

Geometric and Combinatorial Realizations of Crystals of Enveloping Algebras

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ABSTRACT. Kashiwara and Saito have defined a crystal structure on the set of irreducible components of Lusztig's quiver varieties. This gives a geometric realization of the crystal graph of the lower half of the quantum group associated to a simply-laced Kac-Moody algebra. Using an enumeration of the irreducible components of Lusztig's quiver varieties in finite and affine type A by combinatorial data, we compute the geometrically defined crystal structure in terms of this combinatorics. We conclude by comparing the combinatorial realization of the crystal graph thus obtained with other combinatorial models involving Young tableaux and Young walls.

Introduction

Crystal graphs may be viewed as the $q \rightarrow 0$ limit of the theory of quantum groups. In this limit, representation theory becomes combinatorics. A representation is replaced by its crystal graph, from which important information such as its character can be computed. Other information such as the decomposition of tensor products of representations into irreducible ones can also be determined from the crystal graphs of the representations involved. It is thus an important problem to have concrete realizations of crystal graphs.

There are many purely combinatorial realizations of crystal graphs (see [4]). These include (among others) constructions involving Young tableaux and Young walls, Littelmann's path model, and the Kyoto path realization. In [6], Kashiwara and Saito described a geometric realization of the crystals of the lower half $U_q^-(\mathfrak{g})$ of the quantum group corresponding to a simply-laced Kac-Moody algebra \mathfrak{g} (a q -deformation of the lower half of the enveloping algebra of \mathfrak{g}). In [10], Saito gave a similar realization of the crystals of the irreducible integrable highest-weight representations of these quantum groups. These constructions involve the quiver varieties of Lusztig and Nakajima. More specifically, the underlying set of the crystal is the set of irreducible components of certain varieties associated to quivers and the operators defining the crystal structure are described in a natural geometric way.

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In [11], the author described an explicit isomorphism between the geometric realization of crystal graphs of integrable representations and the purely combinatorial approaches involving Young tableaux and Young walls. In the current paper, we turn our attention to the crystals of the lower half of the quantum groups. Enumerating the irreducible components of the quiver varieties of finite and affine type A by combinatorial data, we describe the geometric crystal action in terms of this data. We also describe the relationship between the combinatorial model thus obtained and recently developed models involving Young tableaux and Young walls.

The geometric approach to crystals has several advantages over purely combinatorial ones. First of all, the quiver variety construction works uniformly for all simply-laced Kac-Moody algebras whereas many combinatorial models are often particular to certain subclasses. For instance, the Young tableaux approach works for the semi-simple Lie algebras and the Young wall approach for the affine Lie algebras. Secondly, the geometric viewpoint often yields alternative (frequently simpler) direct proofs of the combinatorial constructions. Once the irreducible components are enumerated by the desired combinatorial data, one simply computes the action of the crystal operators. We know in advance what crystal we have and thus avoid the need to prove that the combinatorial model yields the crystal we want. Finally, and perhaps most importantly, the theory of quiver varieties allows one to “thaw” the crystals. That is, there is a general geometric procedure for constructing the full representations and not just the $q \rightarrow 0$ limit. In this construction, a basis of the representation in question is naturally enumerated by the irreducible components of the quiver variety. From purely combinatorial constructions, it is often a highly non-trivial task to recover the full representation. However, if we know the relationship between the geometric and combinatorial realizations, we can use the general theory of quiver varieties to thaw the combinatorial crystal.

We expect that the results of this paper can be extended to types other than A . For instance, in [11], types A and D were considered for the crystals of highest-weight integrable representations. However, since the proofs in type A are simpler, we restrict ourselves to this case in the current paper for expository reasons.

The organization of this paper is as follows. In Sections 1 and 2 we recall the definition of Lusztig’s quiver varieties and the crystal structure on their irreducible components. We give a combinatorial enumeration of these components in Section 3 and compute the geometric crystal action in terms of this combinatorics in Section 4. Finally, in Section 5, we compare our combinatorial model to the ones in terms of Young tableaux and Young walls.

1. Lusztig’s quiver variety

In this section, we will recount the description given in [9] of Lusztig’s quiver variety and its irreducible components. See this reference for details, including proofs.

Let I be the set of vertices of the Dynkin graph of a symmetric Kac-Moody Lie algebra \mathfrak{g} and let H be the set of pairs consisting of an edge together with an orientation of it. For $h \in H$, let $\text{in}(h)$ (resp. $\text{out}(h)$) be the incoming (resp. outgoing) vertex of h . We define the involution $\bar{} : H \rightarrow H$ to be the function which takes $h \in H$ to the element of H consisting of the same edge with opposite

orientation. An *orientation* of our graph is a choice of a subset $\Omega \subset H$ such that $\Omega \cup \bar{\Omega} = H$ and $\Omega \cap \bar{\Omega} = \emptyset$.

Let \mathcal{V} be the category of finite-dimensional I -graded vector spaces $\mathbf{V} = \bigoplus_{i \in I} \mathbf{V}_i$ over \mathbb{C} with morphisms being linear maps respecting the grading. Then $\mathbf{V} \in \mathcal{V}$ shall denote that \mathbf{V} is an object of \mathcal{V} .

Given $\mathbf{V} \in \mathcal{V}$, let

$$\mathbf{E}_{\mathbf{V}} = \bigoplus_{h \in H} \text{Hom}(\mathbf{V}_{\text{out}(h)}, \mathbf{V}_{\text{in}(h)}).$$

For any subset $H' \subset H$, let $\mathbf{E}_{\mathbf{V}, H'}$ be the subspace of $\mathbf{E}_{\mathbf{V}}$ consisting of all vectors $x = (x_h)$ such that $x_h = 0$ whenever $h \notin H'$. The algebraic group $G_{\mathbf{V}} = \prod_i GL(\mathbf{V}_i)$ acts on $\mathbf{E}_{\mathbf{V}}$ and $\mathbf{E}_{\mathbf{V}, H'}$ by

$$(g, x) = ((g_i), (x_h)) \mapsto (g_{\text{in}(h)} x_h g_{\text{out}(h)}^{-1}).$$

Define the function $\varepsilon : H \rightarrow \{-1, 1\}$ by $\varepsilon(h) = 1$ for all $h \in \Omega$ and $\varepsilon(h) = -1$ for all $h \in \bar{\Omega}$. Let $\langle \cdot, \cdot \rangle$ be the non-degenerate, $G_{\mathbf{V}}$ -invariant, symplectic form on $\mathbf{E}_{\mathbf{V}}$ with values in \mathbb{C} defined by

$$\langle x, y \rangle = \sum_{h \in H} \varepsilon(h) \text{tr}(x_h y_{\bar{h}}).$$

Note that $\mathbf{E}_{\mathbf{V}}$ can be considered as the cotangent space of $\mathbf{E}_{\mathbf{V}, \Omega}$ under this form.

The moment map associated to the $G_{\mathbf{V}}$ -action on the symplectic vector space $\mathbf{E}_{\mathbf{V}}$ is the map $\psi : \mathbf{E}_{\mathbf{V}} \rightarrow \mathfrak{gl}_{\mathbf{V}} = \prod_i \text{End}(\mathbf{V}_i)$ with i -component $\psi_i : \mathbf{E}_{\mathbf{V}} \rightarrow \text{End } \mathbf{V}_i$ given by

$$\psi_i(x) = \sum_{h \in H, \text{in}(h)=i} \varepsilon(h) x_h x_{\bar{h}}.$$

DEFINITION 1.1 ([9]). *An element $x \in \mathbf{E}_{\mathbf{V}}$ is said to be nilpotent if there exists an $N \geq 1$ such that for any sequence h_1, h_2, \dots, h_N in H satisfying $\text{out}(h_1) = \text{in}(h_2)$, $\text{out}(h_2) = \text{in}(h_3)$, \dots , $\text{out}(h_{N-1}) = \text{in}(h_N)$, the composition $x_{h_1} x_{h_2} \dots x_{h_N} : \mathbf{V}_{\text{out}(h_N)} \rightarrow \mathbf{V}_{\text{in}(h_1)}$ is zero.*

DEFINITION 1.2 ([9]). *Let $\Lambda_{\mathbf{V}}$ be the set of all nilpotent elements $x \in \mathbf{E}_{\mathbf{V}}$ such that $\psi_i(x) = 0$ for all $i \in I$.*

We call $\Lambda_{\mathbf{V}}$ *Lusztig's quiver variety*. Since $\Lambda_{\mathbf{V}}$ depends, up to isomorphism, only on the graded dimension of \mathbf{V} , we define

$$\Lambda(\mathbf{v}) = \Lambda_{\mathbf{V}^{\mathbf{v}}}, \quad \text{where } (\mathbf{V}^{\mathbf{v}})_i = \mathbb{C}^{\mathbf{v}_i}.$$

2. Crystal action on quiver varieties

In this section, we review the realization of the crystal graph of integrable highest-weight representations of a Kac-Moody algebra \mathfrak{g} with symmetric Cartan matrix via quiver varieties. For details, we refer the reader to [6].

Let $\mathbf{v}, \bar{\mathbf{v}}, \mathbf{v}' \in (\mathbb{Z}_{\geq 0})^I$ be such that $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'$. Consider the maps

$$(2.1) \quad \Lambda(\mathbf{v}') \times \Lambda(\bar{\mathbf{v}}) \xleftarrow{p_1} \tilde{\Lambda}(\mathbf{v}', \bar{\mathbf{v}}) \xrightarrow{p_2} \Lambda(\mathbf{v}).$$

Here $\tilde{\Lambda}(\mathbf{v}', \bar{\mathbf{v}})$ is the variety of tuples $(x, \phi', \bar{\phi})$ where $x \in \Lambda(\mathbf{v})$ and $\phi' = (\phi'_i)_{i \in I}$, $\bar{\phi} = (\bar{\phi}_i)_{i \in I}$ give an exact sequence

$$0 \rightarrow V^{\mathbf{v}'} \xrightarrow{\phi'} V^{\mathbf{v}} \xrightarrow{\bar{\phi}} V^{\bar{\mathbf{v}}} \rightarrow 0$$

such that $\text{Im } \phi'$ is x -stable, that is, $x_h((\text{Im } \phi')_{\text{out}(h)}) \subseteq (\text{Im } \phi')_{\text{in}(h)}$ for all $h \in H$. Then x induces $x' \in \Lambda(\mathbf{v}')$ and $\bar{x} \in \Lambda(\bar{\mathbf{v}})$. Note that we have used the fact that x is nilpotent if and only if the induced x' and \bar{x} are. The maps in (2.1) are defined by $p_1(x, \phi', \bar{\phi}) = (x', \bar{x})$, $p_2(x, \phi', \bar{\phi}) = x$.

For $i \in I$, define $\varepsilon_i : \Lambda(\mathbf{v}) \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\varepsilon_i(x) = \dim_{\mathbb{C}} \text{Coker} \left(\bigoplus_{h: \text{in}(h)=i} V_{\text{out}(h)} \xrightarrow{(x_h)} V_i \right).$$

Then, for $c \in \mathbb{Z}_{\geq 0}$, let

$$\Lambda(\mathbf{v})_{i,c} = \{x \in \Lambda(\mathbf{v}) \mid \varepsilon_i(x) = c\}.$$

It is easily seen that $\Lambda(\mathbf{v})_{i,c}$ is a locally closed subvariety of $\Lambda(\mathbf{v})$.

From now on, let $\bar{\mathbf{v}} = c\mathbf{e}^i$, where $\mathbf{e}_j^i = \delta_{ij}$. It is obvious that $\Lambda(c\mathbf{e}^i) = \{0\}$ is a point. Then (2.1) becomes

$$\Lambda(\mathbf{v} - c\mathbf{e}^i) \cong \Lambda(\mathbf{v} - c\mathbf{e}^i) \times \Lambda(c\mathbf{e}^i) \xleftarrow{p_1} \tilde{\Lambda}(\mathbf{v} - c\mathbf{e}^i, c\mathbf{e}^i) \xrightarrow{p_2} \Lambda(\mathbf{v}).$$

Assume $\Lambda(\mathbf{v})_{i,c} \neq \emptyset$. Then

$$p_1^{-1}(\Lambda(\mathbf{v} - c\mathbf{e}^i)_{i,0}) = p_2^{-1}(\Lambda(\mathbf{v})_{i,c}).$$

Let

$$\tilde{\Lambda}(\mathbf{v} - c\mathbf{e}^i, c\mathbf{e}^i)_{i,0} = p_1^{-1}(\Lambda(\mathbf{v} - c\mathbf{e}^i)_{i,0}) = p_2^{-1}(\Lambda(\mathbf{v})_{i,c}).$$

We then have the following diagram.

$$(2.2) \quad \Lambda(\mathbf{v} - c\mathbf{e}^i)_{i,0} \xleftarrow{p_1} \tilde{\Lambda}(\mathbf{v} - c\mathbf{e}^i, c\mathbf{e}^i)_{i,0} \xrightarrow{p_2} \Lambda(\mathbf{v})_{i,c}$$

LEMMA 2.1 ([6]). (1) *The map $p_2 : \tilde{\Lambda}(\mathbf{v} - c\mathbf{e}^i, c\mathbf{e}^i)_{i,0} \rightarrow \Lambda(\mathbf{v})_{i,c}$ is a principal fiber bundle with fiber $GL(\mathbb{C}^c) \times \prod_{i \in I} GL(V_i^{\mathbf{v}'})$.*

(2) *The map $p_1 : \tilde{\Lambda}(\mathbf{v} - c\mathbf{e}^i, c\mathbf{e}^i)_{i,0} \rightarrow \Lambda(\mathbf{v} - c\mathbf{e}^i)_{i,0}$ is smooth with fiber a connected rational variety.*

COROLLARY 2.2. *Suppose $\Lambda(\mathbf{v})_{i,c} \neq \emptyset$. Then there is a 1-1 correspondence between the set of irreducible components of $\Lambda(\mathbf{v} - c\mathbf{e}^i)_{i,0}$ and the set of irreducible components of $\Lambda(\mathbf{v})_{i,c}$.*

Let $B(\mathbf{v}, \infty)$ denote the set of irreducible components of $\Lambda(\mathbf{v})$ and let $B_g(\infty) = \bigsqcup_{\mathbf{v}} B(\mathbf{v}, \infty)$. For $X \in B(\mathbf{v}, \infty)$, let $\varepsilon_i(X) = \varepsilon_i(x)$ for a generic point $x \in X$. Then for $c \in \mathbb{Z}_{\geq 0}$ define

$$B(\mathbf{v}, \infty)_{i,c} = \{X \in B(\mathbf{v}, \infty) \mid \varepsilon_i(X) = c\}.$$

Then by Corollary 2.2, $B(\mathbf{v} - c\mathbf{e}^i, \infty)_{i,0} \cong B(\mathbf{v}, \infty)_{i,c}$.

Suppose that $\bar{X} \in B(\mathbf{v} - c\mathbf{e}^i, \infty)_{i,0}$ corresponds to $X \in B(\mathbf{v}, \infty)_{i,c}$ by the above isomorphism. Then we define maps

$$\begin{aligned} \tilde{f}_i^c : B(\mathbf{v} - c\mathbf{e}^i, \mathbf{w})_{i,0} &\rightarrow B(\mathbf{v}, \mathbf{w})_{i,c}, & \tilde{f}_i^c(\bar{X}) &= X, \\ \tilde{e}_i^c : B(\mathbf{v}, \mathbf{w})_{i,c} &\rightarrow B(\mathbf{v} - c\mathbf{e}^i, \mathbf{w})_{i,0}, & \tilde{e}_i^c(X) &= \bar{X}. \end{aligned}$$

We then define the maps

$$\tilde{e}_i, \tilde{f}_i : B_g(\infty) \rightarrow B_g(\infty) \sqcup \{0\}$$

by

$$(2.3) \quad \tilde{e}_i : B(\mathbf{v}, \infty)_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v} - c\mathbf{e}^i, \infty)_{i,0} \xrightarrow{\tilde{f}_i^{c-1}} B(\mathbf{v} - \mathbf{e}^i, \infty)_{i,c-1},$$

$$(2.4) \quad \tilde{f}_i : B(\mathbf{v}, \infty)_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v} - c\mathbf{e}^i, \infty)_{i,0} \xrightarrow{\tilde{f}_i^{c+1}} B(\mathbf{v} + \mathbf{e}^i, \infty)_{i,c+1}.$$

We set $\tilde{e}_i(X) = 0$ for $X \in B(\mathbf{v}, \infty)_{i,0}$. Note that $B(\mathbf{v}, \infty)_{i,c} \neq \emptyset$ implies $B(\mathbf{v}, \infty)_{i,c+1} \neq \emptyset$ and thus $\tilde{f}_i(X)$ is never zero for $X \in B(\mathbf{v}, \infty)_{i,c}$. We also define

$$\begin{aligned} \text{wt} : B_g(\infty) &\rightarrow P, & \text{wt}(X) &= - \sum_{i \in I} \mathbf{v}_i \alpha_i \text{ for } X \in B(\mathbf{v}, \infty), \\ \varphi_i(X) &= \varepsilon_i(X) + \langle h_i, \text{wt}(X) \rangle. \end{aligned}$$

Here the α_i are the simple roots of \mathfrak{g} .

PROPOSITION 2.3 ([6]). *$B_g(\infty)$ is a crystal and is isomorphic to the crystal $B(\infty)$ of $U_q^-(\mathfrak{g})$.*

3. Enumeration of components

In much the same way that Lusztig’s quiver varieties yield a geometric construction of the crystal graphs of $U_q^-(\mathfrak{g})$, Nakajima’s quiver varieties provide one with a geometric realization of the crystal graphs of the irreducible integrable highest-weight representations of $U_q(\mathfrak{g})$ (see [10]). In [11], an enumeration of the irreducible components of Nakajima’s quiver varieties for finite and affine types A and D were given in terms of Young tableaux, Young walls and new objects called Young pyramids. In this section, we describe a natural enumeration of the irreducible components of Lusztig’s quiver varieties in finite and affine type A .

3.1. Type A_n . In this subsection, let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ be the simple Lie algebra of type A_n . A key step in the enumeration of the irreducible components of $\Lambda_{\mathbf{V}}$ is the following.

PROPOSITION 3.1 ([9]). *For \mathfrak{g} a symmetric Lie algebra of finite type (in particular, for $\mathfrak{g} = \mathfrak{sl}_{n+1}$), the irreducible components of $\Lambda_{\mathbf{V}}$ are the closures of the conormal bundles of the various $G_{\mathbf{V}}$ -orbits in $\mathbf{E}_{\mathbf{V},\Omega}$.*

By Proposition 3.1, it suffices to enumerate the $G_{\mathbf{V}}$ -orbits in $\mathbf{E}_{\mathbf{V},\Omega}$. But these are simply the isomorphism classes of the quiver (I, Ω) . By Gabriel’s Theorem, these are in one-to-one correspondence with the positive roots of \mathfrak{g} . We can describe the isomorphism classes explicitly as follows.

Let $I = \{1, \dots, n\}$ be the set of vertices of the Dynkin graph of \mathfrak{g} with the set of oriented edges given by

$$H = \{h_{i,j} \mid i, j \in I, |i - j| = 1\}.$$

For two adjacent vertices i and j , $h_{i,j}$ is the oriented edge from vertex i to vertex j . Thus $\text{out}(h_{i,j}) = i$ and $\text{in}(h_{i,j}) = j$. We define the involution $\bar{\cdot} : H \rightarrow H$ as the function that interchanges $h_{i,j}$ and $h_{j,i}$. Let $\Omega = \{h_{i,i-1} \mid 2 \leq i \leq n\}$.

For two integers k, l such that $1 \leq k \leq l \leq n$, define $\mathbf{V}(k, l) \in \mathcal{V}$ to be the vector space with basis $\{e_r \mid k \leq r \leq l\}$. We require that e_r has degree $r \in I$. Let $x(k, l) \in \mathbf{E}_{\mathbf{V}(k,l),\Omega}$ be defined by $x(k, l) : e_r \mapsto e_{r-1}$ for $k \leq r \leq l$, where $e_{k-1} = 0$. It is clear that $(\mathbf{V}(k, l), x(k, l))$ is an indecomposable representation of our quiver (i.e. element of $\mathbf{E}_{\mathbf{V},\Omega}$). Conversely, any indecomposable finite-dimensional

representation (\mathbf{V}, x) of our quiver is isomorphic to some $(\mathbf{V}(k, l), x(k, l))$. The correspondence guaranteed by Gabriel’s theorem is given by

$$(\mathbf{V}(k, l), x(k, l)) \leftrightarrow \sum_{i=k}^l \alpha_i.$$

Let Z be the set of all pairs (k, l) of integers such that $1 \leq k \leq l \leq n$ and let \tilde{Z} be the set of all functions $Z \rightarrow \mathbf{N}$ with finite support.

It is easy to see that for $\mathbf{V} \in \mathcal{V}$, the set of $G_{\mathbf{V}}$ -orbits in $\mathbf{E}_{\mathbf{V}, \Omega}$ is naturally indexed by the subset $\tilde{Z}_{\mathbf{V}}$ of \tilde{Z} consisting of those $\gamma \in \tilde{Z}$ such that

$$\sum_{(k, l) \in Z : k \leq i \leq l} \gamma(k, l) = \dim \mathbf{V}_i$$

for all $i \in I$. Corresponding to a given γ is the orbit consisting of all representations isomorphic to a sum of the indecomposable representations $(V(k, l), x(k, l))$, each occurring with multiplicity $\gamma(k, l)$. Denote by \mathcal{O}_{γ} the $G_{\mathbf{V}}$ -orbit corresponding to $\gamma \in \tilde{Z}_{\mathbf{V}}$. Let \mathcal{C}_{γ} be the conormal bundle to \mathcal{O}_{γ} and let $\bar{\mathcal{C}}_{\gamma}$ be its closure. We then have the following proposition.

PROPOSITION 3.2. *The map $\gamma \mapsto \bar{\mathcal{C}}_{\gamma}$ is a one-to-one correspondence between the set $\tilde{Z}_{\mathbf{V}}$ and the set of irreducible components of $\Lambda_{\mathbf{V}}$.*

PROOF. This follows immediately from Proposition 3.1. □

3.2. Type $A_n^{(1)}$. In this subsection, we let $\mathfrak{g} = \widehat{\mathfrak{sl}}_{n+1}$ be the affine Lie algebra of type $A_n^{(1)}$. Let $I = \mathbb{Z}/(n+1)\mathbb{Z}$ be the set of vertices of the Dynkin graph of \mathfrak{g} with the set of oriented edges given by

$$H = \{h_{i,j} \mid i, j \in I, i - j \equiv \pm 1 \pmod{n+1}\}.$$

For two adjacent vertices i and j , $h_{i,j}$ is the oriented edge from vertex i to vertex j . Thus $\text{out}(h_{i,j}) = i$ and $\text{in}(h_{i,j}) = j$. We define the involution $\bar{} : H \rightarrow H$ as the function that interchanges $h_{i,j}$ and $h_{j,i}$. Let $\Omega = \{h_{i,i-1} \mid i \in I\}$.

For two integers $k \leq l$, define $\mathbf{V}(k, l) \in \mathcal{V}$ to be the vector space with basis $\{e_r \mid k \leq r \leq l\}$. We require that e_r has degree $r \pmod{n+1}$. Let $x(k, l) \in \mathbf{E}_{\mathbf{V}(k, l), \Omega}$ be defined by $x(k, l) : e_r \mapsto e_{r-1}$ for $k \leq r \leq l$, where $e_{k-1} = 0$. It is clear that $(\mathbf{V}(k, l), x(k, l))$ is an indecomposable representation of our quiver and that $x(k, l)$ is nilpotent. Also, the isomorphism class of this representation does not change when k and l are simultaneously translated by a multiple of $n+1$. Conversely, any indecomposable finite-dimensional representation (\mathbf{V}, x) of our quiver, with x nilpotent, is isomorphic to some $(\mathbf{V}(k, l), x(k, l))$ where k and l are uniquely defined up to a simultaneous translation by a multiple of $n+1$.

Let Z be the set of all pairs $(k \leq l)$ of integers defined up to simultaneous translation by a multiple of $n+1$ and let \tilde{Z} be the set of all functions $Z \rightarrow \mathbf{N}$ with finite support.

It is easy to see that for $\mathbf{V} \in \mathcal{V}$, the set of $G_{\mathbf{V}}$ -orbits on the set of nilpotent elements in $\mathbf{E}_{\mathbf{V}, \Omega}$ is naturally indexed by the subset $\tilde{Z}_{\mathbf{V}}$ of \tilde{Z} consisting of those $\gamma \in \tilde{Z}$ such that

$$\sum_{k \leq l} \gamma(k, l) \cdot \#\{r \mid k \leq r \leq l, r \equiv i \pmod{n+1}\} = \dim \mathbf{V}_i$$

for all $i \in I$. Here the sum is taken over all $k \leq l$ up to simultaneous translation by a multiple of $n+1$. Corresponding to a given γ is the orbit consisting of all representations isomorphic to a sum of the indecomposable representations $(V(k, l), x(k, l))$, each occurring with multiplicity $\gamma(k, l)$. Denote by \mathcal{O}_γ the $G_{\mathbf{V}}$ -orbit corresponding to $\gamma \in \tilde{Z}_{\mathbf{V}}$.

We say that $\gamma \in \tilde{Z}_{\mathbf{V}}$ is *aperiodic* if for any $k \leq l$, not all $\gamma(k, l), \gamma(k+1, l+1), \dots, \gamma(k+n, l+n)$ are greater than zero. For any $\gamma \in \tilde{Z}_{\mathbf{V}}$, let \mathcal{C}_γ be the conormal bundle of \mathcal{O}_γ and let $\bar{\mathcal{C}}_\gamma$ be its closure.

PROPOSITION 3.3 ([9, 15.5]). *Let $\gamma \in \tilde{Z}_{\mathbf{V}}$. The following two conditions are equivalent.*

- (1) \mathcal{C}_γ consists entirely of nilpotent elements.
- (2) γ is aperiodic.

PROPOSITION 3.4 ([9, 15.6]). *The map $\gamma \rightarrow \bar{\mathcal{C}}_\gamma$ is a 1-1 correspondence between the set of aperiodic elements in $\tilde{Z}_{\mathbf{V}}$ and the set of irreducible components of $\Lambda_{\mathbf{V}}$.*

4. Combinatorial crystal graphs

In Section 3, we enumerated the irreducible components of Lusztig’s quiver varieties for finite and affine type A . As noted in Section 2, there is a geometrically defined crystal structure on the set of these irreducible components. We now undertake the task of giving an explicit description of this action in terms of the combinatorial data enumerating irreducible components. The arguments are similar to those used in [11].

4.1. Type A_n . In this subsection, we consider the case where $\mathfrak{g} = \mathfrak{sl}_{n+1}$. For $i \in I = \{1, \dots, n\}$, we define the *i -signature* of $\gamma \in \tilde{Z}$ as follows. Consider the ordering on pairs (k, l) , $1 \leq k \leq l \leq n$, given by

$$(k, l) < (k', l') \text{ if } k < k' \text{ or if } k = k' \text{ and } l > l'.$$

This is the lexicographic ordering where we reverse the usual order on \mathbb{Z} in the second factor. Now, write all the pairs (k, l) in the domain of γ from left to right in increasing order. Below pairs of the form $(k, i-1)$, write $\gamma(k, i-1)$ +’s and below pairs of the form (k, i) , write $\gamma(k, i)$ -’s. Now, from this list of +’s and -’s, cancel all $(+, -)$ pairs. That is, delete a + and - if they are adjacent in this list, with the + to the left, and continue this process until no such configuration exists. What is left is a (possibly empty) sequence of -’s followed by a (possibly empty) sequence of +’s. We call this the *i -signature* of γ .

Let $(k', i-1)$ be the pair corresponding to the leftmost + in the i -signature of γ (that is, the pair under which this + was placed in the above procedure). If there is no + in the i -signature of γ , let $k' = i$. Then define $\gamma^{i,+}$ by

$$(4.1) \quad \gamma^{i,+}(k', i) = \gamma(k', i) + 1,$$

$$(4.2) \quad \gamma^{i,+}(k', i-1) = \gamma(k', i-1) - 1,$$

$$(4.3) \quad \gamma^{i,+}(k, l) = \gamma(k, l), \text{ for } k \neq k' \text{ or } l \neq i, i+1.$$

In the case that $k' = i$, we ignore equation (4.2).

If there is at least one $-$ in the i -signature of γ , let (k'', i) be the pair corresponding to the rightmost $-$. Then define $\gamma^{i,-}$ by

$$(4.4) \quad \gamma^{i,+}(k'', i) = \gamma(k'', i) - 1,$$

$$(4.5) \quad \gamma^{i,+}(k'', i - 1) = \gamma(k'', i - 1) + 1,$$

$$(4.6) \quad \gamma^{i,+}(k, l) = \gamma(k, l), \text{ for } k \neq k'' \text{ or } l \neq i, i - 1.$$

In the case that $k' = i$, we ignore equation (4.5).

THEOREM 4.1. *For $\gamma \in \tilde{Z}$, let $X_\gamma \in B_g(\infty)$ be the element of the crystal corresponding to the irreducible component $\bar{\mathcal{C}}_\gamma$. Then*

$$(4.7) \quad \varepsilon_i(X_\gamma) = \#\{-\text{'s in the } i\text{-signature of } \gamma\},$$

$$(4.8) \quad \text{wt}(X_\gamma) = - \sum_{k \leq i \leq l} f(k, l)\alpha_i,$$

$$(4.9) \quad \varphi_i(X_\gamma) = \varepsilon_i(X_\gamma) + \langle h_i, \text{wt}(X_\gamma) \rangle,$$

$$(4.10) \quad \tilde{e}_i(X_\gamma) = \begin{cases} X_{\gamma^{i,-}} & \text{if } \varepsilon_i(X_\gamma) > 0, \\ 0 & \text{if } \varepsilon_i(X_\gamma) = 0 \end{cases},$$

$$(4.11) \quad \tilde{f}_i(X_\gamma) = X_{\gamma^{i,+}}.$$

The remainder of this section is devoted to the proof of Theorem 4.1.

Consider the irreducible component $\bar{\mathcal{C}}_\gamma$ of $\Lambda_{\mathbf{V}}$. This is the closure of the conormal bundle to the orbit \mathcal{O}_γ . Recall that for a point $x \in \mathcal{O}_\gamma$, \mathbf{V} decomposes into sums of copies of $\mathbf{V}(k, l)$ for various $1 \leq k \leq l \leq n$ corresponding to the decomposition of x into irreducible quiver representations $x(k, l)$. Each copy of the form $\mathbf{V}(k, i)$ corresponds to a $-$ in the construction of the i -signature of γ (before the $(+, -)$ cancelation was performed).

LEMMA 4.2. *For a generic point of the irreducible component $\bar{\mathcal{C}}_\gamma$, $\gamma \in \tilde{Z}_{\mathbf{V}}$, the image of $\oplus_{h: \text{in}(h)=i} x_h$ is spanned by the degree i vectors of those $\mathbf{V}(k, l)$, $k \leq i < l$, appearing in the decomposition of \mathbf{V} and the degree i vectors of those $\mathbf{V}(k, i)$ appearing in this decomposition which correspond to $-$'s that were canceled in the formation of the i -signature of γ .*

PROOF. The proof is almost identical to the proofs of Lemma 6.1 and Proposition 6.2 of [11] and thus is omitted. \square

PROOF OF THEOREM 4.1. It follows from Lemma 4.2 that $\varepsilon_i(X_\gamma)$ is the number of $-$'s in the i -signature of γ and that

$$\tilde{e}_i^{\varepsilon_i(X_\gamma)}(X_\gamma) = X_{\gamma'}$$

where

$$\gamma' = (\dots (\gamma^{i,-})^{i,-} \dots),$$

and the superscript $i, -$ appears $\varepsilon_i(X_\gamma)$ times.

Now, it is easy to see from the definition of the action of \tilde{f}_i on $B_g(\infty)$ that $\tilde{f}_i(X_\gamma)$ is never zero. Then for $\gamma \in \tilde{Z}$ with $\varepsilon_i(X_\gamma) = 0$, we have $\tilde{e}_i^c \tilde{f}_i^c(X_\gamma) = X_\gamma$. Thus,

$$\tilde{f}_i^c(X_\gamma) = X_{\gamma''}$$

where

$$\gamma'' = (\dots (\gamma^{i,+})^{i,+} \dots),$$

and the superscript $i, +$ appears c times. Then equations (4.10) and (4.11) follow immediately. Finally, equations (4.8) and (4.9) follow from the general properties of crystals. \square

4.2. Type $A_n^{(1)}$. We now consider the case $\mathfrak{g} = \widehat{\mathfrak{sl}}_{n+1}$. Fix an i such that $0 \leq i \leq n$. For elements $(k, l) \in Z$ such that $l \equiv i$ or $i - 1 \pmod{n + 1}$, pick the unique representative such that $l = i$ or $i - 1$. Then define an ordering on such elements by

$$(k, l) < (k', l') \text{ if } k < k' \text{ or if } k = k' \text{ and } l > l'.$$

We then mimic the definitions of Section 4.1. Write all the pairs of the form (k, i) or $(k, i - 1)$ (picking a representative of this form for all pairs possible) in the domain of γ from left to right in increasing order. Below pairs of the form $(k, i - 1)$, write $\gamma(k, i - 1)$ $+$'s and below pairs of the form (k, i) , write $\gamma(k, i)$ $-$'s. From this list of $+$'s and $-$'s, cancel all $(+, -)$ pairs. What is left is a (possibly empty) sequence of $-$'s followed by a (possibly empty) sequence of $+$'s. We call this the i -signature of γ .

Let $(k', i - 1)$ be the pair corresponding to the leftmost $+$ in the i -signature of γ . If there is no $+$ in the i -signature of γ , let $k' = i$. Then define $\gamma^{i,+}$ by

$$(4.12) \quad \gamma^{i,+}(k', i) = \gamma(k', i) + 1,$$

$$(4.13) \quad \gamma^{i,+}(k', i - 1) = \gamma(k', i - 1) - 1,$$

$$(4.14) \quad \gamma^{i,+}(k, l) = \gamma(k, l) \text{ for all other } (k, l) \in Z.$$

In the case that $k' = i$, we ignore equation (4.13).

If there is at least one $-$ in the i -signature of γ , let (k'', i) be the pair corresponding to the rightmost $-$. Then define $\gamma^{i,-}$ by

$$(4.15) \quad \gamma^{i,-}(k'', i) = \gamma(k'', i) - 1,$$

$$(4.16) \quad \gamma^{i,-}(k'', i - 1) = \gamma(k'', i - 1) + 1,$$

$$(4.17) \quad \gamma^{i,-}(k, l) = \gamma(k, l) \text{ for all other } (k, l) \in Z.$$

In the case that $k' = i$, we ignore equation (4.16).

THEOREM 4.3. *For an aperiodic $\gamma \in \tilde{Z}$, let $X_\gamma \in B_g(\infty)$ be the element of the crystal corresponding to the irreducible component $\tilde{\mathcal{C}}_\gamma$. Then*

$$(4.18) \quad \varepsilon_i(X_\gamma) = \#\{-\text{'s in the } i\text{-signature of } \gamma\},$$

$$(4.19) \quad \text{wt}(X_\gamma) = - \sum_{(k,l) \in Z} \sum_{i: k \leq i \leq l} f(k, l) \alpha_{\bar{i}},$$

$$(4.20) \quad \varphi_i(X_\gamma) = \varepsilon_i(X_\gamma) + \langle h_i, \text{wt}(X_\gamma) \rangle,$$

$$(4.21) \quad \tilde{e}_i(X_\gamma) = \begin{cases} X_{\gamma^{i,-}} & \text{if } \varepsilon_i(X_\gamma) > 0, \\ 0 & \text{if } \varepsilon_i(X_\gamma) = 0 \end{cases},$$

$$(4.22) \quad \tilde{f}_i(X_\gamma) = X_{\gamma^{i,+}}.$$

In the first sum in Equation (4.19), we chose one representative (k, l) for each element in Z . In this same equation, \bar{i} denotes the unique integer in $\{0, 1, \dots, n\}$ congruent to i modulo $n + 1$.

PROOF. The proof of this Theorem is similar to the proof of Theorem 4.1 (see also [11, Lemma 8.3 and Theorem 8.4]) and thus will be omitted. \square

5. Comparison to Young tableaux and Young walls

In Section 4 we gave explicit formulas for the geometrically defined crystal structure on irreducible components of quiver varieties in terms of the combinatorial enumerations of these components described in Section 3. There are other combinatorial descriptions of the crystal $B(\infty)$ of $U_q^-(\mathfrak{g})$ given in terms of Young tableaux and Young walls (see [3, 7, 8]). In this section we show how our combinatorial description can be translated into the Young tableaux/wall approach and thus provide an alternative geometric proof of these realizations in finite and affine type A .

5.1. Type A_n . Consider the case $\mathfrak{g} = \mathfrak{sl}_{n+1}$. We first recall the realization of the crystal of $U_q^-(\mathfrak{g})$ in terms of Young tableaux (see [3, 8]). A semi-standard tableau is called *large* if it consists of n non-empty rows, and if for $1 \leq i \leq n$, the number of i -entries in the i th row is strictly greater than the number of all boxes in the $(i+1)$ st row. Let \mathcal{T}^L denote the set of all large tableaux.

Two tableaux $T_1, T_2 \in \mathcal{T}^L$ are called *related*, and we write $T_1 \sim T_2$, if for $1 \leq i < j \leq n$, the number of j -entries in the i th rows of T_1 and T_2 are equal. This is an equivalence relation and we define $\mathcal{T}(\infty) = \mathcal{T}^L / \sim$. In [8], a crystal structure on $\mathcal{T}(\infty)$ is defined and it is shown that this crystal is isomorphic to the crystal $B(\infty)$ for $U_q^-(\mathfrak{g})$. The crystal structure on $\mathcal{T}(\infty)$ is inspired by the tableaux realization of irreducible integrable highest-weight representations of \mathfrak{g} (see [2, 5]).

Define a map $\tau : \mathcal{T}^L \rightarrow \tilde{Z}$ by setting $\tau(T)(k, l)$, $1 \leq k \leq l \leq n$, to be the number of $(l+1)$ -entries in the k th row of T . It follows immediately that $\tau(T_1) = \tau(T_2)$ if $T_1 \sim T_2$. Thus, τ descends to a map from $\mathcal{T}(\infty)$ to \tilde{Z} which we also denote by τ .

THEOREM 5.1. *For $\gamma \in \tilde{Z}$, let X_γ be the element of $B_g(\infty)$ corresponding to the irreducible component $\bar{\mathcal{C}}_\gamma$. The map $\mathcal{T}(\infty) \rightarrow B_g(\infty)$ given by $T \mapsto X_{\tau(T)}$ is a crystal isomorphism.*

PROOF. Since the \sim -equivalence classes are uniquely determined by the number of j -entries in the i th row for $j > i$, the map τ is one-to-one. Now, it is easy to see that if $T_1 \sim T_2$ and $1 \leq i \leq n$, then the i -signatures of T_1 and T_2 differ only by the possible addition of $+$'s to the right side. These additional $+$'s come from i -entries in the i th row. The action of \tilde{f}_i will act at one of these entries if and only if, there are no $+$'s in the i -signature of $\gamma = \gamma(T_1) = \gamma(T_2)$. But then the action of \tilde{f}_i on γ is to increase $\gamma(i, i)$ by one. This corresponds to changing an i -entry in the i th row to an $(i+1)$ -entry in T_1 or T_2 , which is precisely how \tilde{f}_i acts on these tableaux. In the case that \tilde{f}_i does not act at one of these additional $+$'s, the fact that the action of \tilde{f}_i commutes with τ follows easily from the definition of the action in terms of the i -signature.

The map ε_i , $1 \leq i \leq n$, commutes with τ since it is simply the number of $-$'s in the i -signature of an element of the crystal. The fact that wt and φ_i , $1 \leq i \leq n$, commute with τ is also straightforward. \square

5.2. Type $A_n^{(1)}$. We now let $\mathfrak{g} = \widehat{\mathfrak{sl}}_{n+1}$. A description of the crystal $\mathcal{B}(\infty)$ is given in [7] in terms of combinatorial objects called Young walls. The set of all Young walls built on the “ground state” wall Y_∞ is denoted by $\mathcal{F}(\infty)$ and the subset consisting of “proper” Young walls is denoted by $\mathcal{Y}(\infty)$. We define a map $\rho : \mathcal{F}(\infty) \rightarrow \tilde{Z}$ as follows.

For $Y \in \mathcal{F}(\infty)$, each block that has been added to Y_∞ to form Y (that is, appears in Y but not Y_∞) sits to the left of a unique block in the rightmost slice which has been added to Y_∞ . For each block in the rightmost slice of Y which has been added to Y_∞ , the number of blocks to its left is finite. For $0 \leq k \leq n$, we define $\rho(Y)(k, l)$ to be the number of $(-k)$ -colored blocks (recall that the set of block colors is $\mathbb{Z}/(n+1)\mathbb{Z}$) in the rightmost slice of Y which have been added to Y_∞ and which have a $l - k$ blocks to their left. Each element of Z has a unique representative (k, l) with $0 \leq k \leq n$ and thus $\rho(Y) \in \tilde{Z}$. If Y is reduced, then $\rho(Y)$ is aperiodic. Thus $\rho(Y)$ restricts to a map from $\mathcal{Y}(\infty)$ to the set of aperiodic elements of \tilde{Z} . This restriction is one-to-one. Now, for a crystal B , define a new crystal B' by the map $\sigma : B \rightarrow B'$ that interchanges the indices i and $-i$ (this is induced by a Dynkin diagram automorphism). That is, $B' = B$ as sets and σ is the identity operator on the level of sets. The crystal structure on B' is given by

$$\begin{aligned} \tilde{f}_i(\sigma(b)) &= \sigma(\tilde{f}_{-i}(b)), \\ \tilde{e}_i(\sigma(b)) &= \sigma(\tilde{e}_{-i}(b)), \\ \text{wt}_i(\sigma(b)) &= \text{wt}_{-i}(b), \\ \varepsilon_i(\sigma(b)) &= \varepsilon_{-i}(b), \\ \varphi_i(\sigma(b)) &= \varphi_{-i}(b). \end{aligned}$$

Recall that $\text{wt}_i(b) = \langle h_i, \text{wt}(b) \rangle$. Note that while B' is a crystal with the above definitions, σ is *not* a morphism of crystals in general. For example, it does not commute with the action of the operators \tilde{e}_i and \tilde{f}_i .

THEOREM 5.2. *For aperiodic $\gamma \in \tilde{Z}$, let X_γ be the element of $B_g(\infty)$ corresponding to the irreducible component $\tilde{\mathcal{C}}_\gamma$. The map $Y \mapsto X_{\rho(Y)}$ is a crystal isomorphism $\mathcal{Y}(\infty) \cong \sigma(B_g(\infty))$.*

PROOF. It is a technical but straightforward calculation to show that the i -signatures of Y and $X_{\rho(Y)}$ agree. Then the fact that the map $Y \mapsto X_{\rho(Y)}$ commutes with the crystal operators follows easily from their definitions in terms of the i -signatures. \square

Note that we could have avoided the appearance of the map σ by choosing the opposite orientation of our quiver. However, we have chosen the orientation to agree with [1, 11]. We also note that while the proof that $\mathcal{Y}(\infty) \cong B(\infty)$ found in [7] involves the Kyoto path realization of $B(\infty)$, the alternative geometric proof presented here is direct in the sense that it avoids any reference to this path realization. One enumerates the irreducible components of the quiver varieties by Young walls and computes the action of the crystal operators on these components directly.

References

- [1] I. B. Frenkel and A. Savage. Bases of representations of type A affine Lie algebras via quiver varieties and statistical mechanics. *Int. Math. Res. Not.*, (28):1521–1547, 2003.
- [2] J. Hong and S.-J. Kang. *Introduction to quantum groups and crystal bases*, volume 42 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.
- [3] J. Hong and H. Lee. Young tableaux and crystal $\mathcal{B}(\infty)$ for finite simple Lie algebras. arXiv:math.QA/0507448.

- [4] M. Kashiwara. Realizations of crystals. In *Combinatorial and geometric representation theory (Seoul, 2001)*, volume 325 of *Contemp. Math.*, pages 133–139. Amer. Math. Soc., Providence, RI, 2003.
- [5] M. Kashiwara and T. Nakashima. Crystal graphs for representations of the q -analogue of classical Lie algebras. *J. Algebra*, 165(2):295–345, 1994.
- [6] M. Kashiwara and Y. Saito. Geometric construction of crystal bases. *Duke Math. J.*, 89(1):9–36, 1997.
- [7] H. Lee. Extended Nakajima’s monomials and realizations of crystals for $\widehat{\mathfrak{sl}}_{n+1}$. Preprint.
- [8] H. Lee. Young tableaux, Nakajima monomials, and crystals for special linear Lie algebras. arXiv:math.QA/0506147.
- [9] G. Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. *J. Amer. Math. Soc.*, 4(2):365–421, 1991.
- [10] Y. Saito. Crystal bases and quiver varieties. *Math. Ann.*, 324(4):675–688, 2002.
- [11] A. Savage. Geometric and combinatorial realizations of crystal graphs. *Algebr. Represent. Theory*, 9(2):161–199, 2006.

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