l_1, \ldots, l_s] \in \mathcal{Y}_n$, let A_Y be the set $\{(1-i, l_i - i) \mid 1 \le i \le s\}$.

Theorem 5.5. The irreducible components of $\mathcal{L}(\boldsymbol{v}, \boldsymbol{w}^0)$ are precisely those X_f, where $f \in \tilde{Z}_V$ such that

$$\left\{(k', k) \mid f(k', k) = 1\right\} = A_Y \tag{5.11}$$

for some $Y \in \mathcal{Y}_n$ and $f(k', k) = 0$ for $(k', k) \notin A_Y$ (up to simultaneous translation of k' and k by $n+1$). Denote the component corresponding to such an f by X_Y. Thus, $Y \leftrightarrow X_Y$ is a natural one-to-one correspondence between the set $\mathcal{Y}_n$ and the irreducible components of $\cup_{\boldsymbol{v}} \mathcal{L}(\boldsymbol{v}, \boldsymbol{w}^0)$. □

Proof. The argument is exactly analogous to that used in the proof of Theorem 5.1. We need only to note that a point in the conormal bundle to the orbit through the point

$$\sum_{i=1}^{s} x\big(k_i', k_i' + l_i - 1\big) \in E_{\oplus_{i=1}^{s} V(k_i', k_i'+l_i-1), \Omega} \tag{5.12}$$

lies in $\Lambda_V(\boldsymbol{v}, \boldsymbol{w}^0)$ if and only if $l_i > l_{i+n}$, for all $i = 1, \ldots, s$ ($l_i = 0$ for $i > s$) by the aperiodicity condition. ∎

Note that Nakajima's construction yields an action of the Lie algebra on the basis $\{g_{X_Y}\}_{Y \in \mathcal{Y}_n}$ of the basic representation. However, this action is not as straightforward to compute as in the A_∞ case and will be considered in a future work.

5.3 $\widehat{\mathfrak{gl}}_{n+1}$ case

We define A_Y for $Y \in \mathcal{Y}$ as in Section 5.1.

Theorem 5.6. The irreducible components of $\tilde{\mathcal{L}}(\boldsymbol{v}, \boldsymbol{w}^0)$ are precisely those X_f, where $f \in \tilde{Z}_V$ such that

$$\big\{(k', k) \mid f(k', k) = 1\big\} = A_Y \tag{5.13}$$

for some $Y \in \mathcal{Y}$ and $f(k', k) = 0$ for $(k', k) \notin A_Y$ (up to simultaneous translation of k' and k by $n + 1$). Denote the component corresponding to such an f by X_Y. Thus, $Y \leftrightarrow X_Y$ is a natural one-to-one correspondence between the set $\mathcal{Y}$ and the irreducible components of $\cup_{\boldsymbol{v}} \tilde{\mathcal{L}}(\boldsymbol{v}, \boldsymbol{w}^0)$. □

Proof. The argument is exactly analogous to that used in the proof of Theorem 5.1. ∎

As noted in Section 2.3, since $\widehat{\mathfrak{gl}}_{n+1}$ is not a Kac-Moody algebra, we need to modify Nakajima's construction of highest weight representations. Note that for any n, the difference between $\widehat{\mathfrak{gl}}_{n+1}$ and $\widehat{\mathfrak{sl}}_{n+1}$ is the same Heisenberg algebra $\widehat{\mathfrak{gl}}_1$. The representations of Heisenberg algebras in the context of geometric representation theory were first constructed by Grojnowski [4] and Nakajima [9] (see [11] for a review). However, it is not obvious how to adapt this representation theory to the new quiver varieties $\tilde{\mathcal{L}}(\boldsymbol{v}, \boldsymbol{w}^0)$, obtaining the desired commutation relations with the generators of $\widehat{\mathfrak{sl}}_{n+1}$. This problem will be considered in a future work.

6 Arbitrary level representations

6.1 Type A_∞

We first recall some definitions from [3]. A *Maya diagram* is a bijection $m : \mathbb{Z} \to \mathbb{Z}$ such that $(m(j))_{j<0}$ and $(m(j))_{j\geq 0}$ are both increasing. For each Maya diagram, there exists a unique $\gamma \in \mathbb{Z}$ such that $m(j) - j = \gamma$ for $|j| \gg 0$. This γ is called the *charge* of m. We denote the set of Maya diagrams of charge γ by $\mathcal{M}[\gamma]$. For $m \in \mathcal{M}[\gamma]$, we let

$$m[r] = \big(m(j) + r\big)_{j\in\mathbb{Z}} \in \mathcal{M}[\gamma + r]. \tag{6.1}$$

We can visualize a Maya diagram by a Young diagram. Consider a lattice on the right half plane with lattice points $\{(i, j) \in \mathbb{Z}^2 \mid i \geq 0\}$. Each edge on the lattice is oriented, starting at (i, j) and ending at $(i + 1, j)$ or $(i, j + 1)$ and is numbered by the integer $i + j$. A *path* on the lattice is a map e from $\mathbb{Z}$ to the set of edges on the lattice such that $e(j)$ has number j and the ending site of $e(j)$ is the starting site of $e(j + 1)$. To each Maya diagram of charge γ, we associate the unique path satisfying the following conditions:

(1) for $j \ll 0$, $e(j)$ is the edge from $(0, j)$ to $(0, j + 1)$;

(2) the edge $e(m(j))$ is vertical (resp., horizontal) if $j < 0$ (resp., $j \geq 0$).

Note that these conditions imply that for $j \gg 0$, $e(j)$ is the edge from $(j-\gamma, \gamma)$ to $(j-\gamma+1, \gamma)$. See Figure 6.1.

Such a path divides the right half plane into two components. The upper half is an *infinite Young diagram* $\mathfrak{Y}$, which consists of a quadrant and a (finite) Young diagram Y attached along a horizontal line at height γ. Thus, the set of Maya diagrams are in one-to-one correspondence with the set of pairs (Y, γ), where $Y \in \mathcal{Y}$ and $\gamma \in \mathbb{Z}$.

Lemma 6.1 (see [3])**.** Let $m \in \mathcal{M}[\gamma]$ and $m' \in \mathcal{M}[\gamma']$, and let $\mathfrak{Y}$ and $\mathfrak{Y}'$ be the corresponding infinite Young diagrams. Then, the following conditions are equivalent:

(1) $m(j) \leq m'(j)$ for $j \geq 0$;

(2) $\gamma \leq \gamma'$ and $m(j - \gamma) \geq m'(j - \gamma')$ for $j < \gamma$;

(3) $\mathfrak{Y} \supset \mathfrak{Y}'$. □

We put a partial ordering on the set of Maya diagrams by letting $m \leq m'$ if the conditions in Lemma 6.1 hold.

Let $\Lambda = \Lambda_{\gamma_1} + \cdots \Lambda_{\gamma_l}$, where $\gamma_1 \leq \cdots \leq \gamma_l$ and the Λ_i are fundamental weights of $\mathfrak{g}$. Let $\boldsymbol{w} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$ (i.e., $\boldsymbol{w}$ is a function from $\mathbb{Z}$ to $\mathbb{Z}_{\geq 0}$) be the vector with ith component equal to the number of γ_j equal to i. Let

$$\mathcal{M}[\Lambda] = \mathcal{M}\big[\gamma_1\big] \times \cdots \times \mathcal{M}\big[\gamma_l\big]. \tag{6.2}$$

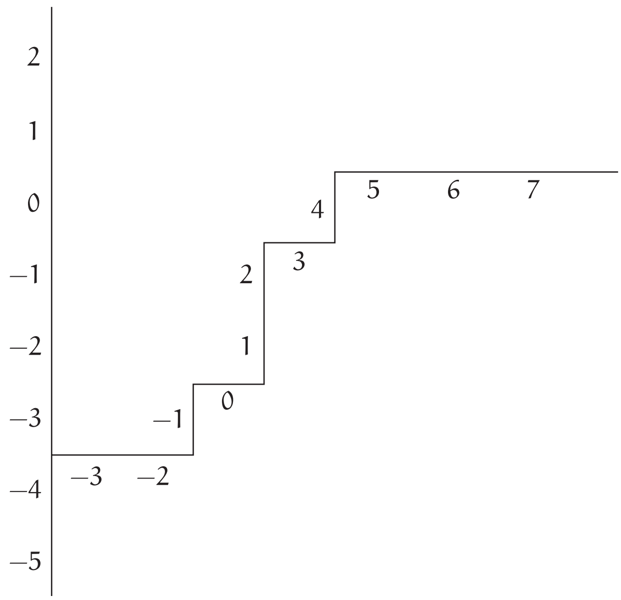

Figure 6.1 The Maya diagram corresponding to $(m(j))_{j\geq 0} = (-3,-2,0,3,5,6,7,\ldots)$ and $(m(j))_{j<0} = (\ldots,-6,-5,-4,,-1,1,2,4)$.

For $Y = [l_1,\ldots,l_s] \in \mathcal{Y}$, let A_Y^γ be the set $\{(\gamma+1-i,\gamma+l_i-i) \mid 1 \leq i \leq s\}$. For $M = ((Y_1,\gamma_1),\ldots,(Y_l,\gamma_l)) \in \mathcal{M}[\Lambda]$, let $A_M = \cup_{i=1}^l A_{Y_i}^{\gamma_i}$ and let $f_M \in \tilde{Z}^\infty$ be the function such that $f(k',k)$ is equal to the number of times (k',k) appears in the set A_M.

Theorem 6.2. The irreducible components of $\mathcal{L}(\boldsymbol{v},\boldsymbol{w})$ are precisely those X_f, where $f = f_M$ for some $M \in \mathcal{M}[\Lambda]$. Denote the component X_{f_M} by X_M. Then $M \leftrightarrow X_M$ is a natural one-to-one correspondence between the set

$$\left\{(m_1,\ldots,m_l) \in \mathcal{M}[\Lambda] \mid m_1 \leq \cdots \leq m_l\right\} \tag{6.3}$$

and the irreducible components of $\cup_{\boldsymbol{v}}\mathcal{L}(\boldsymbol{v},\boldsymbol{w})$. □

Proof. Recall that irreducible components of $\mathcal{L}(\boldsymbol{v},\boldsymbol{w})$ are the closures of the G_V-orbits (or isomorphism classes) in $E_{V,\Omega}$ and that there is a representative of each orbit of the form

$$x = \bigoplus_{(k'\leq k)\in K} x_\infty(k',k) \tag{6.4}$$

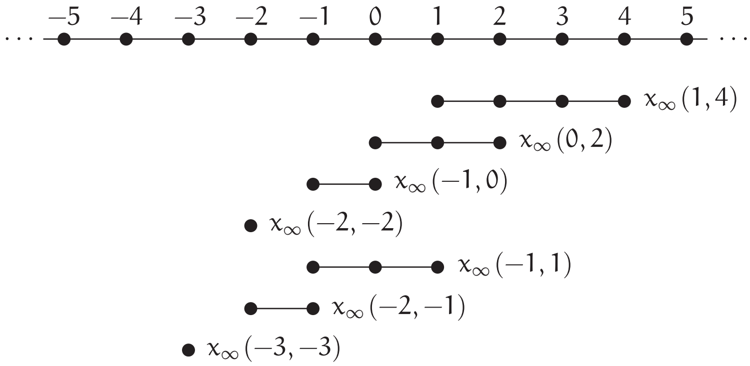

Figure 6.2 The strings associated to some $x \in E_{V,\Omega}$. The top line is the Dynkin diagram of type A_∞.

for some finite set of pairs K. By picturing $x_\infty(k', k)$ as the string of vertices $k', k'+1, \dots, k$, we can represent such an x by a set of finite strings of vertices corresponding to the various $x_\infty(k', k)$ appearing in (6.4). We call the number of vertices in a string its *length*. Each vertex of a string represents a basis vector of $\mathbf{V}$ with degree given by the location of the vertex. The action of x maps each of these basis vectors to the basis vector corresponding to the next (one lower) vertex in the string (or to zero if no such vertex exists). See Figure 6.2.

It is then a straightforward extension of the proof of Theorem 5.1 that the allowable sets of strings are precisely those that can be grouped into subsets, one for each γ_i, such that the subset corresponding to γ_i, when ordered by decreasing leftmost vertex, has weakly decreasing lengths, the first leftmost vertex is γ_i and the leftmost vertices decrease by one as we move through the subset in order (by leftmost, we mean the vertex with the smallest index). This is precisely the first claim of the theorem.

It is easy to see that many different $M \in \mathcal{M}[\Lambda]$ may correspond to the same irreducible component. For example, for $\Lambda = \Lambda_{-1} + \Lambda_1$, both

$$\begin{aligned} M &= \big(([3,2,1],-1),([4,3,2,1],1)\big), \\ M' &= \big(([2,1],-1),([4,3,3,2,1],1)\big) \end{aligned} \tag{6.5}$$

belong to $\mathcal{M}[\Lambda]$ and correspond to the set of strings shown in Figure 6.2 (and hence to the same irreducible component). However, we can associate a unique $M \in \mathcal{M}[\Lambda]$ to each set of strings described above as follows. Associate to γ_1 the longest string with leftmost vertex γ_1 and remove this string from the set. Now, do the same for γ_2 and so forth. After we have associated a string to γ_l, we start again at γ_1, but this time we select the longest string with leftmost vertex $\gamma_1 - 1$, and so forth. If at any point there is no string to associate with a given γ_i, we remove this γ_i from further steps. In this way, we associate to each γ_i a sequence of strings of weakly decreasing length (by our condition on the possible sets of strings) with leftmost vertices decreasing by one. The lengths of the strings associated to γ_i give a Young diagram Y_i, and we set $m_i = (Y_i, \gamma_i)$. By construction, the length of any string associated to γ_i is greater than the length of a string with the same left end point associated to γ_j for $j > i$. This immediately yields the condition $m_1 \leq \cdots \leq m_l$. Our construction thus gives us the one-to-one correspondence asserted in the theorem. ■

Note that the enumeration of the irreducible components given in Theorem 6.2 matches that of [3, Proposition 4.6].

6.2 Type $A_n^{(1)}$

We now consider the case where $\mathfrak{g}$ is of type $A_n^{(1)}$. For an element $M = (m_1, \ldots, m_l) \in \mathcal{M}[\Lambda]$, let R_M be the set (with multiplicity) of pairs (i, l_i), where l_i is the length of a row with top edge having y-coordinate i belonging to one of the m_j. We say that M is n-*reduced* if

$$\{(k+i, l) \mid 0 \leq i \leq n\} \not\subset R_M, \tag{6.6}$$

for all k and l.

Define f_M for $M \in \mathcal{M}[\Lambda]$ as in Section 6.1 (except that now, our pairs are defined only up to simultaneous translation by $n+1$).

Theorem 6.3. The irreducible components of $\mathcal{L}(\boldsymbol{v}, \boldsymbol{w})$ are precisely those X_f, where $f = f_M$ for some n-reduced $M \in \mathcal{M}[\Lambda]$. Denote the component X_{f_M} by X_M. Then, $M \leftrightarrow X_M$ is a natural one-to-one correspondence between the set

$$\{(m_1, \ldots, m_l) \in \mathcal{M}[\Lambda] \mid m_1 \leq \cdots \leq m_l \leq m_1[n+1],\ M \text{ is } n\text{-reduced}\} \tag{6.7}$$

and the irreducible components of $\cup_v \mathcal{L}(\boldsymbol{v}, \boldsymbol{w})$. □

Proof. Each irreducible component corresponds to a set of strings as in the proof of Theorem 6.2 with the added condition that we cannot have $n+1$ strings, each of the same length, whose left endpoints are the $n+1$ vertices of our quiver. That is, we must have that M is n-reduced. Note that the process described in the proof of Theorem 6.2 yields $m_i = (Y_i, \gamma_i)$ satisfying $m_1 \leq \cdots \leq m_l \leq m_1[n+1]$ as desired. The theorem follows. ■

Again, as noted in Section 5.2, Nakajima's construction yields an action of the Lie algebra on the bases $\{g_{X_M}\}$ of the irreducible representations in both the A_∞ and $A_n^{(1)}$ cases, which is more difficult to directly compute than in the level one A_∞ case.

6.3 $\widehat{\mathfrak{gl}}_{n+1}$ case

Theorem 6.4. The irreducible components of $\tilde{\mathcal{L}}(\boldsymbol{v}, \boldsymbol{w})$ are precisely those X_f, where $f = f_M$ for some $M \in \mathcal{M}[\Lambda]$. Denote the component X_{f_M} by X_M. Then, $M \leftrightarrow X_M$ is a natural one-to-one correspondence between the set

$$\{(m_1, \dots, m_l) \in \mathcal{M}[\Lambda] \mid m_1 \le \cdots \le m_l \le m_1[n+1]\} \tag{6.8}$$

and the irreducible components of $\cup_{\boldsymbol{v}} \tilde{\mathcal{L}}(\boldsymbol{v}, \boldsymbol{w})$. □

Proof. The argument is the same as the proof of Theorem 6.3 except that we do not have the aperiodicity condition, and thus do not require that M is n-reduced. ■

Note that the enumeration of the irreducible components of $\cup_{\boldsymbol{v}} \tilde{\mathcal{L}}(\boldsymbol{v}, \boldsymbol{w})$ given by Theorem 6.4 is the same as that given by [3, Proposition 4.7] for a spanning set of the dual to the irreducible highest weight representation of $\widehat{\mathfrak{gl}}_{n+1}$. In order to extend the geometric construction of highest weight representations of $\widehat{\mathfrak{sl}}_{n+1}$ to $\widehat{\mathfrak{gl}}_{n+1}$ for an arbitrary level, one would need a representation of the Heisenberg algebra as discussed in Section 5.3. Here, one might use the construction of the Heisenberg algebra by Baranovsky [1] that generalizes the Grojnowski-Nakajima construction to higher levels.

Remark 6.5. One can also give a geometric interpretation of the full Fock space of [3] with basis indexed by $\mathcal{M}[\Lambda]$ via the "smooth" U_l-instanton moduli space $\sqcup_r \mathcal{M}(r, l)$, which has the same generating function for cohomology (see, e.g., [11, Chapter 5]) as the full Fock space with the basis $\mathcal{M}[\Lambda]$. The types $A_n^{(1)}$ or A_∞ are reflected in the respective action of the groups $\mathbb{Z}/(n+1)\mathbb{Z}$ or $\mathbb{C}^*$ on the moduli space, and $\gamma_1, \dots, \gamma_l$ is the set of one-dimensional representations of these groups that determine this action.

7 A comparison with the path space representation

The authors in [2] constructed the basic representation of $A_n^{(1)}$ on the space of paths. In [3], this path realization is generalized to arbitrary level. We now compare the geometric presentation $L(\boldsymbol{v}, \boldsymbol{w})$ with theirs. We will slightly modify the definitions of [2] to agree with the more general definitions of [3].

7.1 The level one case

A *basic path* is a sequence $p = (\lambda_0, \lambda_1, \dots)$ of integers $\lambda_i \in \{0, 1, \dots, n\}$. The basic path

$$(\bar{j})_{j \geq 0} = (0, 1, \dots, n, 0, 1, \dots, n, \dots) \tag{7.1}$$

is called the *ground state*. Here, $\overline{k}$ for $k \in \mathbb{Z}$ signifies the unique integer such that $0 \leq \overline{k} \leq n$ and $\overline{k} = k \bmod n+1$. Let

$$\mathcal{P}_b = \big\{p = \big(\lambda_0, \lambda_1, \dots\big) \mid \lambda_j = \bar{j} \text{ for all but a finite number of } j\big\}. \tag{7.2}$$

For a basic path $p = (\lambda_0, \lambda_1, \dots) \in \mathcal{P}_b$, let

$$\omega(p) = \sum_{i=1}^{\infty} i\big(H\big(\lambda_i, \lambda_{i+1}\big) - H(\bar{i}, \overline{i+1})\big), \tag{7.3}$$

where

$$H(\lambda, \mu) = \begin{cases} 0 & \text{if } \lambda < \mu, \\ 1 & \text{if } \lambda \geq \mu. \end{cases} \tag{7.4}$$

Basic paths in $\mathcal{P}_b$ can be labeled by Young diagrams as we now describe. The set $\mathcal{M}[0]$ is in one-to-one correspondence with the set of strictly increasing sequences of integers $m = (m(0), m(1), \dots)$ such that $m(j) = j$ for j large and

$$\#\big\{j \mid m(j) < 0\big\} = \#\big(\{0, 1, 2, \dots\} - \big\{m(j) \mid m(j) \geq 0\big\}\big). \tag{7.5}$$

Such a sequence represents the Young diagram of signature $[\cdots 3^{r_3} 2^{r_2} 1^{r_1}]$, where $r_j = m(j) - m(j-1) - 1$ and vice versa. To a Maya diagram $m = (m(0), m(1), \dots) \in \mathcal{Y}_n$, we associate the basic path $p = (\overline{m(0)}, \overline{m(1)}, \dots) \in \mathcal{P}_b$. Then, the ground state corresponds to the empty Young diagram ϕ. In the sequel, we identify n-reduced Young diagrams (i.e., elements of $\mathcal{Y}_n$) and basic paths via the above correspondence.

For $Y = [l_1, \dots, l_s] \in \mathcal{Y}_n$, let

$$\Delta_k(Y) = \delta(k, -s) + \sum_{i=1}^{s} \big(\delta\big(k, l_i - i + 1\big) - \delta\big(k, l_i - i\big)\big). \tag{7.6}$$

Proposition 7.1. One has $H_k g_{X_Y} = \Delta_k(Y) g_{X_Y}$. $\square$

Proof. Let $Y = [l_1, \dots, l_s]$. Then, $g_{X_Y} \in L(\boldsymbol{v}, \boldsymbol{w}^0)$, where

$$\boldsymbol{v} = \dim \bigoplus_{i=1}^{s} \boldsymbol{V}(1-i, l_i - i) = \sum_{i=1}^{s} \sum_{l=1-i}^{l_i - i} \alpha_{\bar{l}}. \tag{7.7}$$

Recall that the weight of the space $L(\boldsymbol{v}, \boldsymbol{w}^0)$ is

$$(u_0, \dots, u_n) = \boldsymbol{w}^0 - C\boldsymbol{v}, \tag{7.8}$$

and thus $H_k g_{X_Y} = u_k g_{X_Y}$ with

$$\begin{aligned} u_k &= \Lambda_0(\alpha_k) - \sum_{i=1}^{s} \sum_{l=1-i}^{l_i-i} \langle \alpha_k, \alpha_{\bar{l}} \rangle \\ &= \delta(k,0) - \sum_{i=1}^{s} \sum_{l=1-i}^{l_i-i} \big(2\delta(k,l) - \delta(k,l-1) - \delta(k,l+1)\big) \\ &= \delta(k,0) - \sum_{i=1}^{s} \big(\delta(k,1-i) - \delta(k,-i) + \delta(k,l_i - i) - \delta(k,l_i - i + 1)\big) \\ &= \delta(k,-s) + \sum_{i=1}^{s} \big(\delta(k,l_i - i + 1) - \delta(k,l_i - i)\big) \\ &= \Delta_k(Y). \end{aligned} \tag{7.9}$$

■

Proposition 7.2. One has $d(g_{X_Y}) = -\omega(Y) g_{X_Y}$. □

Proof. We first compute the left-hand side. It is obvious that

$$d(g_{X_Y}) = -\boldsymbol{v}_0 g_{X_Y}, \tag{7.10}$$

where $X_Y \subset \mathcal{L}(\boldsymbol{v}, \boldsymbol{w}^0)$. Consider the representation $(\boldsymbol{V}(k', k'+l-1), x(k', k'+l-1))$ where $l = (n+1)a + b$ with $0 \le b \le n$. Then,

$$\boldsymbol{v}_0 = \dim \boldsymbol{V}(k', k'+l-1)_0 = a + \begin{cases} 1 & \text{if } \overline{k'-1} + b > n, \\ 0 & \text{if } \overline{k'-1} + b \le n. \end{cases} \tag{7.11}$$

Thus, for $Y = [l_1, \dots, l_s] \in \mathcal{Y}_n$, where $l_i = (n+1)a_i + b_i$ with $0 \le b_i \le n$,

$$\boldsymbol{v}_0 = \sum_{i=1}^{s} \left(a_i + \begin{cases} 1 & \text{if } \overline{-i} + b_i > n \\ 0 & \text{otherwise} \end{cases} \right). \tag{7.12}$$

We now show that this is equal to $\omega(Y)$. Let $Y_i = [l_1, \dots, l_i]$ for $0 \le i \le s$, where $Y_0 = \phi$, and let $(\lambda_0^i, \lambda_1^i, \dots)$ be the corresponding basic path. Then, the first l_i positions of the basic path corresponding to Y_{i-1} are

$$(\overline{1-i}, \overline{1-i}+1, \dots, n, 0, 1, \dots, n, 0, 1, \dots, n, \dots, 0, 1, \dots, n, 0, 1, \dots, \overline{b_i - i}). \qquad (7.13)$$

Here, there are a_i repetitions of $0, 1, \dots, n$ if $\overline{i-1} < b_i$ and $a_i - 1$ repetitions if $\overline{i-1} \ge b_i$.

The first l_i positions of the basic path corresponding to Y_i are simply obtained from the above by lowering all the entries by 1 (interpreting -1 as n). The entries of Y_i and Y_{i-1} numbered $l_i + 1$ and above are equal. Then, by considering the cases $\overline{i-1} < b_i$ and $\overline{i-1} \ge b_i$, we see that

$$\sum_{j=1}^{\infty} j\left(H\left(\lambda_j^i, \lambda_{j+1}^i\right) - H\left(\lambda_j^{i-1}, \lambda_{j+1}^{i-1}\right)\right) = a_i + \begin{cases} 1 & \text{if } \overline{-i} + b_i > n, \\ 0 & \text{otherwise}, \end{cases} \qquad (7.14)$$

and the result follows. ∎

Theorem 7.3. The map $g_{X_Y} \mapsto Y$ is a weight-preserving vector space isomorphism between the geometric presentation $L(\boldsymbol{w}^0)$ of $L(\Lambda_0)$ and the basic path space representation given in [2]. □

Proof. This follows directly from the previous two propositions and the action of the H_i and d given in [2]. ∎

7.2 Arbitrary level

We first recall some definitions from [3]. Let $\epsilon_\mu = (0, \dots, \overset{\mu\text{th}}{1}, \dots, 0)$, for $0 \le \mu \le n$, denote the standard basis vectors of $\mathbb{Z}^{n+1}$. We extend the indices to $\mathbb{Z}$ by setting $\epsilon_{\mu+n+1} = \epsilon_\mu$. Fix a positive integer l (the level of our representation). A *path* is a sequence $\eta = (\eta(k))_{k \ge 0}$ consisting of elements $\eta(k) \in \mathbb{Z}^{n+1}$ of the form $\epsilon_{\mu_1(k)} + \cdots + \epsilon_{\mu_l(k)}$ with $\mu_1(k), \dots, \mu_l(k) \in \mathbb{Z}$. To a level l dominant integral weight $\Lambda = \Lambda_{\gamma_1} + \cdots \Lambda_{\gamma_l}$ is associated the path

$$\eta_\Lambda = \left(\eta_\Lambda(k)\right)_{k \ge 0}, \qquad \eta_\Lambda(k) = \epsilon_{\gamma_1 + k} + \cdots + \epsilon_{\gamma_l + k}. \qquad (7.15)$$

We call η a Λ-path if $\eta(k) = \eta_\Lambda(k)$ for $k \gg 0$. The set of Λ-paths is denoted by $\mathcal{P}(\Lambda)$. Define the weight λ_η of η by

$$\lambda_\eta = \Lambda - \sum_{k \geq 0} \pi\big(\eta(k) - \eta_\Lambda(k)\big) - \omega(\eta)\delta, \tag{7.16}$$

where

$$\omega(\eta) = \sum_{k \geq 1} k\Big(H\big(\eta(k-1), \eta(k)\big) - H\big(\eta_\Lambda(k-1), \eta_\Lambda(k)\big)\Big). \tag{7.17}$$

Here, δ is the null root and π is the $\mathbb{Z}$-linear map from $\mathbb{Z}^{n+1}$ to the weight lattice of the Lie algebra of type $A_n^{(1)}$ such that $\pi(\epsilon_\mu) = \Lambda_{\mu+1} - \Lambda_\mu$ (here, $\Lambda_{n+1} = \Lambda_0$). The function H is defined as follows: if $\alpha = \epsilon_{\mu_1} + \cdots + \epsilon_{\mu_l}$ and $\beta = \epsilon_{\nu_1} + \cdots + \epsilon_{\nu_l}$ $(0 \leq \mu_i, \nu_i \leq n)$, then

$$H(\alpha, \beta) = \min_{\sigma \in S_l} \sum_{i=1}^{l} \theta\big(\mu_i - \nu_{\sigma(i)}\big), \tag{7.18}$$

where S_l is the permutation group on l letters, and

$$\theta(\mu) = \begin{cases} 1 & \text{if } \mu \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7.19}$$

Note that we have redefined the notation ω and H of Section 7.1. However, our new definitions reduce to the old ones in the case $\Lambda = \Lambda_0$ and so, to avoid a proliferation of notation, we denote the new functions by the same symbols.

Let η be a Λ-path. An element $M = (m_1, \ldots, m_l) \in \mathcal{M}[\Lambda]$ is called a *lift* of η if and only if

$$m_1 \leq \cdots \leq m_l \leq m_1[r], \tag{7.20}$$

$$\eta(k) = \epsilon_{m_1(k)} + \cdots + \epsilon_{m_l(k)}. \tag{7.21}$$

If $M = (m_1, \ldots, m_l)$ and $M' = (m_1', \ldots, m_l')$ are lifts of a Λ-path η, then we say $M \geq M'$ if and only if $m_j \geq m_j'$ for $1 \leq j \leq l$.

Recall the definition of R_M given in Section 6.2. For $M, M' \in \mathcal{M}[\Lambda]$, we say that M is an n-*reduction* of M' if R_M is obtained from $R_{M'}$ by the removal of sets of the form $\{(k+i, l) \mid 0 \leq i \leq n\}$ for some k and l.

Proposition 7.4. Suppose that $M = (m_1, \ldots, m_l)$ is an n-reduction of $M' = (m_1', \ldots, m_l')$, $m_1 \leq \cdots \leq m_l \leq m_1[n+1]$, and $m_1' \leq \cdots \leq m_l' \leq m_1'[n+1]$. Then, M and M' are lifts of the same path and $M \geq M'$. $\square$

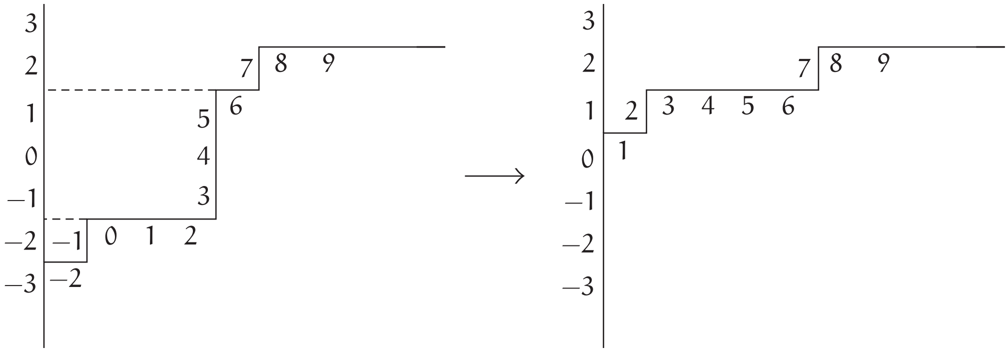

Figure 7.1 Removing an $(n+1) \times l$ square from a Maya diagram (here, $n = 2$ and $l = 4$). Notice that the enumeration of the horizontal edges does not change $\bmod(n+1)$.

Proof. Recall the construction in the proof of Theorem 6.2. Note that choosing arbitrary strings instead of the longest string at each step will not change the values of the right-hand side of (7.21) (for any k). Thus, we form $M'' = (m''_1, \dots, m''_l) \in \mathcal{M}[\Lambda]$ from the same strings comprising M', but where one of the m''_i contains the entire set of strings of the form $\{(k+i, l) \mid 0 \le i \le n\}$, which is removed from $R_{M'}$ to obtain R_M. Now, removing this set of strings from M'' simply amounts to removing this set from m''_i. But, this just cuts an $(n+1) \times l$ square out of the Maya diagram m''_i and shifts the part of the diagram below the cut up $n+1$ units. See Figure 7.1.

Since $\epsilon_{\mu+n+1} = \epsilon_\mu$, the values of right-hand sides of (7.21) for M' and M'' are the same. However, M is simply obtained from M'' by applying the procedure of Theorem 6.2 to the strings of M'', and as mentioned above, this does not change the values of right-hand sides of (7.21). Thus, M and M' are lifts of the same path.

To show that $M \ge M'$, note that by the construction in the proof of Theorem 6.2, M is uniquely determined by R_M. Now, we obtain R_M from $R_{M'}$ by removing a set of the form $\{(k+i, l) \mid 0 \le i \le n\}$ for some k and l. Thus, at each stage in our construction of M, we choose a string of length less than or equal to the string chosen in the construction of M'. Thus, we have that $M \ge M'$. ■

Proposition 7.5 (see [3]). For each Λ-path η, there exists a unique highest lift M of η such that $M \ge M'$ for any lift M' of η. □

Corollary 7.6. The set

$$\left\{M = \left(m_1, \dots, m_l\right) \in \mathcal{M}[\Lambda] \mid M \text{ is } n\text{-reduced},\ m_1 \le \cdots \le m_l \le m_1[n+1]\right\} \quad (7.22)$$

is precisely the set of highest lifts of paths in $\mathcal{P}(\Lambda)$. □

Let M_η be the n-reduced element of $\mathcal{M}[\Lambda]$ corresponding to $\eta \in \mathcal{P}(\Lambda)$, and let $\mathfrak{g}$ be the affine Lie algebra of type $A_n^{(1)}$. Define

$$\mathcal{P}(\Lambda)_\mu = \{\eta \in \mathcal{M}[\Lambda] \mid \lambda_\eta = \mu\}. \tag{7.23}$$

In [3], the authors introduced a basis $\{\xi_\eta \mid \eta \in \mathcal{P}(\Lambda)_\mu\}$ of the μ-weight space of the restricted dual of the highest weight representation of $\mathfrak{g}$ of highest weight Λ [3, Theorem 5.4]. The weight of ξ_η is λ_η [3, Theorem 5.7].

Theorem 7.7. The map $g_{X_{M_\eta}} \mapsto \xi_\eta$ is a weight-preserving vector space isomorphism between the geometric presentation $L(\boldsymbol{w})$ of $L(\Lambda)$ and the path space representation of [3]. □

Proof. The fact that we have a vector space isomorphism follows from Proposition 7.5 and Corollary 7.6. It remains to show that the map is weight-preserving. The definition of a path agrees with the definition of a basic path when $\Lambda = \Lambda_0$, and the weights are the same in this case. Thus, we have the result for $\Lambda = \Lambda_0$ from Section 7.1. Then, the result for arbitrary level-one representations follows easily.

Now, if

$$M_\eta = \big((Y_1, \gamma_1), \dots, (Y_l, \gamma_l)\big) \tag{7.24}$$

and $\mathbf{V}^i$ is the space corresponding to the strings (see the proof of Theorem 6.2) of (Y_i, γ_i) (i.e., its dimension in degree j is equal to the number of vertices of these strings that are numbered j), then the weight of g_{M_η} is

$$\sum_{i=1}^{l} \big(\Lambda_{\gamma_i} - \dim \mathbf{V}^i\big), \tag{7.25}$$

where $\dim \mathbf{V}^i$ is identified with an element of the root lattice as in Section 2. But, this is equal to $\sum_{i=1}^{l} \lambda_{\eta_i}$, where (Y_i, γ_i) is a lift of η_i by the level-one result. By [3, Proposition 5.6], this is λ_η as desired. ■

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