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The tensor product of representations of $U_q(\mathfrak{sl}_2)$ via quivers

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Abstract

Using the tensor product variety introduced in Malkin (Duke Math. J., to appear) and Nakajima (Invent. Math. 146 (2001) 399), the complete structure of the tensor product of a finite number of integrable highest weight modules of $U_q(\mathfrak{sl}_2)$ is recovered. In particular, the elementary basis, Lusztig's canonical basis, and the basis adapted to the decomposition of the tensor product into simple modules are all exhibited as distinguished elements of certain spaces of invariant functions on the tensor product variety. For the latter two bases, these distinguished elements are closely related to the irreducible components of the tensor product variety. The space of intertwiners is also interpreted geometrically.

0. Introduction

The purpose of this paper is to obtain a geometric description of the tensor product of a finite number of integrable highest weight representations of $U_q(\mathfrak{sl}_2)$ using quiver varieties. The definition of a *tensor product variety* corresponding to the tensor product of a finite number of integrable highest weight representations of a Lie algebra $\mathfrak{g}$ of ADE type was introduced in [6,9] (see also [10] for a geometric description of the tensor product). There it is demonstrated that the set of irreducible components of the tensor product variety can be equipped with the structure of a $\mathfrak{g}$-crystal isomorphic to the crystal of the canonical basis in the tensor product representation.

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In this paper, we consider the specific case $\mathfrak{g} = \mathfrak{sl}_2$ and recover the entire structure (as opposed to the crystal structure alone) of $U_q(\mathfrak{sl}_2)$ via the tensor product variety. Our definition of the tensor product variety differs slightly from that of [6,9] in that we consider our varieties over the finite field $\mathbb{F}_{q^2}$ with q^2 elements (or its algebraic closure $\bar{\mathbb{F}}_{q^2}$) rather than over $\mathbb{C}$. The reader who is only interested in representations of $\mathfrak{sl}_2$, rather than its associated quantum group, may replace $\mathbb{F}_{q^2}$ by $\mathbb{C}$ and set $q = 1$ everywhere. With a few obvious modifications, the arguments of the paper still hold. Let $\mathbf{d} \in (\mathbb{Z}_{\geqslant 0})^k$. We find three distinct spaces, $\mathscr{T}_0(\mathbf{d})$, $\mathscr{T}_c(\mathbf{d})$, and $\mathscr{T}_s(\mathbf{d})$, of invariant (with respect to a natural group action) functions on the tensor product variety $\mathfrak{T}(\mathbf{d})$, each isomorphic to $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$. In each space we define a natural basis. These three bases, $\mathscr{B}_e$, $\mathscr{B}_c$, and $\mathscr{B}_s$, correspond, respectively, to the elementary basis, Lusztig's canonical basis [4], and a basis compatible with the decomposition of $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ into a direct sum of irreducible modules. The two bases $\mathscr{B}_c$ and $\mathscr{B}_s$ are characterized by their relation to the irreducible components of $\mathfrak{T}(\mathbf{d})$. We define the irreducible components of $\mathfrak{T}(\mathbf{d})$ (defined over $\mathbb{F}_{q^2}$) to be the $\mathbb{F}_{q^2}$ points of the irreducible components of $\mathfrak{T}(\mathbf{d})'$ (the corresponding variety defined over $\bar{\mathbb{F}}_{q^2}$). We then define the *dense points* of an irreducible component of $\mathfrak{T}(\mathbf{d})$ to be the $\mathbb{F}_{q^2}$ points of a certain dense subset of the corresponding irreducible component of $\mathfrak{T}(\mathbf{d})'$. Distinct elements of the basis $\mathscr{B}_c$ and $\mathscr{B}_s$ are supported on distinct irreducible components of $\mathfrak{T}(\mathbf{d})$ and equal to a non-zero constant on the set of dense points of that irreducible component (see Theorems 2.6.1 and 3.3.2). However, the supports of the elements of $\mathscr{B}_s$ are disjoint whereas the supports of the elements of $\mathscr{B}_c$ are not. We also find a geometric description of the space of intertwiners $H^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k} = \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}, V_\mu)$. A natural basis $\mathscr{B}_I$ of this space is again characterized by its relation to the irreducible components of $\mathfrak{T}(\mathbf{d})$.

An important tool used in the development and proof of the results of this paper is the graphical calculus of intertwiners of $U_q(\mathfrak{sl}_2)$ introduced by Penrose, Kauffman and others. This graphical calculus is expanded in [1] and used to prove various results concerning Lusztig's canonical basis. The present paper can be considered a "geometrization" of these results.

In Section 2.7 we conjecture a characterization of the basis $\mathscr{B}_c$ as the image of certain intersection cohomology sheaves of $\mathfrak{T}(\mathbf{d})$ under a particular functor from the space of constructible semisimple perverse sheaves on $\mathfrak{T}(\mathbf{d})$ to the space of invariant functions on $\mathfrak{T}(\mathbf{d})$. Since the definition of $\mathscr{T}_c(\mathbf{d})$ relies on the graphical calculus of intertwiners of $U_q(\mathfrak{sl}_2)$ (and no such graphical calculus exists for more general Lie algebras), this conjecture should play a key role in the possible extension of the results of this paper to a more general set of Lie algebras (for instance, those of type ADE).

This paper is organized as follows. Section 1 contains a review of $U_q(\mathfrak{sl}_2)$ and its representations, Nakajima's quiver varieties, and the graphical calculus of intertwiners of $U_q(\mathfrak{sl}_2)$. The tensor product variety is defined in Section 2 where the spaces $\mathscr{T}_0(\mathbf{d})$ and $\mathscr{T}_c(\mathbf{d})$ are introduced, an isomorphism between the two is given, and various results concerning these spaces and their distinguished bases $\mathscr{B}_e$ and $\mathscr{B}_c$ are proved. Section 3 is concerned with a geometric realization of the space of intertwiners and the decomposition of the tensor product representation into a direct sum of irreducible modules (via the space $\mathscr{T}_s(\mathbf{d})$ and the distinguished basis $\mathscr{B}_s$). It is concluded with the discussion of an isomorphism between the spaces $\mathscr{T}_c(\mathbf{d})$ and $\mathscr{T}_s(\mathbf{d})$.

The notation used in the description of quiver varities is not standardized. Lusztig denotes the fixed vector space by D and the subspace by V while Nakajima denotes these objects by W and V, respectively. Since we wish to use the notation V_n for certain $U_q(\mathfrak{sl}_2)$ modules (to agree with the notation of [1]), we denote the fixed vector space by D and the subspace by W. We hope that this will not cause confusion among those readers familiar with the work of Lusztig and Nakajima.

Throughout this paper the topology is the Zariski topology and the ground field is $\bar{\mathbb{F}}_{q^2}$ unless otherwise specified. However, we will usually deal with varieties defined over $\mathbb{F}_{q^2}$ and consider the corresponding set of $\mathbb{F}_{q^2}$-rational points. Thus, for instance, $\mathbb{P}^n = \mathbb{P}^n\mathbb{F}_{q^2}$ and a vector space is an $\mathbb{F}_{q^2}$ vector space. A function on an algebraic variety is a function into $\mathbb{C}(q)$, the field of rational functions in an indeterminate q. The span of a set of such functions is their $\mathbb{C}(q)$-span. The support of a function f is defined to be the set $\{x \mid f(x) \neq 0\}$ and *not* the closure of this set.

1. The quantum group $U_q(\mathfrak{sl}_2)$ and its representations

1.1. The Hopf algebra structure of $U_q(\mathfrak{sl}_2)$

Let $\mathbb{C}(q)$ be the field of rational functions in an indeterminate q and define $^{-}: \mathbb{C}(q) \to \mathbb{C}(q)$ to be the $\mathbb{C}$-algebra involution such that $\overline{q^n} = q^{-n}$ for all n. The quantum group $U_q(\mathfrak{sl}_2)$ (which we will denote by $\mathbf{U}_q$) is the associative algebra over $\mathbb{C}(q)$ with generators E, F, K, K^{-1} and relations

$$KK^{-1} = K^{-1}K,$$

$$KE = q^2 EK,$$

$$KF = q^{-2} FK,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

The comultiplication and counit of the Hopf algebra structure of $\mathbf{U}_q$ are given by

$$\Delta K^{\pm 1} = K^{\pm 1} \otimes K^{\pm 1},$$

$$\Delta E = E \otimes 1 + K \otimes E,$$

$$\Delta F = F \otimes K^{-1} + 1 \otimes F$$

and

$$\eta(K^{\pm 1}) = 1,$$

$$\eta(E) = \eta(F) = 0,$$

respectively. Although an explicit expression for the antipode exists, we will not need it in this paper.

Let us introduce two involutions of $\mathbf{U}_q$. The first one is the *Cartan involution*, denoted by ω, which acts as follows:

$$\omega(E) = F, \quad \omega(F) = E, \quad \omega(K^{\pm 1}) = K^{\pm 1}, \quad \omega(q^{\pm 1}) = q^{\pm 1},$$

$$\omega(xy) = \omega(y)\omega(x), \quad x, y \in \mathbf{U}_q.$$

The second, denoted by σ, is called the "*bar*" *involution* and is defined by

$$\sigma(E) = E, \quad \sigma(F) = F, \quad \sigma(K^{\pm 1}) = K^{\mp 1}, \quad \sigma(q^{\pm 1}) = q^{\mp 1},$$

$$\sigma(xy) = \sigma(x)\sigma(y), \quad x, y \in \mathbf{U}_q.$$

Using σ we can define a second comultiplication $\bar{\Delta}$ by

$$\bar{\Delta}(x) = (\sigma \otimes \sigma)\Delta(\sigma(x)), \quad x \in \mathbf{U}_q$$

which implies

$$\bar{\Delta}K^{\pm 1} = K^{\pm 1} \otimes K^{\pm 1},$$

$$\bar{\Delta}E = E \otimes 1 + K^{-1} \otimes E,$$

$$\bar{\Delta}F = F \otimes K + 1 \otimes F.$$

1.2. Irreducible representations of $U_q(\mathfrak{sl}_2)$

Any finite-dimensional irreducible $\mathbf{U}_q$-module V is generated by a highest weight vector, v, of weight εq^d where $\varepsilon = \pm 1$ and $d = \dim(V) - 1$ [2]. In this paper we consider those representations with $\varepsilon = +1$. Let $v_{d-2k} = F^k v/[k]!$ where

$$[k] = (q^k - q^{-k})/(q - q^{-1}) = q^{-k+1} + q^{-k+3} + \cdots + q^{k-1},$$

$$[k]! = [1][2]\cdots[k].$$

Then $v_{d-2k}=0$ for $k>d$ and $\{v=v_d, v_{d-2}, \dots, v_{-d}\}$ is a basis of V. We denote this representation by V_d. The action of $\mathbf{U}_q$ on V_d is given by

$$K^{\pm 1} v_m = q^{\pm m} v_m,$$

$$Ev_m = \left[\frac{d+m}{2}+1\right] v_{m+2},$$

$$Fv_m = \left[\frac{d-m}{2}+1\right] v_{m-2}. \tag{1}$$

Define a bilinear symmetric pairing on V_d by requiring

$$\langle xu, v\rangle = \langle u, \omega(x)v\rangle, \quad \langle v_d, v_d\rangle = 1, \quad u, v \in V_d \text{ and } x \in \mathbf{U}_q.$$

It follows that

$$\langle v_{d-2k}, v_{d-2l}\rangle = \delta_{k,l} \begin{bmatrix} d \\ k \end{bmatrix},$$

where

$$\begin{bmatrix} d \\ k \end{bmatrix} = \frac{[d]!}{[k]![d-k]!}.$$

Let $\{v^{d-2k}\}_{k=0}^{d}$ be the basis dual to $\{v_{d-2k}\}_{k=0}^{d}$ with respect to the form $\langle\, ,\, \rangle$. Then

$$v^{d-2k} = \begin{bmatrix} d \\ k \end{bmatrix}^{-1} v_{d-2k}$$

and the action of $\mathbf{U}_q$ in the dual basis is

$$K^{\pm 1} v_m = q^{\pm m} v_m,$$

$$Ev^m = \left[\frac{d-m}{2}\right] v^{m+2},$$

$$Fv^m = \left[\frac{d+m}{2}\right] v^{m-2}.$$

1.3. Geometric realization of irreducible representations of $U_q(\mathfrak{sl}_2)$

We recall here Nakajima's quiver variety construction of finite-dimensional irreducible representations of Kac–Moody algebras associated to symmetric Cartan matrices [7,8] in the specific case of $U_q(\mathfrak{sl}_2)$. In order to introduce the quantum parameter q, some of our definitions differ slightly from those in [7,8]. Since the Dynkin diagram of $\mathfrak{sl}_2$ consists of a single vertex and no edges, the definition of the quiver variety simplifies considerably. Fix vector spaces W and D of dimensions w and d, respectively, and consider the variety

$$\mathbf{M}(w,d) = \mathrm{Hom}(D, W) \oplus \mathrm{Hom}(W, D).$$

The two components of an element of $\mathbf{M}(w,d)$ will be denoted by f_1 and f_2, respectively. $\mathrm{GL}(W)$ acts on $\mathbf{M}(w,d)$ by

$$(f_1, f_2) \mapsto g(f_1, f_2) \stackrel{\mathrm{def}}{=} (gf_1, f_2 g^{-1}), \ \ g \in \mathrm{GL}(W).$$

Define the map $\mu : \mathbf{M}(w,d) \to \mathrm{End}\, W$ by

$$\mu(f_1, f_2) = f_1 f_2.$$

Let $\mu^{-1}(0)$ be the algebraic variety defined as the zero set of μ. We say a point (f_1, f_2) of $\mu^{-1}(0)$ is *stable* if f_2 is injective. The *quiver variety* is then given by

$$\{(f_1, f_2) \in \mu^{-1}(0) \mid (f_1, f_2) \text{ is stable}\}/\mathrm{GL}(W).$$

Via the map $(f_1, f_2) \mapsto (\mathrm{im}\, f_2, f_2 f_1)$, this variety is seen to be isomorphic to the variety

$$\mathfrak{M}(w,d) = \{(W,t) \mid W \subset D, \ \dim W = w, \ t \in \mathrm{End}\, D, \ \mathrm{im}\, t \subset W \subset \ker t\}$$

Note that the condition $\mathrm{im}\, t \subset W \subset \ker t$ implies $t^2 = 0$. Let

$$\mathfrak{M}(d) = \bigcup_w \mathfrak{M}(w,d) = \{(W,t) \mid W \subset D, \ t \in \mathrm{End}\, D, \ \mathrm{im}\, t \subset W \subset \ker t\}$$

and

$$\mathfrak{M}(w, w+1, d) = \{(U, W, t) \mid t \in \mathrm{End}\, D, \ \mathrm{im}\, t \subset U \subset W \subset \ker t, \ \dim U = w, \ \dim W = w+1\}.$$

We then have the projections

$$\mathfrak{M}(d) \stackrel{\pi_1}{\leftarrow} \bigcup_w \mathfrak{M}(w, w+1, d) \stackrel{\pi_2}{\rightarrow} \mathfrak{M}(d)$$

given by $\pi_1(U, W, t) = (U, t)$ and $\pi_2(U, W, t) = (W, t)$.

For a subset Y of a variety A, let $\mathbf{1}_Y$ denote the function on A which takes the value 1 on Y and 0 elsewhere. Note that since our varieties are defined over $\mathbb{F}_{q^2}$, they consist of a finite number of ($\mathbb{F}_{q^2}$-rational) points. Let $\chi_q(Y)$ denote the Euler characteristic of the algebraic variety Y, which is merely the number of points in Y. For a map π between algebraic varieties A and B, let $\pi_!$ [5] denote the map between the abelian groups of functions on A and B given by

$$\pi_!(f)(x) = \sum_{y \in \pi^{-1}(x)} f(y) \Rightarrow \pi_!(\mathbf{1}_Y)(x) = \chi_q(\pi^{-1}(x) \cap Y), \quad Y \subset A$$

and let π^* be the pullback map from functions on B to functions on A acting as $\pi^* f(x) = f(\pi(x))$.

We then define the action of E, F and $K^{\pm 1}$ on the set of functions on $\mathfrak{M}(d)$ by

$$\begin{aligned} Ef &= q^{-\dim(\pi_1^{-1}(\cdot))}(\pi_1)_! \pi_2^* f, \\ Ff &= q^{-\dim(\pi_2^{-1}(\cdot))}(\pi_2)_! \pi_1^* f, \\ K^{\pm 1} f &= q^{\pm(d - 2\dim(\cdot))} f, \end{aligned} \tag{2}$$

where the notation means that for a function f on $\mathfrak{M}(d)$ and $(W, t) \in \mathfrak{M}(d)$,

$$\begin{aligned} Ef(W,t) &= q^{-\dim(\pi_1^{-1}(W,t))}(\pi_1)_! \pi_2^* f(W,t), \\ Ff(W,t) &= q^{-\dim(\pi_2^{-1}(W,t))}(\pi_2)_! \pi_1^* f(W,t), \\ K^{\pm 1} f(W,t) &= q^{\pm(d - 2\dim W)} f(W,t). \end{aligned} \tag{3}$$

Let

$$\begin{aligned} \mathfrak{M}^r(d) &= \{(W,t) \in \mathfrak{M}(d) \mid \operatorname{rank} t = r\}, \\ \mathfrak{M}^r(w,d) &= \{(W,t) \in \mathfrak{M}(w,d) \mid \operatorname{rank} t = r\}, \\ \mathcal{M}^r(w,d) &= \mathbb{C}(q) \mathbf{1}_{\mathfrak{M}^r(w,d)}, \\ \mathcal{M}^r(d) &= \bigoplus_w \mathcal{M}^r(w,d), \\ \mathcal{M}(w,d) &= \bigoplus_r \mathcal{M}^r(w,d), \\ \mathcal{M}(d) &= \bigoplus_w \mathcal{M}(w,d). \end{aligned}$$

Also, let us introduce the following notation for Grassmanians:

$$\mathrm{Gr}_w^d = \{W \subset (\mathbb{F}_{q^2})^d \mid \dim W = w\}.$$

Proposition 1.3.1. *The action of* $\mathbf{U}_q$ *defined by* (2) *endows* $\mathscr{M}^r(d)$ (*and hence* $\mathscr{M}(d)$) *with the structure of a* $\mathbf{U}_q$*-module and the map* $\mathbf{1}_{\mathfrak{M}^r(w,d)} \mapsto v_{d-2w}$ (*extended by linearity*) *is an isomorphism* $\mathscr{M}^r(d) \cong V_{d-2r}$ *of* $\mathbf{U}_q$*-modules.*

To prove this proposition, we will need the following lemmas.

Lemma 1.3.1. *For vector spaces* $W \subset D$,

$$\{U \mid W \subset U \subset D,\ \dim U = u\} \cong Gr^{\dim D - \dim W}_{u - \dim W}.$$

Proof. This follows immediately from the fact that

$$\{U \mid W \subset U \subset D,\ \dim U = u\} \cong \{U' \mid U' \subset D/W,\ \dim U' = u - \dim W\}$$

via the map $U \mapsto U' = U/W$. $\square$

Lemma 1.3.2. $\chi_q(\mathbb{P}^n) = \sum_{i=0}^n q^{2i}$.

Proof. This follows from simply counting the number of possible one dimensional subspaces of $\mathbb{P}^n$. $\square$

Proof of Proposition 1.3.1. If $(W,t) \in \mathfrak{M}^r(w,d)$ then

$$\begin{aligned}
E\mathbf{1}_{\mathfrak{M}^r(w+1,d)}(W,t) &= q^{-\dim(\pi_1^{-1}(W,t))}(\pi_1)_! \pi_2^* \mathbf{1}_{\mathfrak{M}^r(w+1,d)}(W,t) \\
&= q^{-\dim(\{U \mid W \subset U \subset \ker t,\, \dim U = w+1\})}(\pi_1)_! \mathbf{1}_{\mathfrak{M}^r(w,w+1,d)}(W,t) \\
&= q^{-\dim(Gr_1^{d-w-r})}\chi_q(\pi_1^{-1}(W,t) \cap \mathfrak{M}^r(w,w+1,d)) \\
&= q^{-\dim(\mathbb{P}^{d-w-r-1})}\chi_q(\{U \mid W \subset U \subset \ker t,\ \dim U = w+1\}) \\
&= q^{-(d-w-r-1)}\chi_q(Gr_1^{d-w-r}) \\
&= q^{-(d-w-r-1)}\chi_q(\mathbb{P}^{d-w-r-1}) \\
&= q^{-(d-w-r-1)} \sum_{i=0}^{d-w-r-1} q^{2i} \\
&= q^{-(d-w-r-1)} + q^{-(d-w-r-1)+2} + \cdots + q^{d-w-r-1} \\
&= [d-w-r]
\end{aligned}$$

and $E\mathbf{1}_{\mathfrak{M}^r(w+1,d)}(W,t) = 0$ otherwise. So $E\mathbf{1}_{\mathfrak{M}^r(w+1,d)} = [d-w-r]\mathbf{1}_{\mathfrak{M}^r(w,d)}$. Similarly, if $(W,t)\in\mathfrak{M}^r(w+1,d)$,

$$\begin{aligned}
F\mathbf{1}_{\mathfrak{M}^r(w,d)}(W,t) &= q^{-\dim(\pi_2^{-1}(W,t))}(\pi_2)_!\pi_1^*\mathbf{1}_{\mathfrak{M}^r(w,d)}(W,t)\\
&= q^{-\dim(\{U | \mathrm{im}\, t\subset U\subset W, \dim U = w\})}(\pi_2)_!\mathbf{1}_{\mathfrak{M}^r(w,w+1,d)}(W,t)\\
&= q^{-\dim(Gr_{w-r}^{w+1-r})}\chi_q(\pi_2^{-1}(W,t)\cap\mathfrak{M}^r(w,w+1,d))\\
&= q^{-\dim(\mathbb{P}^{w-r})}\chi_q(\{U \mid \mathrm{im}\, t\subset U\subset W,\ \dim U = w\})\\
&= q^{-(w-r)}\chi_q(\mathbb{P}^{w-r})\\
&= q^{-(w-r)}\sum_{i=0}^{w-r} q^{2i}\\
&= q^{-(w-r)} + q^{-(w-r)+2} + \cdots + q^{w-r}\\
&= [w+1-r]
\end{aligned}$$

and $F\mathbf{1}_{\mathfrak{M}^r(w,d)}(W,t) = 0$ otherwise. So $F\mathbf{1}_{\mathfrak{M}^r(w,d)} = [w+1-r]\mathbf{1}_{\mathfrak{M}^r(w+1,d)}$. It is obvious that

$$K^{\pm 1}\mathbf{1}_{\mathfrak{M}^r(w,d)} = q^{\pm(d-2w)}\mathbf{1}_{\mathfrak{M}^r(w,d)}. \tag{4}$$

Now, $\mathfrak{M}^r(w,d) = \emptyset$ unless $r\leqslant w\leqslant d-r$ due to the requirement $\mathrm{im}\, t\subset W\subset \ker t$ in the definition of $\mathfrak{M}^r(w,d)$. Thus $\mathscr{M}^r(d) = \oplus_{w=r}^{w=d-r}\mathscr{M}^r(w,d)$.

Comparing the above calculations to (1), the result follows. □

So $\mathscr{M}(d)$ is isomorphic to the direct sum of the irreducible representations of highest weight $d-2r$ where $0\leqslant r\leqslant d/2$ since these are the possible ranks of t (recall that $t^2=0$).

Let $\mathfrak{L}(d) = \mathfrak{M}^0(d)$. Then $\mathfrak{L}(d)$ is isomorphic to the algebraic variety of all subspaces $W\subset D$, which is a union of Grassmanians. Let

$$\mathfrak{L}(w,d) = \mathfrak{M}^0(w,d) = \{W\subset D \mid \dim W = w\}\cong Gr_w^d$$

and

$$\mathscr{L}(w,d) = \mathscr{M}^0(w,d) = \mathbb{C}(q)\mathbf{1}_{\mathfrak{L}(w,d)},\quad \mathscr{L}(d) = \mathscr{M}^0(d) = \bigoplus_{w=1}^{d}\mathscr{L}(w,d).$$

We see from Proposition 1.3.1 that the action of $\mathbf{U}_q$ defined by (2) endows $\mathscr{L}(d)$ with the structure of the irreducible module V_d via the isomorphism $\mathbf{1}_{\mathfrak{L}(w,d)}\mapsto v_{d-2w}$ (extended by linearity). Note that for $(W,t)\in\mathfrak{M}(d)$, we can think of t as belonging to $\mathrm{Hom}(D/W, W)$ and thus $\mathfrak{M}(d)$ is the cotangent bundle of $\mathfrak{L}(d)$.

1.4. Tensor products and the graphical calculus of intertwiners

We define the bilinear pairing of $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ with $V_{\mathbf{d}_k} \otimes \cdots \otimes V_{\mathbf{d}_1}$ by

$$\langle v_{i_1} \otimes \cdots \otimes v_{i_k}, v^{l_k} \otimes \cdots \otimes v^{l_1} \rangle = \delta_{i_1}^{l_1} \dots \delta_{i_k}^{l_k}.$$

Then

$$\langle \Delta^{n-1}(x) v_{i_1} \otimes \cdots \otimes v_{i_k}, v^{l_k} \otimes \cdots \otimes v^{l_1} \rangle$$
$$= \langle v_{i_1} \otimes \cdots \otimes v_{i_k}, \bar{\Delta}^{n-1}(\omega(x)) v^{l_k} \otimes \cdots \otimes v^{l_1} \rangle.$$

Lusztig's canonical basis of the tensor product is described in [4]. We refer the reader to this article or the overview in [1, Section 1.5], for the definition of this basis. As in [1,4], we denote the elements of Lusztig's canonical basis by $v_{i_1} \diamondsuit \cdots \diamondsuit v_{i_k}$ and their dual by $v_{i_1} \heartsuit \cdots \heartsuit v_{i_k}$. The dual is defined with respect to the form $\langle \, , \, \rangle$:

$$\langle v_{i_1} \diamondsuit \cdots \diamondsuit v_{i_k}, v^{l_k} \heartsuit \cdots \heartsuit v^{l_1} \rangle = \delta_{i_1}^{l_1} \dots \delta_{i_k}^{l_k}.$$

When we wish to make explicit to which representation a vector belongs, we use the notation ${}^d v_k, {}^d v^k \in V_d$.

To simplify notation, we make the following definitions:

$$\otimes^{\mathrm{d}} v_{\mathbf{w}} = {}^{\mathbf{d}_1} v_{\mathbf{d}_1 - 2\mathbf{w}_1} \otimes \cdots \otimes {}^{\mathbf{d}_k} v_{\mathbf{d}_k - 2\mathbf{w}_k},$$

$$\diamondsuit^{\mathrm{d}} v_{\mathbf{w}} = {}^{\mathbf{d}_1} v_{\mathbf{d}_1 - 2\mathbf{w}_1} \diamondsuit \cdots \diamondsuit {}^{\mathbf{d}_k} v_{\mathbf{d}_k - 2\mathbf{w}_k},$$

$$\otimes^{\mathrm{d}} v^{\mathbf{w}} = {}^{\mathbf{d}_1} v^{\mathbf{d}_1 - 2\mathbf{w}_1} \otimes \cdots \otimes {}^{\mathbf{d}_k} v^{\mathbf{d}_k - 2\mathbf{w}_k},$$

$$\heartsuit^{\mathrm{d}} v^{\mathbf{w}} = {}^{\mathbf{d}_1} v^{\mathbf{d}_1 - 2\mathbf{w}_1} \heartsuit \cdots \heartsuit {}^{\mathbf{d}_k} v^{\mathbf{d}_k - 2\mathbf{w}_k},$$

where $\mathbf{d}, \mathbf{w} \in (\mathbb{Z}_{\geqslant 0})^k$.

We can extend the bar involution σ to tensor products of irreducible representations as follows. Define

$$\sigma(f(q)(\otimes^{\mathbf{d}} v_{\mathbf{w}})) = f(q^{-1})(\otimes^{\mathbf{d}} v_{\mathbf{w}})$$

and extend by $\mathbb{C}$-linearity. Then σ is an isomorphism from $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ to itself and

$$\sigma(\Delta^{(k-1)}(x)(v)) = ((\sigma \otimes \cdots \otimes \sigma)(\Delta^{(k-1)} x))(\sigma v) \tag{5}$$

for $x \in \mathbf{U}_q$ and $v \in V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$.

We now recall some results on the graphical calculus of tensor products and intertwiners. For a more complete treatment, see [1]. In the graphical calculus, V_d is depicted by a box marked d with d vertices. To depict $\mathrm{CM}^{\mathbf{a}_1,\dots,\mathbf{a}_l}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, we place the boxes representing the $V_{\mathbf{d}_i}$ on a horizontal line and the boxes representing the $V_{\mathbf{a}_i}$ on another horizontal line lying above the first one. $\mathrm{CM}^{\mathbf{a}_1,\dots,\mathbf{a}_l}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ is then the set of non-intersecting curves (up to isotopy) connecting the vertices of the boxes such that the following conditions are satisfied:

1. Each curve connects exactly two vertices.
2. Each vertex is the endpoint of exactly one curve.
3. No curve joins a box to itself.
4. The curves lie inside the box bounded by the two horizontal lines and the vertical lines through the extreme right and left points.

An example is given in Fig. 1. We call the curves joining two lower boxes *lower curves*, those joining two upper boxes *upper curves* and those joining a lower and an upper box *middle curves*. We define the set of oriented crossingless matches $\mathrm{OCM}^{\mathbf{a}_1,\dots,\mathbf{a}_l}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ to be the set of elements of $\mathrm{CM}^{\mathbf{a}_1,\dots,\mathbf{a}_l}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ along with an orientation of the curves such that all upper and lower curves are oriented to the left and all middle curves are oriented so that those oriented down are to the right of those oriented up. See Fig. 2.

As shown in [1], the set of crossingless matches $\mathrm{CM}^{\mathbf{a}_1,\dots,\mathbf{a}_l}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ is in one to one correspondence with a basis of the set of intertwiners

$$H^{\mathbf{a}_1,\dots,\mathbf{a}_l}_{\mathbf{d}_1,\dots,\mathbf{d}_k} = \mathrm{Hom}_{\mathbf{U}_q}(V_{\mathbf{d}_1}\otimes\cdots\otimes V_{\mathbf{d}_k}, V_{\mathbf{a}_1}\otimes\cdots\otimes V_{\mathbf{a}_l}).$$

The matrix coefficients of the intertwiner associated to a particular crossingless match are given by Theorem 2.1 of [1]. Note that these are intertwiners in the dual basis and thus commute with the action of $\mathbf{U}_q$ on the tensor product given by $\bar{\Delta}^{(k-1)}$. Let $\tilde{\gamma}$ be such an intertwiner and define $\gamma = \sigma\tilde{\gamma}\sigma$. Then for $x \in \mathbf{U}_q$

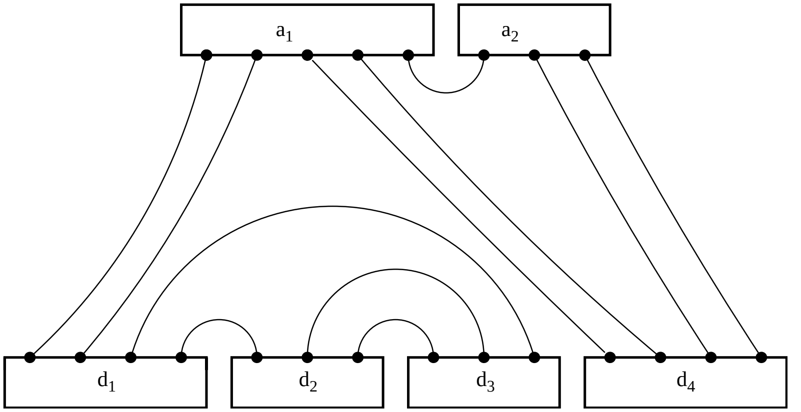

Fig. 1. A crossingless match.

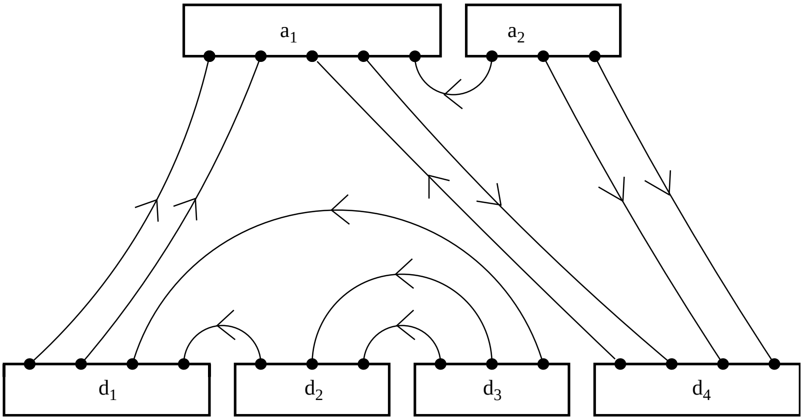

Fig. 2. An oriented crossingless match.

and $v \in V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$,

$$\begin{aligned}\gamma\Delta^{(k-1)}(x)(v) &= \sigma\tilde{\gamma}\sigma\Delta^{(k-1)}(x)(v)\\ &= \sigma\tilde{\gamma}((\sigma\otimes\cdots\otimes\sigma)\Delta^{(k-1)}(x))(\sigma v)\\ &= \sigma\tilde{\gamma}\bar{\Delta}^{(k-1)}(\sigma x)(\sigma v)\\ &= \sigma\bar{\Delta}^{(k-1)}(\sigma x)\tilde{\gamma}(\sigma v)\\ &= \sigma((\sigma\otimes\cdots\otimes\sigma)\Delta^{(k-1)}(x))\sigma\gamma(v)\\ &= \Delta^{(k-1)}(x)\gamma(v).\end{aligned}$$

Thus γ is an intertwiner in the usual basis commuting with the action of $\mathbf{U}_q$ given by $\Delta^{(k-1)}$.

We will also need to define the set of *lower crossingless matches* $\mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ and *oriented lower crossingless matches* $\mathrm{OLCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$. Elements of $\mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ and $\mathrm{OLCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ are obtained from elements of $\mathrm{CM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ and $\mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ (respectively) by removing the upper boxes (thus converting lower endpoints of upper curves to unmatched vertices). For the case of $\mathrm{OLCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, unmatched vertices will still have an orientation (indicated by an arrow attached to the vertex). As for middle curves in the case of $\mathrm{OCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}^{\mathbf{a}_1,\dots,\mathbf{a}_l}$, the unmatched vertices in an element of $\mathrm{OLCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ must be arranged to that those oriented down are to the right of those oriented up. See Fig. 3.

Let $\mathbf{a} \in (\mathbb{Z}_{\geqslant 0})^k$ be such that $\mathbf{a}_i \leqslant \mathbf{d}_i$ for $i = 1, 2, \dots, k$. We associate an oriented lower crossingless match to $\mathbf{a}$ as follows. For each i, place down arrows on the rightmost $\mathbf{a}_i$

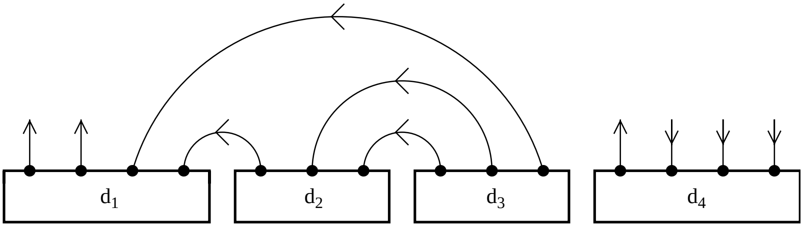

Fig. 3. An oriented lower crossingless match.

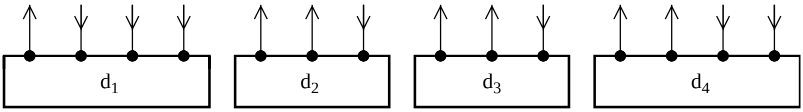

Fig. 4. $\mathbf{d} = (4, 3, 3, 4)$, $\mathbf{a} = (3, 1, 1, 2)$.

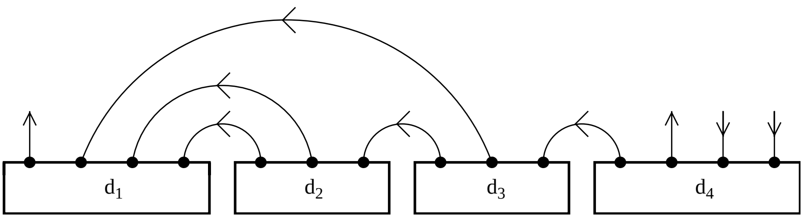

Fig. 5. Oriented lower crossingless match associated to $\mathbf{d} = (4, 3, 3, 4)$, $\mathbf{a} = (3, 1, 1, 2)$.

vertices of the box representing $V_{\mathbf{d}_i}$. Place up arrows on the remaining vertices. See Fig. 4. There is a unique way to form an oriented lower crossingless match such that the orientation of any curve agrees with the direction of the arrows at its endpoints. Namely, starting from the right connect each down arrow to the first unmatched up arrow to its right (if there is any). Note that this produces an oriented lower crossingless match where the unmatched vertices are arranged so that all those with down arrows are to the right of those with up arrows (otherwise, we could have matched more vertices). See Fig. 5. So to each $\mathbf{a}$ there is an associated element of $\mathrm{OLCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$. Conversely, given an element of $\mathrm{OLCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, there is exactly one $\mathbf{a}$ which produces it. So we have a one to one correspondence between the set of elements $\mathbf{a} \in (\mathbb{Z}_{\geqslant 0})^k$ such that $\mathbf{a}_i \leqslant \mathbf{d}_i$ and oriented lower crossingless matches $\mathrm{OLCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$. We will denote the oriented lower crossingless match associated to $\mathbf{a}$ by $M(\mathbf{d}, \mathbf{a})$.

We can put a partial ordering on the sets $\mathrm{CM}^{\mathbf{a}_1,\dots,\mathbf{a}_l}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, $\mathrm{OCM}^{\mathbf{a}_1,\dots,\mathbf{a}_l}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, $\mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ and $\mathrm{OLCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ as follows. For any two elements S_1 and S_2 of one of these sets, $S_1 \leqslant S_2$ if the set of lower curves of S_1 is a subset of the set of lower curves of S_2.

Given the geometrization of irreducible representations of $\mathbf{U}_q$ (Section 1.3), it is natural to seek a geometrization of the tensor product and the space of intertwiners. This geometric realization is the focus of Sections 2 and 3.

2. Geometric realization of the tensor product

2.1. Definition of the tensor product variety $\mathfrak{T}(\mathbf{d})$

We now describe a variety (introduced in [6,9]) corresponding to the tensor product of the irreducible representations $V_{\mathbf{d}_1}, V_{\mathbf{d}_2}, \dots, V_{\mathbf{d}_k}$. This construction will yield three distinct bases of the tensor product in a natural way.

Fix a d-dimensional vector space D and let $\mathbf{d} \in (\mathbb{Z}_{\geqslant 0})^k$ be such that $\sum_{i=1}^k \mathbf{d}_i = d$. Define

$$\mathfrak{T}_0(\mathbf{d}) = \{(\mathbf{D} = \{\mathbf{D}_i\}_{i=0}^k, W) \mid 0 = \mathbf{D}_0 \subset \mathbf{D}_1 \subset \cdots \subset \mathbf{D}_k = D,\\ W \subset D, \dim \mathbf{D}_i/\mathbf{D}_{i-1} = \mathbf{d}_i\}. \tag{6}$$

$\mathfrak{T}_0(\mathbf{d})$ admits a natural $GL(D)$ action. Namely,

$$g \cdot (\{\mathbf{D}_i\}_{i=0}^k, W) = (\{g\mathbf{D}_i\}_{i=0}^k, gW)$$

for $g \in GL(D)$ and $(\mathbf{D}, W) \in \mathfrak{T}_0(\mathbf{d})$. Now let

$$\mathfrak{T}(\mathbf{d}) \stackrel{\text{def}}{=} \{(\mathbf{D} = \{\mathbf{D}_i\}_{i=0}^k, W, t) \mid 0 = \mathbf{D}_0 \subset \mathbf{D}_1 \subset \cdots \subset \mathbf{D}_k = D, W \subset D,\\ t \in \operatorname{End} D,\ t(\mathbf{D}_i) \subset \mathbf{D}_{i-1}, \dim \mathbf{D}_i/\mathbf{D}_{i-1} = \mathbf{d}_i,\ \operatorname{im} t \subset W \subset \ker t\}. \tag{7}$$

We call $\mathfrak{T}(\mathbf{d})$ the *tensor product variety*. We say a flag $\mathbf{D} = (0 = \mathbf{D}_0 \subset \mathbf{D}_1 \subset \cdots \subset \mathbf{D}_k = D)$ is *t-stable* if $t(\mathbf{D}_i) \subset \mathbf{D}_{i-1}$ for $i = 1, \dots, k$.

If we consider the corresponding varieties $\mathfrak{T}_0(\mathbf{d})'$ and $\mathfrak{T}(\mathbf{d})'$ defined over $\bar{\mathbb{F}}_{q^2}$, a straightforward computation shows that $\mathfrak{T}(\mathbf{d})'$ is the union of the conormal bundles of the orbits of the action of $GL(D)$ on $\mathfrak{T}_0(\mathbf{d})'$.

We define the action of $E,\ F$ and $K^{\pm 1}$ on the set of functions on $\mathfrak{T}(\mathbf{d})$ just as for the other spaces considered so far. Namely, let

$$\mathfrak{T}(w; \mathbf{d}) = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}) \mid \dim W = w\},$$

$$\mathfrak{T}(w, w+1; \mathbf{d}) = \{(\mathbf{D}, U, W, t) \mid (\mathbf{D}, U, t), (\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}),\ U \subset W,\\ \dim U = w,\ \dim W = w+1\}.$$

We then have the projections

$$\mathfrak{T}(\mathbf{d}) \stackrel{\pi_1}{\leftarrow} \bigcup_w \mathfrak{T}(w, w+1; \mathbf{d}) \stackrel{\pi_2}{\rightarrow} \mathfrak{T}(\mathbf{d}), \tag{8}$$

where $\pi_1(\mathbf{D}, U, W, t) = (\mathbf{D}, U, t)$ and $\pi_2(\mathbf{D}, U, W, t) = (\mathbf{D}, W, t)$. The action of $E,\ F$ and $K^{\pm 1}$ is defined by (2) as usual. Of course, the notation for the action of $K^{\pm 1}$ now means that

$$(K^{\pm 1}f)(\mathbf{D}, W, t) = q^{\pm(d-2\dim W)}f(\mathbf{D}, W, t). \tag{9}$$

2.2. A set of basic functions on the tensor product variety

We now describe a set of basic functions on $\mathfrak{T}(\mathbf{d})$ which will be used to form spaces of functions isomorphic to $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$. As usual, fix a d-dimensional vector space D. For a flag $\mathbf{D} = (0 = \mathbf{D}_0 \subset \cdots \subset \mathbf{D}_k = D)$ and a subspace $W \subset D$, define $\alpha(W, \mathbf{D}) \in (\mathbb{Z}_{\geqslant 0})^k$ by

$$\alpha(W, \mathbf{D})_i = \dim(W \cap \mathbf{D}_i)/(W \cap \mathbf{D}_{i-1}).$$

For $\mathbf{w}, \mathbf{r}, \mathbf{n} \in (\mathbb{Z}_{\geqslant 0})^k$, define

$$A_{\mathbf{w},\mathbf{r},\mathbf{n}} = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}) \mid \alpha(W, \mathbf{D}) = \mathbf{w}, \alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r}, \alpha(\ker t, \mathbf{D}) = \mathbf{n}\}. \tag{10}$$

Note that the non-empty sets $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ are precisely the orbits of the action of $GL(D)$ given by

$$g \cdot (\{\mathbf{D}_i\}_{i=0}^k, W, t) = (\{g\mathbf{D}_i\}_{i=0}^k, gW, gtg^{-1}), \quad g \in GL(D).$$

From now on, the term *constructible* will mean constructible with respect to the stratification given by these sets. We say that a function f on $\mathfrak{T}(\mathbf{d})$ is *invariant* if it is invariant under the action of $\mathrm{GL}(D)$ given by

$$(g \cdot f)(x) = f(g^{-1}x), \quad g \in \mathrm{GL}(D).$$

Let $\mathscr{T}(\mathbf{d})$ denote the space of invariant functions on $\mathfrak{T}(\mathbf{d})$. We will also use the notation

$$\mathbf{a}^{(j,l)} = \sum_{i=j}^{l} \mathbf{a}_i, \quad |\mathbf{a}| = \sum_{i=1}^{k} \mathbf{a}_i$$

for $\mathbf{a} \in (\mathbb{Z}_{\geqslant 0})^k$ and we will let δ^j denote the element of $(\mathbb{Z}_{\geqslant 0})^k$ such that $\delta^j_j = 1$ and $\delta^j_i = 0$ for all $i \neq j$.

Let

$$k_{\mathbf{w},\mathbf{r},\mathbf{n}} = q^{\sum_{i<j}(\mathbf{r}_i\mathbf{w}_j + \mathbf{w}_i\mathbf{n}_j - \mathbf{w}_i\mathbf{w}_j)} \tag{11}$$

and define

$$f_{\mathbf{w},\mathbf{r},\mathbf{n}} = k_{\mathbf{w},\mathbf{r},\mathbf{n}} \mathbf{1}_{A_{\mathbf{w},\mathbf{r},\mathbf{n}}}. \tag{12}$$

Then it is easy to see that

$$\mathscr{T}(\mathbf{d}) = \operatorname{Span}\{f_{\mathbf{w},\mathbf{r},\mathbf{n}}\}_{\mathbf{w},\mathbf{r},\mathbf{n}}.$$

We will call the $f_{\mathbf{w},\mathbf{r},\mathbf{n}}$ *basic functions*. Note that $f_{\mathbf{w},\mathbf{r},\mathbf{n}} = \mathbf{1}_{A_{\mathbf{w},\mathbf{r},\mathbf{n}}}$ if $q = 1$. As will be seen below, the factor of $k_{\mathbf{w},\mathbf{r},\mathbf{n}}$ is necessary in order for the $f_{\mathbf{w},\mathbf{r},\mathbf{n}}$ to correspond to certain vectors in the tensor product. Note that $f_{\mathbf{w},\mathbf{r},\mathbf{n}} \equiv 0$ unless $\mathbf{r} \leqslant \mathbf{w} \leqslant \mathbf{n}$ where we define the partial ordering such that for $\mathbf{a}, \mathbf{b} \in (\mathbb{Z}_{\geqslant 0})^k$,

$$\mathbf{a} \leqslant \mathbf{b} \;\Leftrightarrow\; \sum_{i=1}^{j} \mathbf{a}_i \leqslant \sum_{i=1}^{j} \mathbf{b}_i \quad \text{for } 1 \leqslant j \leqslant k$$

$$4\mathbf{a} < \mathbf{b} \;\Leftrightarrow\; \mathbf{a} \leqslant \mathbf{b},\; \mathbf{a} \neq \mathbf{b}. \tag{13}$$

Also, $f_{\mathbf{w},\mathbf{r},\mathbf{n}} \equiv 0$ unless $|\mathbf{r}| + |\mathbf{n}| = |\mathbf{d}| = d$.

Theorem 2.2.1. *The action of* $\mathbf{U}_q$ *described in Section* 2.1 *endows* $\mathscr{T}(\mathbf{d})$ *with the structure of a* $\mathbf{U}_q$*-module and the map*

$$\eta_{\mathbf{r},\mathbf{n}}\colon \mathrm{Span}\{f_{\mathbf{w},\mathbf{r},\mathbf{n}}\}_{\mathbf{w}} \to V_{\mathbf{n}_1-\mathbf{r}_1} \otimes \cdots \otimes V_{\mathbf{n}_k-\mathbf{r}_k}$$

given by

$$\eta_{\mathbf{r},\mathbf{n}}(f_{\mathbf{w},\mathbf{r},\mathbf{n}}) = \otimes^{\mathbf{n}-\mathbf{r}} v_{\mathbf{w}-\mathbf{r}} \tag{14}$$

(*and extended by linearity*) *is a* $\mathbf{U}_q$*-module isomorphism.*

Proof. Fix a $(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})$ such that $\alpha(W, \mathbf{D}) = \mathbf{w} - \delta^j$ for some j (it is easy to see that $Ef_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t) = 0$ unless W satisfies this property). Then

$$\begin{aligned} Ef_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t) &= q^{-\dim(\pi_1^{-1}(\mathbf{D},W,t))}(\pi_1)_! \pi_2^* f_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t) \\ &= k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-\dim(\pi_1^{-1}(\mathbf{D},W,t))} \chi_q(\pi_1^{-1}(\mathbf{D}, W, t) \cap \pi_2^{-1}(A_{\mathbf{w},\mathbf{r},\mathbf{n}})). \end{aligned}$$

Now,

$$\begin{aligned} \pi_1^{-1}(\mathbf{D}, W, t) &\cong \{U \mid W \subset U \subset \ker t,\ \dim U = \dim W + 1\} \\ &\cong \mathbb{P}^{\dim(\ker t) - \dim W - 1} \\ &= \mathbb{P}^{|\mathbf{n}| - (|\mathbf{w}| - 1) - 1} \\ &= \mathbb{P}^{|\mathbf{n}| - |\mathbf{w}|}. \end{aligned}$$

So $\dim(\pi_1^{-1}(\mathbf{D}, W, t)) = |\mathbf{n}| - |\mathbf{w}|$ and

$$\begin{aligned}
&\pi_1^{-1}(\mathbf{D}, W, t) \cap \pi_2^{-1}(A_{\mathbf{w},\mathbf{r},\mathbf{n}}) \\
&\quad \cong \{U \mid W \subset U \subset \ker t, \alpha(U, \mathbf{D}) = \mathbf{w}\} \\
&\quad \cong \{U \mid (W \cap \mathbf{D}_j) \subset U \subset (\ker t \cap \mathbf{D}_j), \\
&\qquad \dim(U \cap \mathbf{D}_{j-1}) = \mathbf{w}^{(1,j-1)}, \dim U = \mathbf{w}^{(1,j)}\} \\
&\quad \cong \{U \mid U \subset (\ker t \cap \mathbf{D}_j)/(W \cap \mathbf{D}_j), \\
&\qquad U \not\subset (\ker t \cap \mathbf{D}_{j-1})/(W \cap \mathbf{D}_{j-1}), \dim U = 1\} \\
&\quad \cong \mathbb{P}^{\dim(\ker\, t \cap \mathbf{D}_j)/(W \cap \mathbf{D}_j)-1} - \mathbb{P}^{\dim(\ker\, t \cap \mathbf{D}_{j-1})/(W \cap \ker\, \mathbf{D}_{j-1})-1} \\
&\quad = \mathbb{P}^{\mathbf{n}^{(1,j)} - (\mathbf{w}-\delta^j)^{(1,j)} - 1} - \mathbb{P}^{\mathbf{n}^{(1,j-1)} - (\mathbf{w}-\delta^j)^{(1,j-1)} - 1} \\
&\quad = \mathbb{P}^{\mathbf{n}^{(1,j)} - \mathbf{w}^{(1,j)}} - \mathbb{P}^{\mathbf{n}^{(1,j-1)} - \mathbf{w}^{(1,j-1)} - 1}.
\end{aligned}$$

Thus

$$\begin{aligned}
Ef_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t) &= k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-(|\mathbf{n}|-|\mathbf{w}|)} \left(\sum_{i=0}^{\mathbf{n}^{(1,j)} - \mathbf{w}^{(1,j)}} q^{2i} - \sum_{i=0}^{\mathbf{n}^{(1,j-1)} - \mathbf{w}^{(1,j-1)} - 1} q^{2i} \right) \\
&= k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{|\mathbf{w}|-|\mathbf{n}|} \sum_{\mathbf{n}^{(1,j-1)} - \mathbf{w}^{(1,j-1)}}^{\mathbf{n}^{(1,j)} - \mathbf{w}^{(1,j)}} q^{2i} \\
&= k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{|\mathbf{w}|-|\mathbf{n}|+2(\mathbf{n}^{(1,j-1)} - \mathbf{w}^{(1,j-1)})} \sum_{i=0}^{\mathbf{n}_j - \mathbf{w}_j} q^{2i} \\
&= k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-\mathbf{w}^{(1,j-1)} + \mathbf{w}^{(j+1,k)} + \mathbf{n}^{(1,j-1)} - \mathbf{n}^{(j+1,k)}} [\mathbf{n}_j - \mathbf{w}_j + 1].
\end{aligned}$$

Now,

$$k_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}} = k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-\mathbf{r}^{(1,j-1)} - \mathbf{n}^{(j+1,k)} + \mathbf{w}^{1,j-1} + \mathbf{w}^{j+1,k}}$$

So

$$k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-\mathbf{w}^{(1,j-1)} + \mathbf{w}^{(j+1,k)} + \mathbf{n}^{(1,j-1)} - \mathbf{n}^{(j+1,k)}} = k_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}} q^{\mathbf{r}(1,j-1) + \mathbf{n}^{(1,j-1)} - 2\mathbf{w}^{(1,j-1)}}$$

and thus

$$Ef_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t) = k_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}} q^{\mathbf{r}^{(1,j-1)} + \mathbf{n}^{(1,j-1)} - 2\mathbf{w}^{(1,j-1)}} [\mathbf{n}_j - \mathbf{w}_j + 1].$$

Therefore,

$$\begin{aligned} Ef_{\mathbf{w},\mathbf{r},\mathbf{n}} &= \sum_{j=1}^{k} q^{\mathbf{r}^{(1,j-1)}+\mathbf{n}^{(1,j-1)}-2\mathbf{w}^{(1,j-1)}}[\mathbf{n}_j - \mathbf{w}_j + 1]k_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}}\mathbf{1}_{A_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}}} \\ &= \sum_{j=1}^{k} q^{\mathbf{r}^{(1,j-1)}+\mathbf{n}^{(1,j-1)}-2\mathbf{w}^{(1,j-1)}}[\mathbf{n}_j - \mathbf{w}_j + 1]f_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}} \\ &= \sum_{j=1}^{k} q^{\sum_{i=1}^{j-1} (\mathbf{n}_i-\mathbf{r}_i-2(\mathbf{w}_i-\mathbf{r}_i))}[\mathbf{n}_j - \mathbf{w}_j + 1]f_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}}. \end{aligned} \tag{15}$$

Similarly

$$Ff_{\mathbf{w},\mathbf{r},\mathbf{n}} = \sum_{j=1}^{k} q^{-\sum_{i=j+1}^{k}(\mathbf{n}_i-\mathbf{r}_i-2(\mathbf{w}_i-\mathbf{r}_i))}[\mathbf{w}_j - \mathbf{r}_j + 1]f_{\mathbf{w}+\delta^j,\mathbf{r},\mathbf{n}}. \tag{16}$$

It follows immediately from (9) that

$$\begin{aligned} K^{\pm 1}f_{\mathbf{w},\mathbf{r},\mathbf{n}} &= q^{\pm(d-2|\mathbf{w}|)}f_{\mathbf{w},\mathbf{r},\mathbf{n}} \\ &= q^{\pm\sum_{i=1}^{k}(\mathbf{n}_i-\mathbf{r}_i-2(\mathbf{w}_i-\mathbf{r}_i))}f_{\mathbf{w},\mathbf{r},\mathbf{n}} \end{aligned} \tag{17}$$

since $|\mathbf{r}| + |\mathbf{n}| = |d|$.

Now recall that $x \in \mathbf{U}_q$ acts on $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ as $\Delta^{(k-1)}(x)$. In particular,

$$\Delta^{(k-1)}E = \sum_{i=1}^{k} K \otimes \cdots \otimes K \otimes E \otimes 1 \otimes \cdots \otimes 1,$$

$$\Delta^{(k-1)}F = \sum_{i=1}^{k} 1 \otimes \cdots \otimes 1 \otimes F \otimes K^{-1} \otimes \cdots \otimes K^{-1},$$

$$\Delta^{(k-1)}K^{\pm 1} = K^{\pm 1} \otimes \cdots \otimes K^{\pm 1}, \tag{18}$$

where in the first two equations, the E or F appears in the ith position. Comparing (18) and (1) to (15)–(17) the result follows. □

2.3. The space $\mathscr{T}_0(\mathbf{d})$ and the elementary basis $\mathscr{B}_e$

Note that if $t = 0$, then $\mathbf{r} = \mathbf{0}$ and $\mathbf{n} = \mathbf{d}$. Let $\mathscr{B}_e = \{f_{\mathbf{w},\mathbf{0},\mathbf{d}}\}_{\mathbf{w}}$. Then Span $\mathscr{B}_e$ is the space of invariant functions on $\mathfrak{T}_0(\mathbf{d})$ which we shall denote by $\mathscr{T}_0(\mathbf{d})$. We see from Theorem 2.2.1 that the map

$$\eta_{\mathbf{0},\mathbf{d}} : f_{\mathbf{w},\mathbf{0},\mathbf{d}} \mapsto \otimes^{\mathbf{d}} v_{\mathbf{w}}$$

(extended by linearity) is an isomorphism

$$\mathcal{T}_0(\mathbf{d}) \cong V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}.$$

We have therefore exhibited the elementary basis as the set of invariant functions on the variety $\mathfrak{T}_0 \subset \mathfrak{T}(\mathbf{d})$.

2.4. The space $\mathcal{T}_c(\mathbf{d})$

The goal of this section is to develop a natural way to extend invariant functions on $\mathfrak{T}_0(\mathbf{d})$ to invariant functions on $\mathfrak{T}(\mathbf{d})$ with larger supports. Recall that $\mathcal{T}_0(\mathbf{d})$ and $\mathcal{T}(\mathbf{d})$ are the spaces of invariant functions on $\mathfrak{T}_0(\mathbf{d})$ and $\mathfrak{T}(\mathbf{d})$, respectively. The action of E, F and $K^{\pm 1}$ defined by (2) gives both $\mathcal{T}_0(\mathbf{d})$ and $\mathcal{T}(\mathbf{d})$ the structure of a $\mathbf{U}_q$-module as can be seen from Theorem 2.2.1. We will call a $\mathbf{U}_q$-module map $\varepsilon : f \mapsto f^e$ from $\mathcal{T}_0(\mathbf{d})$ to $\mathcal{T}(\mathbf{d})$ an *extension*. Assuming an extension ε exists, $\eta_{\mathbf{r},\mathbf{n}} \circ \varepsilon \circ (\eta_{\mathbf{0},\mathbf{d}})^{-1}$ is an intertwiner from $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ to $V_{\mathbf{n}_1 - \mathbf{r}_1} \otimes \cdots \otimes V_{\mathbf{n}_k - \mathbf{r}_k}$. Conversely, each such set of intertwiners determines an extension. Namely, given a set of intertwiners

$$\{\gamma_{\mathbf{r},\mathbf{n}} : V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k} \to V_{\mathbf{n}_1 - \mathbf{r}_1} \otimes \cdots \otimes V_{\mathbf{n}_k - \mathbf{r}_k}\}_{\mathbf{r},\mathbf{n}},$$

we extend a function $f \in \mathcal{T}_0(\mathbf{d})$ to a function $f^e \in \mathcal{T}(\mathbf{d})$ by defining

$$f^e = \sum_{\mathbf{r},\mathbf{n}} (\eta_{\mathbf{r},\mathbf{n}})^{-1} \circ \gamma_{\mathbf{r},\mathbf{n}} \circ \eta_{\mathbf{0},\mathbf{d}}(f). \tag{19}$$

From Section 1.4 we know that a basis for the space of intertwiners between two tensor product representations of $\mathbf{U}_q$ is given by the corresponding crossingless matches. Now, a lower curve represents a particular action of $t \in \operatorname{End} W$. A lower curve connecting $V_{\mathbf{d}_i}$ and $V_{\mathbf{d}_j}$ with $i<j$ represents the fact that t sends a vector in $\mathbf{D}_j - \mathbf{D}_{j-1}$ to a vector in $\mathbf{D}_i - \mathbf{D}_{i-1}$. So for any lower crossingless match S, fix a basis of D compatible with the flag $\mathbf{D}$ and let t be the map whose matrix in this basis has (i,j) component equal to 1 if $i<j$ and S has an curve connecting the ith and jth vertices and is equal to zero otherwise. Then let $\mathbf{r}^S$ and $\mathbf{n}^S$ be defined as $\alpha(\operatorname{im} t, \mathbf{D})$ and $\alpha(\ker t, \mathbf{D})$. Thus, $\mathbf{r}_i^S$ is the number of left endpoints of the lower curves contained in $V_{\mathbf{d}_i}$ and $\mathbf{n}_i^S$ is $\mathbf{d}_i$ minus the number of right endpoints of the lower curves contained in $V_{\mathbf{d}_i}$. See Fig. 6. Then complete S to a crossingless match to $V_{\mathbf{n}_1^S - \mathbf{r}_1^S} \otimes \cdots \otimes V_{\mathbf{n}_k^S - \mathbf{r}_k^S}$ as in Fig. 7 (there is a unique way to do this). Let $\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S}$ be the corresponding intertwiner in the dual basis (that is, commuting with the action of $\mathbf{U}_q$ given by $\bar{\Delta}^{(k-1)}$). Note that $\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S}$ is well defined since the map $S \mapsto (\mathbf{r}^S, \mathbf{n}^S)$ described above is injective. Now let $\gamma_{\mathbf{r}^S,\mathbf{n}^S} = \sigma \tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S} \sigma$. As noted in Section 1.4, $\gamma_{\mathbf{r}^S,\mathbf{n}^S} : V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k} \to V_{\mathbf{n}_1^S - \mathbf{r}_1^S} \otimes \cdots \otimes V_{\mathbf{n}_k^S - \mathbf{r}_k^S}$ is an intertwiner in the usual basis (that is, it commutes with the action of $\mathbf{U}_q$ given by $\Delta^{(k-1)}$). For all $(\mathbf{r}, \mathbf{n})$ not of the form $(\mathbf{r}^S, \mathbf{n}^S)$ for some

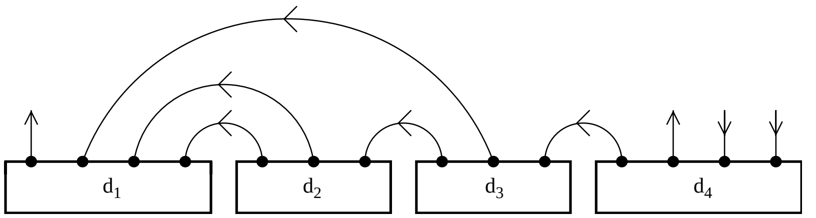

Fig. 6. Oriented lower crossingless match S. $\mathbf{r}^S = (3,1,1,0)$, $\mathbf{n}^S = (4,1,1,3)$.

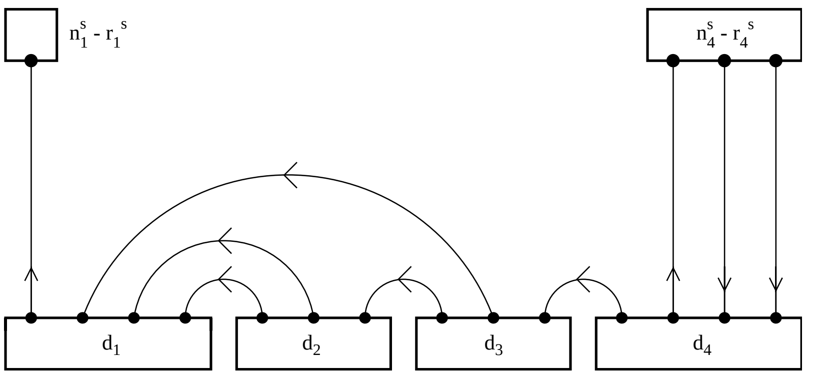

Fig. 7. Completion of Fig. 6 to an oriented crossingless match to $V_{\mathbf{n}_1^S - \mathbf{r}_1^S} \otimes \cdots \otimes V_{\mathbf{n}_k^S - \mathbf{r}_k^S} = V_1 \otimes V_0 \otimes V_0 \otimes V_3$.

lower crossingless match S, let $\gamma_{\mathbf{r},\mathbf{n}} = 0$. Then let $\varepsilon : f \mapsto f^e$ be the map defined by (19).

Proposition 2.4.1. *The extension* ε *is an isomorphism onto its image and*

$$f^e|_{\mathfrak{T}_0(\mathbf{d})} = f.$$

Proof. This follows immediately from Theorem 2.2.1. $\square$

Let

$$\mathscr{T}_c(\mathbf{d}) = \varepsilon(\mathscr{T}_0(\mathbf{d})) \subset \mathscr{T}(\mathbf{d}).$$

It follows from Proposition 2.4.1 and Theorem 2.2.1 that $\mathscr{T}_c(\mathbf{d}) \cong V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$. And it follows from Proposition 2.4.1 that $\varepsilon : \mathscr{T}_0(\mathbf{d}) \to \mathscr{T}_c(\mathbf{d})$ is an isomorphism of $\mathbf{U}_q$-modules with inverse given by restriction to $\mathfrak{T}_0(\mathbf{d})$. We will find a distinguished basis of $\mathscr{T}_c(\mathbf{d})$ related to the irreducible components of $\mathfrak{T}(\mathbf{d})'$. Before we do this, we must first examine these irreducible components.

2.5. The irreducible components of the tensor product variety

For the remainder of this section we consider varieties defined over $\bar{\mathbb{F}}_{q^2}$. To avoid confusion, we denote the corresponding varieties by $\mathfrak{T}_0(\mathbf{d})'$ and $\mathfrak{T}(\mathbf{d})'$. For $\mathbf{w}\in(\mathbb{Z}_{\geqslant 0})^k$ such that $\mathbf{w}_i\leqslant\mathbf{d}_i$, let

$$Z'_{\mathbf{w}}=\{(\mathbf{D},W,t)\in\mathfrak{T}(\mathbf{d})'|\alpha(W,\mathbf{D})=\mathbf{w}\}.$$

We then have the following.

Theorem 2.5.1. *$\{\overline{Z'_{\mathbf{w}}}\}_{\mathbf{w}}$ are the irreducible components of $\mathfrak{T}(\mathbf{d})'$.*

Proof. It is obvious that $\sqcup_{\mathbf{w}}Z'_{\mathbf{w}}=\mathfrak{T}(\mathbf{d})'$ (where $\sqcup$ denotes disjoint union). Also, the connected components of $\mathfrak{T}(\mathbf{d})'$ are given by fixing the dimension of W. Thus, since $|\mathbf{w}|=\dim W$, it suffices to prove that the $Z'_{\mathbf{w}}$ are irreducible and locally closed and that $\dim Z'_{\mathbf{w}}$ is independent of $\mathbf{w}$ for fixed $|\mathbf{w}|$. Consider the maps

$$Z'_{\mathbf{w}}\xrightarrow{p_1}{}^1Z'_{\mathbf{w}}\xrightarrow{p_2}{}^2Z'_{\mathbf{w}},$$

where

$${}^1Z'_{\mathbf{w}}=\{(\mathbf{D},W)\mid(\mathbf{D},W,t)\in Z'_{\mathbf{w}}\text{ for some }t\},$$

$${}^2Z'_{\mathbf{w}}=\{\mathbf{D}\mid(\mathbf{D},W)\in{}^1Z'_{\mathbf{w}}\text{ for some }W\},$$

$$p_1(\mathbf{D},W,t)=(\mathbf{D},W),$$

$$p_2(\mathbf{D},W)=\mathbf{D}.$$

Then p_1 and p_2 are locally trivial fibrations. Now

$${}^2Z'_{\mathbf{w}}=\{\mathbf{D}=\{\mathbf{D}_i\}_{i=0}^k\mid 0=\mathbf{D}_0\subset\mathbf{D}_1\subset\cdots\subset\mathbf{D}_k=D,\ \dim\mathbf{D}_i/\mathbf{D}_{i-1}=\mathbf{d}_i\}$$

is simply a flag manifold. It is a homogeneous space as follows. $\mathrm{GL}(D)$ acts transitively on ${}^2Z'_{\mathbf{w}}$ with stabilizer isomorphic to the set of matrices

$$G_0=\left\{\left.\begin{pmatrix}M_1&*&\cdots&*\\0&M_2&\ddots&\vdots\\\vdots&\ddots&\ddots&*\\0&\cdots&0&M_k\end{pmatrix}\right|M_i\in\mathrm{GL}(\mathbf{d}_i)\right\}.$$

Thus,

$$\dim {}^2Z'_{\mathbf{w}} = \dim \mathrm{GL}(D) - \dim G_0 = \sum_{i<j} \mathbf{d}_i \mathbf{d}_j. \quad (20)$$

Now, the fiber of p_2 over a point $\mathbf{D} \in {}^2Z'_{\mathbf{w}}$ is

$$F_2 = \{W \subset D \mid \alpha(W, \mathbf{D}) = \mathbf{w}\}.$$

The group G_0 acts transitively on this space and the stabilizer is isomorphic to the set of matrices

$$G_1 = \left\{ \begin{pmatrix} M_1 & * & * & * & * & \cdots & \cdots & * \\ 0 & N_1 & 0 & * & 0 & \cdots & \cdots & * \\ 0 & 0 & M_2 & * & * & & & * \\ 0 & 0 & 0 & N_2 & 0 & & & * \\ 0 & 0 & 0 & 0 & \ddots & \ddots & & * \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & M_k & * \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & N_k \end{pmatrix} \middle| \begin{array}{c} M_i \in GL(\mathbf{w}_i), \\ N_i \in GL(\mathbf{d}_i - \mathbf{w}_i) \end{array} \right\}.$$

So

$$\dim F_2 = \dim G_0 - \dim G_1 = \sum_{j \leqslant i} \mathbf{w}_i (\mathbf{d}_j - \mathbf{w}_j). \quad (21)$$

The fiber of p_1 over a point $(\mathbf{D}, W) \in {}^1Z'_{\mathbf{w}}$ is

$$F_1 = \{t \in \mathrm{End}\, D \mid t(\mathbf{D}_i) \subset \mathbf{D}_{i-1},\ \mathrm{im}\, t \subset W \subset \ker t\}. \quad (22)$$

Pick a basis $\{u_i\}_{i=1}^d$ of D such that $\{u_i\}_{i=1}^{\mathbf{d}_1 + \cdots + \mathbf{d}_j}$ is a basis for $\mathbf{D}_j$ and

$$\bigcup_{l=0}^{j} \{u_i\}_{i=\mathbf{d}^{(1,l-1)}+1}^{\mathbf{d}^{(1,l-1)}+\mathbf{w}_l}$$

(where $\mathbf{d}^{(1,0)} = 0$) is a basis for $W \cap \mathbf{D}_j$. Then by considering the matrices of t in this basis it is easy to see that F_1 is an affine space of dimension

$$\dim F_1 = \sum_{i<j} \mathbf{w}_i (\mathbf{d}_j - \mathbf{w}_j). \quad (23)$$

So from Eqs. (20)–(23) we see that

$$\dim Z'_{\mathbf{w}} = \sum_{i<j} \mathbf{d}_i \mathbf{d}_j + \sum_{i,j=1}^{k} \mathbf{w}_i(\mathbf{d}_j - \mathbf{w}_j)$$
$$= \sum_{i<j} \mathbf{d}_i \mathbf{d}_j + |\mathbf{w}|(d - |\mathbf{w}|) \tag{24}$$

and thus $\dim Z'_{\mathbf{w}}$ is independent of $\mathbf{w}$ (for a fixed value of $|\mathbf{w}|$).

Now, ${}^2Z'_{\mathbf{w}}, F_1$ and F_2 are all smooth and connected, hence irreducible. Also, ${}^2Z'_{\mathbf{w}}$ and F_1 are closed while F_2 is locally closed. The latter statement follows from the fact that F_2 is equal to the closed set $\{W \subset D \mid \alpha(W, \mathbf{D}) \geqslant \mathbf{w}\}$ minus the finite collection of closed sets $\{W \subset D \mid \alpha(W, \mathbf{D}) \geqslant \mathbf{a}\}_{\mathbf{a} > \mathbf{w}}$. Thus each $Z'_{\mathbf{w}}$ is irreducible and locally closed. □

Let

$$A'_{\mathbf{w},\mathbf{r},\mathbf{n}} = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})' \mid \alpha(W, \mathbf{D}) = \mathbf{w}, \alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r}, \alpha(\ker t, \mathbf{D}) = \mathbf{n}\}. \tag{25}$$

We will need the following two propositions in the sequel.

Proposition 2.5.1. *Let* $\mathbf{w} \in (\mathbb{Z}_{\geqslant 0})^k$ *with* $\mathbf{w}_i \leqslant \mathbf{d}_i$ *for all* i *and let* $M = M(\mathbf{d}, \mathbf{w})$. *Then* $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ *is an open dense subset of* $\overline{Z'_{\mathbf{w}}}$. *In particular,* $\overline{A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}} = \overline{Z'_{\mathbf{w}}}$.

Proof. It is enough to show that $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ is dense in $Z'_{\mathbf{w}}$ (it is obvious from the definitions that $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M} \subset Z'_{\mathbf{w}}$). Since $\mathbf{r}^M \leqslant \mathbf{w} \leqslant \mathbf{n}^M$ by construction of M, we have that the projection of $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ onto ${}^1Z'_{\mathbf{w}}$ is all of ${}^1Z'_{\mathbf{w}}$. Thus it suffices to show that $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ is dense in each fiber. Fix $(\mathbf{D}, W) \in {}^1Z'_{\mathbf{w}}$. The fiber, F_1, of the projection p_1 is given by (22). The intersection of F_1 with $(p_1|_{A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}})^{-1}(\mathbf{D}, W)$ is isomorphic to

$$B = \{t \in \operatorname{End} D \mid t(\mathbf{D}_i) \subset \mathbf{D}_{i-1}, \operatorname{im} t \subset W \subset \ker t, \alpha(\ker t, \mathbf{D}) = \mathbf{n}^M, \alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r}^M\}.$$

Choose a basis β of D compatible with the flag $\mathbf{D}$ and subspace W (that is, there exist bases for W and each $\mathbf{D}_i$ which are subsets of β). Now, since $\operatorname{im} t \subset W \subset \ker t$, t can be factored through D/W and considered as a map into W. Each t is uniquely determined by the corresponding $\bar{t} \in \operatorname{End}(D/W, W)$. Consider the matrix of $\bar{t}$ in the basis of D/W given by the projection of the basis β under the natural map $D \to D/W$ and the basis of W which is a subset of β. It must be of the following form:

$$C_t = \begin{pmatrix} 0 & A_{1,2} & A_{1,3} & \cdots & A_{1,k} \\ \vdots & 0 & A_{2,3} & \cdots & A_{2,k} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & A_{k-1,k} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{26}$$

where $A_{i,j}$ is a $(\mathbf{w}_i) \times (\mathbf{d}_j - \mathbf{w}_j)$ matrix. Then $t \in B$ if and only if each submatrix

$$C_t^{i,j} = \begin{pmatrix} A_{i,i+1} & A_{i,i+2} & \cdots & A_{i,j+1} \\ 0 & A_{i+1,i+2} & \cdots & A_{i+1,j+1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{j,j+1} \end{pmatrix}, \quad 1 \leqslant i \leqslant j \leqslant k-1 \tag{27}$$

has maximal rank. To see this, consider the diagram M' of non-crossing-oriented curves connecting the $V_{\mathbf{d}_i}$ associated to a $t \in F_1$. That is, the number of down-oriented vertices among those associated to $V_{\mathbf{d}_i}$ is given by $\mathbf{w}_i$ and the number of left and right endpoints of curves of M' in $V_{\mathbf{d}_i}$ are given by $\alpha(\operatorname{im} t, \mathbf{D})_i$ and $\mathbf{d}_i - \alpha(\ker t, \mathbf{D})_i$, respectively. A priori, this is not an oriented lower crossingless match (for instance, the unmatched vertices of M' might not be arranged so that those oriented down are to the right of those oriented up). The requirement that $C_t^{i,j}$ has maximal rank is equivalent to the requirement that M' has the maximum possible number of curves connecting $V_{\mathbf{d}_i}$, $V_{\mathbf{d}_{i+1}}, \ldots$, and $V_{\mathbf{d}_{j+1}}$. Thus, referring to the definition of $M(\mathbf{d}, \mathbf{w})$ given in Section 1.4, we see that the condition that all the $C_t^{i,j}$ have maximal rank is equivalent to the condition that $M' = M$ (so M' is indeed an oriented crossingless match) and thus equivalent to $\alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r}^{M'} = \mathbf{r}^M$ and $\alpha(\ker t, \mathbf{D}) = \mathbf{n}^{M'} = \mathbf{n}^M$ or $t \in B$. Note that this argument also allows us to see that B is not empty since it contains the element t given by the matrix whose (i,j) entry is 1 if $i<j$ and M contains a curve connecting the ith and jth vertices and zero otherwise. In fact, this is a canonical form of any $t \in B$. That is, by a change of basis (preserving the flag $\mathbf{D}$), we can transform the matrix of any $t \in B$ to this form.

Assume we know that the subset $N_{m,n}$ of the set $M_{m,n}$ of $m \times n$ matrices given by

$$N_{m,n} = \{A \in M_{m,n} \mid A \text{ has maximal rank}\}$$

is an open subset of the set $M_{m,n}$. Then $N_{m,n}$ is given by the non-vanishing of a finite collection of polynomials in the matrix elements of $M_{m,n}$ (recall we are working in the Zariski topology). Thus, the requirement that the submatrices $C_t^{i,j}$ have maximal rank is equivalent to the non-vanishing of a finite number of polynomials in the matrix elements of the $C_t^{i,j}$ (and hence of C_t). Therefore, we will have shown that B is the intersection of a finite number of open subsets of F_1 and hence is open (and thus dense since it is not empty) in F_1.

So it remains to show that $N_{m,n}$ is dense in $M_{m,n}$. But if we let $r = \min(m,n)$, then

$$N_{m,n} = \{A \in M_{m,n} \mid \text{At least one } r \times r \text{ submatrix of } A \text{ has rank } r\}$$

which is a union of open subsets of $M_{m,n}$ (since an $r \times r$ matrix has rank r if and only if its determinant is non-zero) and hence open (and dense) in $M_{m,n}$. $\square$

Proposition 2.5.2. *With the notation of Proposition* 2.5.1, $\overline{A'_{\mathbf{a},\mathbf{r}^S,\mathbf{n}^S}} \subset \overline{Z'_{\mathbf{w}}}$ for all $S \leqslant M$, $\mathbf{a} \geqslant \mathbf{w}$, $|\mathbf{a}| = |\mathbf{w}|$.

Proof. It suffices to show that $A'_{\mathbf{a},\mathbf{r}^S,\mathbf{n}^S}$ is contained in $\overline{Z'_{\mathbf{w}}}$. The image of $A'_{\mathbf{a},\mathbf{r}^S,\mathbf{n}^S}$ under the projection p_1 is $\{(\mathbf{D}, W) \mid \alpha(W, \mathbf{D}) = \mathbf{a}\}$ which is contained in $\overline{{}^1Z'_{\mathbf{w}}}$ since $\mathbf{a} \geqslant \mathbf{w}$ and $|\mathbf{a}| = |\mathbf{w}|$. The fiber of the projection p_1 (restricted to $A'_{\mathbf{a},\mathbf{r}^S,\mathbf{n}^S}$) over a point $(\mathbf{D}, W)$ is

$$\{t \mid (\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}), \alpha(\ker t, \mathbf{D}) = \mathbf{n}^S, \alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r}^S\}$$

and this is in the closure of the set

$$\{t \mid (\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}), \alpha(\ker t, \mathbf{D}) = \mathbf{n}^M, \alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{r}^M\}$$

since $S \leqslant M$. So $A'_{\mathbf{a},\mathbf{r}^S,\mathbf{n}^S} \subset \overline{Z'_{\mathbf{w}}}$. $\square$

We now define the irreducible components of $\mathfrak{T}(\mathbf{d})$ to be the $\mathbb{F}_{q^2}$ points of the irreducible components of $\mathfrak{T}(\mathbf{d})'$. Let $\overline{Z_{\mathbf{w}}}$ denote the set of $\mathbb{F}_{q^2}$ points of the irreducible component $\overline{Z'_{\mathbf{w}}}$ of $\mathfrak{T}(\mathbf{d})'$. We also define the *dense points* of an irreducible component $\overline{Z_{\mathbf{w}}}$ of $\mathfrak{T}(\mathbf{d})$ to be the $\mathbb{F}_{q^2}$ points of the dense subset $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ (where $M = M(\mathbf{d}, \mathbf{w})$) of the corresponding irreducible component $\overline{Z'_{\mathbf{w}}}$ of $\mathfrak{T}(\mathbf{d})'$. However, the $\mathbb{F}_{q^2}$ points of $A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$ are exactly the elements of $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$. Thus, the dense points of the irreducible component $\overline{Z_{\mathbf{w}}}$ of $\mathfrak{T}(\mathbf{d})$ are just the points of $A_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$.

2.6. Geometric realization of the canonical basis

We are now ready to describe the set of functions mentioned at the end of Section 2.4. Define

$$h^{\mathbf{d}}_{\mathbf{w}} = \eta^{-1}_{\mathbf{0},\mathbf{d}}(\Diamond^{\mathbf{d}} v_{\mathbf{w}}),$$

$$g^{\mathbf{d}}_{\mathbf{w}} = (h^{\mathbf{d}}_{\mathbf{w}})^e,$$

$$\mathscr{B}_c = \{g^{\mathbf{d}}_{\mathbf{w}}\}_{\mathbf{w}}. \tag{28}$$

For a vector $w_1 \otimes \cdots \otimes w_k \in V_{\mathbf{a}_1} \otimes \cdots V_{\mathbf{a}_k}$, let $(w_1 \otimes \cdots \otimes w_k)^r = w_k \otimes \cdots \otimes w_1 \in V_{\mathbf{a}_k} \otimes \cdots \otimes V_{\mathbf{a}_1}$. For an intertwiner $\gamma : V_{\mathbf{a}_1} \otimes \cdots \otimes V_{\mathbf{a}_k} \to V_{\mathbf{b}_1} \otimes \cdots \otimes V_{\mathbf{b}_l}$ corresponding to a crossingless match S, let $\gamma^\dagger : V_{\mathbf{b}_l} \otimes \cdots \otimes V_{\mathbf{b}_1} \to V_{\mathbf{a}_k} \otimes \cdots \otimes V_{\mathbf{a}_1}$ denote the intertwiner corresponding to the crossingless match S rotated 180°. It follows easily from the graphical calculus described in [1] that

$$\langle \gamma(v), w \rangle = \langle v, (\sigma\gamma^\dagger\sigma)(w) \rangle = \langle v, (\tilde{\gamma})^\dagger(w) \rangle$$

for any $v \in V_{\mathbf{a}_1} \otimes \cdots \otimes V_{\mathbf{a}_k}$ and $w \in V_{\mathbf{b}_l} \otimes \cdots \otimes V_{\mathbf{b}_1}$.

We will need the following results.

Lemma 2.6.1

$$\gamma_{\mathbf{r}^S,\mathbf{n}^S}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}) = \begin{cases} \diamondsuit^{\mathbf{n}^S-\mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S} & \text{if } S \leqslant M(\mathbf{d},\mathbf{w}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is apparent from the graphical calculus of [1] that if $S \leqslant M(\mathbf{d},\mathbf{w})$, then $(\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S})^\dagger((\heartsuit^{\mathbf{n}^S-\mathbf{r}^S} v^{\mathbf{w}-\mathbf{r}^S})^r) = (\heartsuit^{\mathbf{d}} v^{\mathbf{w}})^r$ and that $(\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S})^\dagger$ sends other dual canonical basis elements $(\heartsuit^{\mathbf{n}^S-\mathbf{r}^S} v^{\mathbf{a}})^r$, $\mathbf{a} \neq \mathbf{w} - \mathbf{r}^S$, to elements of the form $(\heartsuit^{\mathbf{d}} v^{\mathbf{a}'})^r$ with $\mathbf{a}' \neq \mathbf{w}$. Therefore

$$\begin{aligned} \langle \gamma_{\mathbf{r}^S,\mathbf{n}^S}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}), (\heartsuit^{\mathbf{n}^S-\mathbf{r}^S} v^{\mathbf{w}-\mathbf{r}^S})^r \rangle &= \langle \diamondsuit^{\mathbf{d}} v_{\mathbf{w}}, (\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S})^\dagger((\heartsuit^{\mathbf{n}^S-\mathbf{r}^S} v^{\mathbf{w}-\mathbf{r}^S})^r) \rangle \\ &= \langle \diamondsuit^{\mathbf{d}} v_{\mathbf{w}}, (\heartsuit^{\mathbf{d}} v^{\mathbf{w}})^r \rangle \\ &= 1 \end{aligned}$$

and

$$\langle \gamma_{\mathbf{r}^S,\mathbf{n}^S}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}), (\heartsuit^{\mathbf{n}^S-\mathbf{r}^S} v^{\mathbf{a}})^r \rangle = 0$$

for all $\mathbf{a} \neq \mathbf{w} - \mathbf{r}^S$. Thus $\gamma_{\mathbf{r}^S,\mathbf{n}^S}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}) = \diamondsuit^{\mathbf{n}^S-\mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S}$. A similar argument demonstrates that $\gamma_{\mathbf{r}^S,\mathbf{n}^S}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}) = 0$ if $S \not\leqslant M(\mathbf{d},\mathbf{w})$ since then the image of $(\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S})^\dagger$ is spanned by $\heartsuit^{\mathbf{d}} v^{\mathbf{a}}$ with $\mathbf{a} \neq \mathbf{w}$. □

Proposition 2.6.1.

$$g^{\mathbf{d}}_{\mathbf{w}} = \sum_{S \leqslant M(\mathbf{d},\mathbf{w})} (\eta_{\mathbf{r}^S \mathbf{n}^S})^{-1} (\diamondsuit^{\mathbf{n}^S-\mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S}).$$

Proof. This follows immediately from Lemma 2.6.1. □

Proposition 2.6.2. *$\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}$ is equal to $\otimes^{\mathbf{d}} v_{\mathbf{w}}$ plus a linear combination of elements $\otimes^{\mathbf{d}} v_{\mathbf{a}}, \mathbf{a} > \mathbf{w}, |\mathbf{a}| = |\mathbf{w}|$, with coefficients in $q^{-1}\mathbb{N}[q^{-1}]$.*

Proof. This follows from Sections 1.5 and 1.6 of [1]. □

We can now prove one of our main results.

Theorem 2.6.1. *$g^{\mathbf{d}}_{\mathbf{w}}$ is the unique element of $\mathcal{T}_c(\mathbf{d})$, up to a multiplicative constant, satisfying the following conditions*:

1. *$g^{\mathbf{d}}_{\mathbf{w}}$ is equal to a non-zero constant on the set of dense points $A_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ of the irreducible component $\overline{Z_{\mathbf{w}}}$ (where $M = M(\mathbf{d},\mathbf{w})$).*
2. *The support of $g^{\mathbf{d}}_{\mathbf{w}}$ lies in $\overline{Z_{\mathbf{w}}}$.*

Furthermore, the set $\{g_{\mathbf{w}}^{\mathbf{d}}\}_{\mathbf{w}}$ *is a basis of* $\mathcal{T}_c(\mathbf{d})$ *and the map*

$$\diamondsuit^{\mathbf{d}} v_{\mathbf{w}} \mapsto g_{\mathbf{w}}^{\mathbf{d}}$$

(*extended by linearity*) *is a* $\mathbf{U}_q$*-module isomorphism* $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k} \cong \mathcal{T}_c(\mathbf{d})$.

Proof. In this proof, to simplify notation in calculations, we will suppress the isomorphism $\eta_{\mathbf{r},\mathbf{n}}$ defined by (14) and identify the vector $\otimes^{\mathbf{n}-\mathbf{r}} v_{\mathbf{w}-\mathbf{r}}$ with the function $f_{\mathbf{w},\mathbf{r},\mathbf{n}}$. Assume that $g_{\mathbf{w}}^{\prime\mathbf{d}}$ satisfies the above conditions and let $g_{\mathbf{w}}^{\prime\mathbf{d}} = (h_{\mathbf{w}}^{\prime\mathbf{d}})^e$. The value of $g_{\mathbf{w}}^{\prime\mathbf{d}}$ on $A_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ is given by $k_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ times the coefficient of $f_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ when $g_{\mathbf{w}}^{\prime\mathbf{d}}$ is written as a linear combination of the basic functions. This coefficient is equal to

$$\langle \gamma_{\mathbf{r}^M,\mathbf{n}^M}(h_{\mathbf{w}}^{\prime\mathbf{d}}), (\otimes^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M})^r \rangle.$$

.

Therefore, since $k_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M} \neq 0$, condition 1 is equivalent to

$$\langle \gamma_{\mathbf{r}^M,\mathbf{n}^M}(h_{\mathbf{w}}^{\prime\mathbf{d}}), (\otimes^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M})^r \rangle \neq 0 \iff \langle h_{\mathbf{w}}^{\prime\mathbf{d}}, (\tilde{\gamma}_{\mathbf{r}^M,\mathbf{n}^M})^{\dagger}((\otimes^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M})^r) \rangle \neq 0.$$

Now, since $M = M(\mathbf{d}, \mathbf{w})$ is the oriented lower crossingless match associated to $\mathbf{w}$, $M(\mathbf{n}^M - \mathbf{r}^M, \mathbf{w} - \mathbf{r}^M)$ has no lower curves and all down arrows are to the right of all up arrows. So after being rotated by 180° (but keeping the original orientation of unmatched vertices—for example, those oriented up remain oriented up), this diagram has all down arrows to the left of all up arrows. Thus, by Section 2.3 of [1], $(\otimes^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M})^r = (\heartsuit^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M})^r$. It also follows from the graphical calculus of [1] that

$$(\tilde{\gamma}_{\mathbf{r}^M,\mathbf{n}^M})^{\dagger}((\heartsuit^{\mathbf{n}^M-\mathbf{r}^M} v^{\mathbf{w}-\mathbf{r}^M})^r) = (\heartsuit^{\mathbf{d}} v^{\mathbf{w}})^r.$$

Therefore condition 1 is equivalent to

$$\langle h_{\mathbf{w}}^{\prime\mathbf{d}}, (\heartsuit^{\mathbf{d}} v^{\mathbf{w}})^r \rangle \neq 0. \tag{29}$$

Next we consider condition 2. In order for this condition to be satisfied, $g_{\mathbf{w}}^{\prime\mathbf{d}}$ must be equal to zero on $A_{\mathbf{w}',\mathbf{r}^{M'},\mathbf{n}^{M'}}$ for all $\mathbf{w}' \neq \mathbf{w}$ (where $M' = M(\mathbf{d}, \mathbf{w}')$). By an argument analogous to that given above, this is equivalent to the condition

$$\langle h_{\mathbf{w}}^{\prime\mathbf{d}}, (\heartsuit^{\mathbf{d}} v^{\mathbf{w}'})^r \rangle = 0 \tag{30}$$

for all $\mathbf{w}' \neq \mathbf{w}$. Therefore, by (29) and (30), we must have

$$h_{\mathbf{w}}^{\prime\mathbf{d}} = c_{\mathbf{w}}^{\mathbf{d}} \cdot \diamondsuit^{\mathbf{d}} v_{\mathbf{w}} = c_{\mathbf{w}}^{\mathbf{d}} \cdot h_{\mathbf{w}}^{\mathbf{d}}$$

for some non-zero constant $c_{\mathbf{w}}^{\mathbf{d}}$ which proves uniqueness up to a multiplicative constant. It still remains to show that $g_{\mathbf{w}}^{\mathbf{d}}$ satisfies the given conditions.

Now, by Proposition 2.6.1, the value of $g_{\mathbf{w}}^{\mathbf{d}}$ on $A_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$, where $M = M(\mathbf{d},\mathbf{w})$, is equal to $k_{\mathbf{w},\mathbf{r},\mathbf{n}}$ times the coefficient of $\otimes^{\mathbf{n}^M-\mathbf{r}^M} v_{\mathbf{w}-\mathbf{r}^M}$ in the expression of $\diamondsuit^{\mathbf{n}^M-\mathbf{r}^M} v_{\mathbf{w}-\mathbf{r}^M}$ as a linear combination of elementary basis elements. By Proposition 2.6.2, this coefficient is equal to 1. So $g_{\mathbf{w}}^{\mathbf{d}}$ is equal to a non-zero constant on $A_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$. Also, by Propositions 2.6.1 and 2.6.2, $g_{\mathbf{w}}^{\mathbf{d}}$ is equal to a linear combination of functions of the form $(\eta_{\mathbf{r}^S,\mathbf{n}^S})^{-1}(\otimes^{\mathbf{n}^S-\mathbf{r}^S} v_{\mathbf{a}}) = f_{\mathbf{a}+\mathbf{r}^S,\mathbf{r}^S,\mathbf{n}^S}$ with $S \leqslant M, |\mathbf{a}| = |\mathbf{w}-\mathbf{r}^S|$ $(\Rightarrow |\mathbf{a}+\mathbf{r}^S| = |\mathbf{w}|)$, and $\mathbf{a} \geqslant \mathbf{w}-\mathbf{r}^S$ $(\Rightarrow \mathbf{a}+\mathbf{r}^s \geqslant \mathbf{w})$. Thus, by Proposition 2.5.2, the support of $g_{\mathbf{w}}^{\mathbf{d}}$ lies in $\overline{Z_{\mathbf{w}}}$. So we have demonstrated that the functions $g_{\mathbf{w}}^{\mathbf{d}}$ are the unique functions, up to a multiplicative constant, satisfying conditions 1 and 2.

The last two statements of the theorem follow from the fact that the map $\eta_{\mathbf{0},\mathbf{d}} : \diamondsuit^{\mathbf{d}} v_{\mathbf{w}} \mapsto h_{\mathbf{w}}^{\mathbf{d}}$ (extended by linearity) is a $\mathbf{U}_q$-module isomorphism $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k} \cong \mathscr{T}_0(\mathbf{d})$ and the fact that ε is an isomorphism onto its image. □

2.7. A conjectured characterization of $\mathscr{T}_c(\mathbf{d})$ and $\mathscr{B}_c$

We present here a conjecture concerning an alternative characterization of the basis $\mathscr{B}_c$. Let $\mathscr{P}$ be the category of semisimple perverse sheaves on $\mathfrak{T}(\mathbf{d})'$ constructible with respect to the stratification given by the $A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$ and let $\mathscr{D} : \mathscr{P} \to \mathscr{P}$ be the operation of Verdier Duality. For $\mathbf{B}^\bullet \in \mathscr{P}$ and $x \in \mathfrak{T}(\mathbf{d})'$, $\mathbf{B}_x^\bullet$ denotes the stalk complex at the point x. We define the action of the involution $\Psi^{(k)}$ on $\mathscr{T}(\mathbf{d})$ by

$$\Psi^{(k)}(f_{\mathbf{w},\mathbf{r},\mathbf{n}}) = (\eta_{\mathbf{r},\mathbf{n}})^{-1} \Psi^{(k)} \eta_{\mathbf{r},\mathbf{n}}(f_{\mathbf{w},\mathbf{r},\mathbf{n}}),$$

where on the right-hand side $\Psi^{(k)}$ is the involution used to characterize the canonical basis (see [1, Section 1.6]). In particular, the canonical basis is invariant under the action of $\Psi^{(k)}$. Now let $\theta : \mathscr{P} \to \mathscr{T}(\mathbf{d})$ be the map such that

$$(\theta(\mathbf{B}^\bullet))(x) = \sum_i (-1)^i q^i \dim H_i(\mathbf{B}_x^\bullet) \quad \text{for } x \in \mathfrak{T}(\mathbf{d}) \text{ and } \mathbf{B}^\bullet \in \mathscr{P}.$$

For each irreducible component $\overline{Z'_{\mathbf{w}}}$ of $\mathfrak{T}(\mathbf{d})'$ there is an intersection sheaf complex $IC_{\mathbf{w}}^\bullet$ associated to the local system which is the constant sheaf $\mathbb{C}$ (in degree zero) on the dense subset $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$ where $M = M(\mathbf{d},\mathbf{w})$ (see [3] for details).

Conjecture 2.7.1. $\theta(IC_{\mathbf{w}}^\bullet) = k_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}^{-1} g_{\mathbf{w}}^{\mathbf{d}}$.

The factor of $k_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}^{-1}$ arises from the fact that $\theta(IC_{\mathbf{w}}^\bullet)$ is equal to one on the set $A'_{\mathbf{w},\mathbf{r}^M,\mathbf{n}^M}$. The proof of Conjecture 2.7.1 would most likely center around the idea that the action of Verdier Duality in $\mathscr{P}$ should correspond to the action of $\Psi^{(k)}$ in $\mathscr{T}(\mathbf{d})$. The precise statement is the following:

Conjecture 2.7.2. $\theta\mathscr{D} = \Psi^{(k)}\theta$.

3. Geometric realization of the intertwiners

3.1. Defining the intertwiners

The goal of this section is to decompose $\mathfrak{T}(\mathbf{d})$ into subsets corresponding to a basis for the space of intertwiners

$$H^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k} = \mathrm{Hom}_{\mathbf{U}_q}(V_{\mathbf{d}_1}\otimes\cdots\otimes V_{\mathbf{d}_k}, V_\mu). \tag{31}$$

Note that $H^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k}=0$ unless $\mu = d-2r$ for some $0\leqslant r\leqslant d/2$ (where $d=|\mathbf{d}|$). Thus, the intertwiners will be maps from $\mathscr{T}(\mathbf{d})$ to $\mathscr{M}(d)$ since these V_μ are precisely the representations appearing in $\mathscr{M}(d)$ (see Section 1.3).

Let Y be a constructible subset of $\mathfrak{T}(\mathbf{d})$. Define $R_Y:\mathscr{T}(\mathbf{d})\to\mathscr{T}(\mathbf{d})$ to be the map which restricts functions to their values on Y. That is, for $f\in\mathscr{T}(\mathbf{d})$, $R_Yf=\mathbf{1}_Yf$ (where the multiplication of functions is pointwise).

Consider the map $p:\mathfrak{T}(\mathbf{d})\to\mathfrak{M}(d)$ such that $p(\mathbf{D},W,t)=(W,t)$. Let $T_Y=p_!R_Y$. Then T_Y is a map from $\mathscr{T}(\mathbf{d})$ to $\mathscr{M}(d)$.

Proposition 3.1.1. If $Y\subset\mathfrak{T}(\mathbf{d})$ *satisfies* $\pi_1\pi_2^{-1}(Y)\subset Y$ *and* $\pi_2\pi_1^{-1}(Y)\subset Y$ *where* π_1 *and* π_2 *are the maps from* (8) *then* T_Y *is an intertwiner.*

Proof. It suffices to show that T_Y commutes with the action of E, F and $K^{\pm 1}$ since these elements generate $\mathbf{U}_q$. Note that the condition $\pi_1\pi_2^{-1}(Y)\subset Y$ implies $\pi_2^{-1}(Y)\subset\pi_1^{-1}(Y)$ and the condition $\pi_2\pi_1^{-1}(Y)\subset Y$ implies $\pi_1^{-1}(Y)\subset\pi_2^{-1}(Y)$. Thus $\pi_1^{-1}(Y)=\pi_2^{-1}(Y)$. We first show that $T_YE=ET_Y$. Now $T_YE=p_!R_YE$ and $ET_Y=Ep_!R_Y$. Thus it suffices to show that $R_YE=ER_Y$ and $p_!E=Ep_!$. Since $\mathscr{T}(\mathbf{d})$ is spanned by functions of the form $\mathbf{1}_A$ where A is a subvariety of $\mathfrak{T}(\mathbf{d})$, we need only check that actions agree on such functions. For $x=(\mathbf{D},W,t)\in\mathfrak{T}(\mathbf{d})$

$$\begin{aligned}
R_YE\mathbf{1}_A(x) &= \mathbf{1}_Y(x)(E\mathbf{1}_A)(x)\\
&= \mathbf{1}_Y(x)q^{-\dim(\pi_1^{-1}(x))}((\pi_1)_!\pi_2^*\mathbf{1}_A)(x)\\
&= q^{-\dim(\pi_1^{-1}(x))}\mathbf{1}_Y(x)((\pi_1)_!\mathbf{1}_{\pi_2^{-1}(A)})(x)\\
&= q^{-\dim(\pi_1^{-1}(x))}\mathbf{1}_Y(x)\chi_q(\pi_1^{-1}(x)\cap\pi_2^{-1}(A))\\
&= q^{-\dim(\pi_1^{-1}(x))}\chi_q(\pi_1^{-1}(x\cap Y)\cap\pi_2^{-1}(A))\\
&= q^{-\dim(\pi_1^{-1}(x))}\chi_q(\pi_1^{-1}(x)\cap\pi_1^{-1}(Y)\cap\pi_2^{-1}(A))\\
&= q^{-\dim(\pi_1^{-1}(x))}\chi_q(\pi_1^{-1}(x)\cap\pi_2^{-1}(Y)\cap\pi_2^{-1}(A))\\
&= q^{-\dim(\pi_1^{-1}(x))}\chi_q(\pi_1^{-1}(x)\cap\pi_2^{-1}(Y\cap A))
\end{aligned}$$

$$= q^{-\dim(\pi_1^{-1}(x))}(\pi_1)_! \mathbf{1}_{\pi_2^{-1}(Y\cap A)}(x)$$

$$= q^{-\dim(\pi_1^{-1}(x))}(\pi_1)_! \pi_2^* \mathbf{1}_{Y\cap A}(x)$$

$$= q^{-\dim(\pi_1^{-1}(x))}(\pi_1)_! \pi_2^* (\mathbf{1}_Y \mathbf{1}_A)(x)$$

$$= ER_Y \mathbf{1}_A(x),$$

where the fifth equality holds from consideration of the two cases $x \in Y$ and $x \notin Y$.

It remains to show that $p_! E = Ep_!$. For the purposes of this demonstration, we introduce the map

$$p' : \bigcup_w \mathfrak{T}(w, w+1; \mathbf{d}) \to \bigcup_w \mathfrak{M}(w, w+1, d)$$

which acts as $p'(\mathbf{D}, U, W, t) = (U, W, t)$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{T}(\mathbf{d}) & \xrightarrow{p} & \mathfrak{M}(d) \\ \uparrow \pi_2 & & \uparrow \pi_2 \\ \bigcup_w \mathfrak{T}(w, w+1; \mathbf{d}) & \xrightarrow{p'} & \bigcup_w \mathfrak{M}(w, w+1, d) \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ \mathfrak{T}(\mathbf{d}) & \xrightarrow{p} & \mathfrak{M}(d) \end{array}$$

As before we use the notation π_1 and π_2 to denote several different, but analogous maps.

Note that $p^{-1}\pi_2 = \pi_2(p')^{-1}$ (both are the map $(U, W, t) \mapsto \{(\mathbf{D}, W', t') \in \mathfrak{T}(\mathbf{d}) \mid W' = W, t' = t\}$). Using this fact we show that $(p')_! \pi_2^* = \pi_2^* p_!$. Let $x \in \mathfrak{T}(\mathbf{d})$. Then

$$(\pi_2^* p_! \mathbf{1}_A)(x) = (p_! \mathbf{1}_A)(\pi_2(x))$$

$$= \chi_q(p^{-1}(\pi_2(x)) \cap A)$$

$$= \chi_q(\pi_2((p')^{-1}(x)) \cap A)$$

$$= \chi_q(\pi_2((p')^{-1}(x) \cap \pi_2^{-1}(A)))$$

$$= \chi_q((p')^{-1}(x) \cap \pi_2^{-1}(A))$$

$$= (p')_! \mathbf{1}_{\pi_2^{-1}(A)}(x)$$

$$= (p')_! \pi_2^* \mathbf{1}_A(x),$$

where in the fourth equality we used the general fact that $\pi_2(B) \cap A = \pi_2(B \cap \pi_2^{-1}(A))$ and in the fifth equality we used the fact that if $x = (U', W', t')$ then

$$\begin{aligned}(p')^{-1}(x) \cap \pi_2^{-1}(A) &= \{(\mathbf{D}, U, W, t) \mid p'(\mathbf{D}, U, W, t) = (U', W', t'), \pi_2(\mathbf{D}, U, W, t) \in A\} \\ &= \{(\mathbf{D}, U, W, t) \mid U = U', W = W', t = t', (\mathbf{D}, W, t) \in A\} \\ &\cong \{(\mathbf{D}, W, t) \mid W = W', t = t', (\mathbf{D}, W, t) \in A\} \\ &= \pi_2((p')^{-1}(x) \cap \pi_2^{-1}(A)).\end{aligned}$$

We also have that $\pi_1 p' = p\pi_1$ (both are the map $(\mathbf{D}, U, W, t) \mapsto (U, t)$). Thus

$$\begin{aligned}Ep_! &= q^{-\dim(\pi_1^{-1}(\cdot))}(\pi_1)_! \pi_2^* p_! \\ &= q^{-\dim(\pi_1^{-1}(\cdot))}(\pi_1)_!(p')_! \pi_2^* \\ &= q^{-\dim(\pi_1^{-1}(\cdot))}(\pi_1 p')_! \pi_2^* \\ &= q^{-\dim(\pi_1^{-1}(\cdot))}(p\pi_1)_! \pi_2^* \\ &= q^{-\dim(\pi_1^{-1}(\cdot))} p_!(\pi_1)_! \pi_2^* \\ &= p_! q^{-\dim(\pi_1^{-1}(\cdot))}(\pi_1)_! \pi_2^* \\ &= p_! E,\end{aligned}$$

where we have used the fact that the map $f \mapsto f_!$ is functorial [5].

Thus, we have shown that $T_Y E = E T_Y$. The proof that $T_Y F = F T_Y$ is analogous. Also,

$$\begin{aligned}K^{\pm 1} T_Y f(\mathbf{D}, W, t) &= q^{\pm(d - 2\dim W)} T_Y f(\mathbf{D}, W, t) \\ &= T_Y q^{\pm(d - 2\dim W)} f(\mathbf{D}, W, t) \\ &= T_Y K^{\pm 1} f(\mathbf{D}, W, t). \qquad \square\end{aligned}$$

3.2. A basis $\mathscr{B}_I$ for the space of intertwiners

We see from Section 1.4 that a basis for the space of intertwiners $H^{\mu}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$ is in one-to-one correspondence with the set of crossingless matches $\mathrm{CM}^{\mu}_{\mathbf{d}_1,\ldots,\mathbf{d}_k}$. Note that crossingless matches of the form $\mathrm{CM}^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ (i.e. with only one box on the top vertical line) are in one to one correspondence with elements of $\mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$. For a given element S of $\mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ simply set μ equal to the number of unmatched vertices of S and join the unmatched vertices to the upper box. Recall that elements $\mathbf{a}\in(\mathbb{Z}_{\geqslant 0})^n$ such that $\mathbf{a}_i\leqslant\mathbf{d}_i$ are in one to one correspondence with the elements of $\mathrm{OLCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$. Given such an $\mathbf{a}$, consider its associated oriented lower crossingless match $M(\mathbf{d},\mathbf{a})$. Note that $|\mathbf{a}|$ is the number of vertices (both matched and unmatched) in $M(\mathbf{d},\mathbf{a})$ which are oriented down.

For any flag $\mathbf{D}$ and $t\in\operatorname{End} D$ let $\alpha(t,\mathbf{D})=\alpha(\ker t,\mathbf{D})$. Then let

$$\begin{aligned} Y_{\mathbf{a}} &= \{(\mathbf{D},W,t)\in\mathfrak{T}(\mathbf{d}) \mid \alpha(t,\mathbf{D})=\mathbf{n}^{M(\mathbf{d},\mathbf{a})}, \dim W=|\mathbf{a}|\} \\ &= \bigcup_{\mathbf{w}:|\mathbf{w}|=|\mathbf{a}|}\ \bigcup_{\mathbf{r}} A_{\mathbf{w},\mathbf{r},\mathbf{n}^{M(\mathbf{d},\mathbf{a})}}. \end{aligned} \tag{32}$$

Now, note that $\mathbf{n}^{M(\mathbf{d},\mathbf{a})}$ depends only on the lower curves of $\mathbf{a}$ and not on the orientation of the unmatched vertices. Thus, if $\bar{\mathbf{a}}$ denotes the (unoriented) lower crossingless match associated to $\mathbf{a}$, we can unambiguously define $\mathbf{n}^{\bar{\mathbf{a}}}=\mathbf{n}^{M(\mathbf{d},\mathbf{a})}$. Then if b is an unoriented crossingless match, we define

$$\begin{aligned} Y_b &= \{(\mathbf{D},W,t)\in\mathfrak{T}(\mathbf{d}) \mid \alpha(t,\mathbf{D})=\mathbf{n}^{b}\} \\ &= \bigcup_{\mathbf{a}:\bar{\mathbf{a}}=b} Y_{\mathbf{a}}. \end{aligned} \tag{33}$$

The last equality arises from the fact that if $(\mathbf{D},W,t)\in\mathfrak{T}(\mathbf{d})$ then $\operatorname{im} t\subset W\subset\ker t$, so $r\leqslant\dim W\leqslant d-r$ (where $r=\operatorname{rank} t$). Thus, since $(\mathbf{D},W,t)\in Y_b$ implies that $r=\operatorname{rank} t$ is the number of lower curves in b, the values $r,r+1,\dots,d-r$ are precisely the number of down arrows (that is, the $|\mathbf{a}|$) in the various $\mathbf{a}$ such that $\bar{\mathbf{a}}=b$. We also have the following:

Proposition 3.2.1. $\bigsqcup_b Y_b=\bigsqcup_{\mathbf{a}} Y_{\mathbf{a}}=\mathfrak{T}(\mathbf{d})$.

Proof. It is obvious that the $Y_{\mathbf{a}}$ are disjoint. Thus, from Eq. (33) we see that it suffices to prove that for every $(\mathbf{D},W,t)\in\mathfrak{T}(\mathbf{d}),\alpha(t,\mathbf{D})=\mathbf{n}^{M(\mathbf{d},\mathbf{a})}$ for some crossingless match $\mathbf{a}$. Fix an $(\mathbf{D},W,t)\in\mathfrak{T}(\mathbf{d})$ and let $\mathbf{a}=\alpha(t,\mathbf{D})$. Now, down arrows of $\mathbf{a}$ represent dimensions of the kernel of t while up arrows of $\mathbf{a}$ represent dimensions of $D/\ker t$. Let c denote the i^{th} up arrow from the left. Since $\operatorname{im} t\subset\ker t$ and $t(\mathbf{D}_j)\subset(\mathbf{D}_{j-1})$, there must be at least i down arrows to the left of c. Since this holds for all i, it follows that each up arrow of $M(\mathbf{d},\mathbf{a})$ is matched. Thus, since $\mathbf{n}^{M(\mathbf{d},\mathbf{a})}$ is obtained from $\mathbf{a}$ by forcing all unmatched vertices to be oriented down, we have that $\mathbf{n}^{M(\mathbf{d},\mathbf{a})}=\mathbf{a}=\alpha(t,\mathbf{D})$. $\square$

Define

$$\mathscr{B}_I = \left\{ T_{Y_b} \middle| b \in \bigcup_\mu CM^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k} \right\}. \tag{34}$$

Proposition 3.2.2. *Each element of* $\mathscr{B}_I$ *is an intertwiner and*

$$T_{Y_b}(\mathscr{T}(\mathbf{d})) \subset \mathscr{M}^r(d) \cong V_\mu$$

for $b \in \mathrm{CM}^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ *and* $r = (d-\mu)/2$.

Proof. According to Proposition 3.1.1, to show that T_{Y_b} is an intertwiner we need only check that $\pi_2\pi_1^{-1}(Y_b) \subset Y_b$ and $\pi_1\pi_2^{-1}(Y_b) \subset Y_b$ for all $b \in \mathrm{CM}^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$. If we denote by t^x and $\mathbf{D}^x$ the map t and flag $\mathbf{D}$ of the point $x \in \mathfrak{T}(\mathbf{d})$ (so $x = (\mathbf{D}^x, W, t^x)$ for some W), then $t^y = t^x$ and $\mathbf{D}^y = \mathbf{D}^x$ for all $y \in \pi_2\pi_1^{-1}(x)$. Thus $\alpha(t^x, \mathbf{D}^x) = \alpha(t^y, \mathbf{D}^y)$ for all $y \in \pi_2\pi_1^{-1}(x)$ which implies that $\pi_2\pi_1^{-1}(Y_b) \subset Y_b$ for all b. Similarly $\pi_1\pi_2^{-1}(Y_b) \subset Y_b$ for all b. Now, the image of T_{Y_b} consists of functions on $\mathfrak{M}^r(d)$ where r is the number of lower curves in b. In fact, it is easy to see that for $f \in \mathscr{T}(\mathbf{d})$, $T_{Y_b}(f)(W,t)$ depends only on the dimension of W and the rank of t. So the image of T_{Y_b} is contained in $\mathscr{M}^r(d)$. Recall from Section 1.3 that $\mathscr{M}^r(d) \cong V_{d-2r}$. Since r is equal to the number of lower curves in b, $d-2r$ is equal to the number of middle curves and hence $d - 2r = \mu$. So T_{Y_b} is an intertwiner into the representation V_μ as it should be. □

3.3. *The space* $\mathscr{T}_s(\mathbf{d})$ *and the basis* $\mathscr{B}_s$

For the purposes of this section we will identify the sets $\mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ and $\bigcup_\mu \mathrm{CM}^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ as in Section 3.2. Also, to simplify notation, we shall identify elements $\mathbf{a} \in (\mathbb{Z}_{\geqslant 0})^k$ such that $\mathbf{a}_i \leqslant \mathbf{d}_i$ with their associated oriented lower crossingless matches $M(\mathbf{d}, \mathbf{a})$.

Let $\mathscr{T}_s(\mathbf{d})$ be the space of all functions $f \in \mathscr{T}(\mathbf{d})$ such that

$$\dim W = \dim W',\ \alpha(t, \mathbf{D}) = \alpha(t', \mathbf{D}') \Rightarrow f(\mathbf{D}, W, t) = f(\mathbf{D}', W', t').$$

It is obvious that if we define

$$\mathscr{B}_s = \left\{ \mathbf{1}_{Y_\mathbf{a}} \middle| \mathbf{a} \in \bigcup_\mu \mathrm{OCM}^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k} \right\}, \tag{35}$$

then

$$\mathscr{T}_s(\mathbf{d}) = \mathrm{Span}\, \mathscr{B}_s.$$

Theorem 3.3.1. *$\mathcal{T}_s(\mathbf{d})$ is isomorphic as a $\mathbf{U}_q$-module to $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$ and $\mathcal{B}_s$ is a basis for $\mathcal{T}_s(\mathbf{d})$ adapted to its decomposition into a direct sum of irreducible representations. That is, for a given $b \in \mathrm{CM}^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, the space $\mathrm{Span}\{\mathbf{1}_{Y_\mathbf{a}} \mid \bar{\mathbf{a}} = b\}$ is isomorphic to the irreducible representation V_μ via the map*

$$\mathbf{1}_{Y_\mathbf{a}} \mapsto {}^{\mu}v_{\mu - 2(\#\ \text{of unmatched down arrows in}\ \mathbf{a})}$$

(*extended by linearity*).

Proof. For $\mathbf{a} \in \mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ such that $\mathbf{a}$ has at least one unmatched up arrow, let $\mathbf{a}^+$ be the element of $\mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ obtained from $\mathbf{a}$ by switching the orientation of the rightmost unmatched up arrow. Thus $\overline{\mathbf{a}^+} = \bar{\mathbf{a}}$ and $\mathbf{a}^+$ has one more unmatched down arrow than $\mathbf{a}$. Similarly, if $\mathbf{a} \in \mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ has at least one unmatched down arrow, let $\mathbf{a}^-$ be the element of $\mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ obtained from $\mathbf{a}$ by switching the orientation of the leftmost unmatched down arrow. Recall from the proofs of Propositions 3.1.1 and 3.2.2 that $\pi_1^{-1}(Y_b) = \pi_2^{-1}(Y_b)$. It follows from this and the fact that $Y_b = \bigcup_{\mathbf{a}:\bar{\mathbf{a}}=b} Y_\mathbf{a}$ that $\pi_2\pi_1^{-1}(Y_\mathbf{a}) = Y_{\mathbf{a}^+}$ if $\mathbf{a}$ has at least one unmatched up arrow and $\pi_2\pi_1^{-1}(Y_\mathbf{a}) = \emptyset$ otherwise. Similarly, $\pi_1\pi_2^{-1}(Y_\mathbf{a}) = Y_{\mathbf{a}^-}$ if $\mathbf{a}$ has at least one unmatched down arrow and $\pi_1\pi_2^{-1}(Y_\mathbf{a}) = \emptyset$ otherwise.

Now, for $x \in \mathfrak{T}(\mathbf{d})$,

$$\begin{aligned} F\mathbf{1}_{Y_\mathbf{a}}(x) &= q^{-\dim(\pi_2^{-1}(x))}(\pi_2)_! \pi_1^* \mathbf{1}_{Y_\mathbf{a}}(x) \\ &= q^{-\dim(\pi_2^{-1}(x))}(\pi_2)_! \mathbf{1}_{\pi_1^{-1}(Y_\mathbf{a})}(x) \\ &= q^{-\dim(\pi_2^{-1}(x))}\chi_q(\pi_2^{-1}(x) \cap \pi_1^{-1}(Y_\mathbf{a})). \end{aligned}$$

Now, we already know from the above discussion that $\pi_2^{-1}(x) \cap \pi_1^{-1}(Y_\mathbf{a}) = \emptyset$ if $x \notin Y_{\mathbf{a}^+}$. So assuming $x = (\mathbf{D}, W, t) \in Y_{\mathbf{a}^+}$, let $r = \operatorname{rank} t$. Then

$$\begin{aligned} F\mathbf{1}_{Y_\mathbf{a}}(\mathbf{D}, W, t) &= q^{-\dim(\pi_2^{-1}(\mathbf{D}, W, t))}\chi_q(\pi_2^{-1}(\mathbf{D}, W, t) \cap \pi_1^{-1}(Y_\mathbf{a})) \\ &= q^{-\dim \mathbb{P}^{|\mathbf{a}^+|-r-1}}\chi_q(\mathbb{P}^{|\mathbf{a}^+|-r-1}) \\ &= q^{-(|\mathbf{a}^+|-r-1)} \sum_{i=0}^{|\mathbf{a}^+|-r-1} q^{2i} \\ &= [|\mathbf{a}^+| - r] \\ &= [(\#\ \text{down arrrows in}\ \mathbf{a}^+) - (\#\ \text{lower curves in}\ \mathbf{a}^+)] \\ &= [\#\ \text{unmatched down arrows in}\ \mathbf{a}^+]. \end{aligned}$$

Thus,

$$F\mathbf{1}_{Y_\mathbf{a}} = [\#\text{ unmatched down arrows in }\mathbf{a}^+]\mathbf{1}_{Y_{\mathbf{a}^+}}. \tag{36}$$

Now,

$$\begin{aligned} E\mathbf{1}_{Y_\mathbf{a}}(x) &= q^{-\dim(\pi_1^{-1}(x))}(\pi_1)_!\pi_2^*\mathbf{1}_{Y_\mathbf{a}}(x) \\ &= q^{-\dim(\pi_1^{-1}(x))}(\pi_1)_!\mathbf{1}_{\pi_2^{-1}(Y_\mathbf{a})}(x) \\ &= q^{-\dim(\pi_1^{-1}(x))}\chi_q(\pi_1^{-1}(x)\cap\pi_2^{-1}(Y_\mathbf{a})). \end{aligned}$$

We know that $\pi_1^{-1}(x)\cap\pi_2^{-1}(Y_\mathbf{a}) = \emptyset$ if $x\notin Y_{\mathbf{a}^-}$. So assuming $x = (\mathbf{D}, W, t)\in Y_{\mathbf{a}^-}$, let $r = \operatorname{rank} t$. Then

$$\begin{aligned} E\mathbf{1}_{Y_\mathbf{a}}(\mathbf{D}, W, t) &= q^{-\dim(\pi_1^{-1}(\mathbf{D},W,t))}\chi_q(\pi_1^{-1}(\mathbf{D}, W, t)\cap\pi_2^{-1}(Y_\mathbf{a})) \\ &= q^{-\dim\mathbb{P}^{d-r-|\mathbf{a}^-|-1}}\chi_q(\mathbb{P}^{d-r-|\mathbf{a}^-|-1}) \\ &= [d-r-|\mathbf{a}^-|] \\ &= [d-(\#\text{ lower curves in }\mathbf{a}^-)-(\#\text{ down arrows in }\mathbf{a}^-)] \\ &= [(\#\text{ up arrows in }\mathbf{a}^-)-(\#\text{ lower curves in }\mathbf{a}^-)] \\ &= [\#\text{ unmatched up arrows in }\mathbf{a}^-]. \end{aligned}$$

Thus,

$$E\mathbf{1}_{Y_\mathbf{a}} = [\#\text{ unmatched up arrows in }\mathbf{a}^-]\mathbf{1}_{Y_{\mathbf{a}^-}}. \tag{37}$$

Finally, it is easy to see that

$$\begin{aligned} K\mathbf{1}_{Y_\mathbf{a}} &= q^{\pm(d-2|\mathbf{a}|)}\mathbf{1}_{Y_\mathbf{a}} \\ &= q^{\pm(\mu-2(\#\text{ unmatched down arrows in }\mathbf{a}))}\mathbf{1}_{Y_\mathbf{a}}, \end{aligned} \tag{38}$$

where μ is the total number of unmatched arrows in $\mathbf{a}$. Using the fact that μ is the total number of middle curves of b (and hence the total number of unmatched vertices in any $\mathbf{a}$ such that $\bar{\mathbf{a}} = b$), the second statement of the theorem now follows easily from a comparison with (1).

Since we know from Section 1.4 that the set $\mathrm{CM}^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ is in one to one correspondence with the set of intertwiners $H^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, we have that

$$\mathcal{T}_s(\mathbf{d})\cong\bigoplus_\mu H^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}\otimes V_\mu\cong V_{\mathbf{d}_1}\otimes\cdots\otimes V_{\mathbf{d}_k}$$

which proves the first statement of the theorem. □

Now, like the canonical basis, the basis $\mathscr{B}_s$ we have constructed here is closely related to the irreducible components of $\mathfrak{T}(\mathbf{d})$. To see this, we first need a proposition. Consider the varieties $Y_\mathbf{a}$ and Y_b defined over $\bar{\mathbb{F}}_{q^2}$. To avoid confusion, denote these by $Y'_\mathbf{a}$ and Y'_b. Then

Proposition 3.3.1. $\overline{Y'_\mathbf{a}} = \overline{Z'_\mathbf{a}}$.

Proof. Since the $Y'_\mathbf{a}$ are smooth and connected, they are irreducible. Also, from an argument analogous to the one given in the proof of Proposition 3.2.1, we know that $\sqcup_\mathbf{a} Y'_\mathbf{a} = \mathfrak{T}(\mathbf{d})'$. Thus, since the cardinality of the sets $\{\overline{Y'_\mathbf{a}}\}$ and $\{\overline{Z'_\mathbf{a}}\}$ are the same, $\{\overline{Y'_\mathbf{a}}\}$ must be the set of irreducible components of $\mathfrak{T}(\mathbf{d})'$. Now, $Y'_\mathbf{a} \cap Z'_\mathbf{a} = \bigcup_\mathbf{r} A'_{\mathbf{a},\mathbf{r},\mathbf{n}^{M(\mathbf{d},\mathbf{a})}}$. But, by Proposition 2.5.1, $\overline{A'_{\mathbf{a},\mathbf{r}^{M(\mathbf{d},\mathbf{a})},\mathbf{n}^{M(\mathbf{d},\mathbf{a})}}} = \overline{Z'_\mathbf{a}}$. Therefore we must have $\overline{Y'_\mathbf{a}} = \overline{Z'_\mathbf{a}}$. $\square$

Since $Y_\mathbf{a}$ is precisely the set of $\mathbb{F}_{q^2}$ points of $Y'_\mathbf{a}$, we have the following characterization of the basis $\mathscr{B}_s$.

Theorem 3.3.2. *The elements $\mathbf{1}_{Y_\mathbf{a}}$ of the basis $\mathscr{B}_s$ are the unique elements of $\mathscr{T}_s(\mathbf{d})$ equal to one on the dense points of the irreducible component $\overline{Z_\mathbf{a}}$ of $\mathfrak{T}(\mathbf{d})$ with support contained in this irreducible component.*

So, like the elements of $\mathscr{B}_c$, the elements of $\mathscr{B}_s$ are equal to a non-zero constant on the set of dense points of an irreducible component of $\mathfrak{T}(\mathbf{d})$ with supports contained in distinct irreducible components. However, unlike $\mathscr{B}_c$, the elements of $\mathscr{B}_s$ have disjoint supports.

3.4. *The multiplicity variety $\mathfrak{S}(\mathbf{d})$*

We briefly describe here the relation between $\mathscr{B}_I$ and $\mathscr{B}_s$ and the multiplicity variety [6]. Let $\mathbf{d} \in (\mathbb{Z}_{\geqslant 0})^k$ and let D be a $|\mathbf{d}|$-dimensional $\bar{\mathbb{F}}_{q^2}$ vector space. The *multiplicity variety* is the variety (defined over $\bar{\mathbb{F}}_{q^2}$)

$$\mathfrak{S}(\mathbf{d})' = \{(\mathbf{D}, t) \mid (\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})' \text{ for some } W \subset D\}.$$

Define the projection $\pi : \mathfrak{T}(\mathbf{d})' \to \mathfrak{S}(\mathbf{d})'$ by $\pi(\mathbf{D}, W, t) = (\mathbf{D}, t)$. It follows easily from the above results that the irreducible components of $\mathfrak{S}(\mathbf{d})'$ are given by the closures of the sets

$$\mathscr{Y}'_b = \{(\mathbf{D}, t) \mid \alpha(t, \mathbf{D}) = \mathbf{n}^b\}, \quad b \in \mathrm{LCM}_{\mathbf{d}_1,\dots,\mathbf{d}_k},$$

and that these irreducible components are in one to one correspondence with the irreducible modules in the direct sum decomposition of $V_{\mathbf{d}_1} \otimes \cdots \otimes V_{\mathbf{d}_k}$. Then $Y'_b =$

$\pi_{-1}(\mathscr{Y}'_b)$ and $\{Y_{\mathbf{a}} \mid \bar{\mathbf{a}} = b\}$ yields a decomposition of the $\mathbb{F}_{q^2}$ points of the fiber of $\pi|_{Y'_b}$ isomorphic to the decomposition of $\mathfrak{M}^r(d)$ into the subsets $\mathfrak{M}^r(w,d)$ where r is the number of lower curves in b. Thus the bases $\mathscr{B}_I$ and $\mathscr{B}_s$ have natural geometric interpretations in terms of the multiplicity variety and the projection π.

3.5. The action of the intertwiners on $\mathscr{T}_s(\mathbf{d})$

We will now determine how our intertwiners act on the space $\mathscr{T}_s(\mathbf{d})$. For $\mathbf{a} \in (Z_{\geqslant 0})^k$, let $\mathbf{a}^j = \mathbf{a}^{(1,j)}$. We will need the following two technical lemmas.

Lemma 3.5.1. *If $\mathbf{D} = (0 = \mathbf{D}_0 \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset \cdots \subset \mathbf{D}_k = D)$ is a flag with $\mathbf{d} = \alpha(D, \mathbf{D})$ and $\mathbf{a} \in (\mathbb{Z}_{\geqslant 0})^k$ with $\mathbf{a}_i \leqslant \mathbf{d}_i$, then*

$$\chi_q(\{W \mid W \subset D, \alpha(W, \mathbf{D}) = \mathbf{a}\}) = c_{\mathbf{d},\mathbf{a}} \stackrel{\text{def}}{=} \sum_{\mathbf{b} \in C_{\mathbf{a}}} q^{2\sum_{1 \leqslant j < i \leqslant d} \mathbf{b}_i(1-\mathbf{b}_j)},$$

where

$$C_{\mathbf{a}} = \{\mathbf{b} \in (\mathbb{Z}_{\geqslant 0})^d | \mathbf{b}_i \in \{0,1\} \forall i, \mathbf{b}^{(\mathbf{d}_{j-1}+1, \mathbf{d}_j)} = \mathbf{a}_j\}$$

and we set $\mathbf{d}_0 = 0$.

Proof. Complete $\mathbf{D}$ to a flag $\mathbf{F} = (0 \subset \mathbf{F}_1 \subset \mathbf{F}_2 \subset \cdots \subset \mathbf{F}_d = D)$ such that $\dim \mathbf{F}_i = i$ and $\mathbf{F}_{\mathbf{d}^i} = \mathbf{D}_i$ where $d = |\mathbf{d}|$. This gives a decomposition of $\mathrm{Gr}^d_{|\mathbf{a}|}$ into cells, each isomorphic to $(\mathbb{F}_{q^2})^j$ for some j. The cells are given by $\{W \mid W \subset D, \alpha(W, \mathbf{F}) = \mathbf{b}\}$ for a fixed $\mathbf{b}$. The number of points in such a cell is equal to

$$q^{2\sum_{1 \leqslant j < i \leqslant d} \mathbf{b}_i(1-\mathbf{b}_j)}.$$

Our variety is the union of those cells such that $\mathbf{b}^{(\mathbf{d}_{j-1}+1, \mathbf{d}_j)} = \mathbf{a}_j$. The result follows. $\square$

Specializing to $q = 1$ yields

Lemma 3.5.2.

$$c_{\mathbf{d},\mathbf{a}}|_{q=1} = \prod_{i=1}^{k} \binom{\mathbf{d}_i}{\mathbf{a}_i}.$$

Proof. This follows immediately from Lemma 3.5.1 since $\mathbf{b}_i \in \{0,1\}$ for each cell. $\square$

Theorem 3.5.1. *The set $\mathscr{B}_I$ acting on $\mathscr{T}_s(\mathbf{d})$ spans the space of intertwiners $\bigoplus_\mu H^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$. In particular, for $b \in CM^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, T_{Y_b} acts on the basis $\mathscr{B}_s$ of $\mathscr{T}_s(\mathbf{d})$ as*

$$T_{Y_b}\mathbf{1}_{Y_\mathbf{a}} = \begin{cases} c_b \mathbf{1}_{\mathfrak{M}^r(|\mathbf{a}|,d)} \in \mathscr{M}^r(d) \cong V_{d-2r} = V_\mu & \text{if } \bar{\mathbf{a}} = b, \\ 0 & \text{if } \bar{\mathbf{a}} \neq b, \end{cases}$$

where r is the number of lower curves in b, $d = |\mathbf{d}|$ and c_b is non-zero constant.

Proof. Recall that $T_Y = p_! R_Y$. It is obvious from the fact that $Y_b = \bigcup_{\mathbf{a}:\bar{\mathbf{a}}=b} Y_\mathbf{a}$ that

$$R_{Y_b}\mathbf{1}_{Y_\mathbf{a}} = \mathbf{1}_{Y_b}\mathbf{1}_{Y_\mathbf{a}} = \begin{cases} \mathbf{1}_{Y_\mathbf{a}} & \text{if } \bar{\mathbf{a}} = b, \\ 0 & \text{if } \bar{\mathbf{a}} \neq b. \end{cases}$$

So we need only determine $p_!\mathbf{1}_{Y_\mathbf{a}}$ for $\bar{\mathbf{a}} = b$. Now, for $x = (W^x, t^x) \in \mathfrak{M}(d)$,

$$p_!\mathbf{1}_{Y_\mathbf{a}}(x) = \chi_q(p^{-1}(x) \cap Y_\mathbf{a}).$$

Recall that p is the map $(\mathbf{D}, W, t) \mapsto (W, t)$ and

$$Y_\mathbf{a} = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}) | \alpha(t, \mathbf{D}) = \mathbf{n}^b, \dim W = |\mathbf{a}|\}.$$

Thus,

$$p^{-1}(x) \cap Y_\mathbf{a} \cong \{\mathbf{D} \mid \dim(\mathbf{D}_i/\mathbf{D}_{i-1}) = \mathbf{d}_i, t^x(\mathbf{D}_i) \subset \mathbf{D}_{i-1}, \alpha(t^x, \mathbf{D}) = \mathbf{n}^b\} \tag{39}$$

if $\dim W^x = |\mathbf{a}|$ and $p^{-1}(x) \cap Y_\mathbf{a} = \emptyset$ otherwise. Note that this variety depends only on the dimension of the kernel of t^x (or equivalently, the rank of t^x) and the dimension of W^x. The variety is empty unless $r = \operatorname{rank} t^x$ is equal to the number of lower curves in $\mathbf{a}$. Thus, $T_{Y_b}\mathbf{1}_{Y_\mathbf{a}}$ is a constant function on $\mathfrak{M}^r(|\mathbf{a}|, d)$. Moreover, this constant c_b, equal to the number of points in the variety in (39), depends only on $\bar{\mathbf{a}} = b$ and not on the orientation of $\mathbf{a}$. As long as c_b is non-zero, we know that T_{Y_b} is a non-zero intertwiner. Moreover, it is obvious that if all the Y_b are non-zero then the intertwiners T_{Y_b} are linearly independent.

To show that $c_b \neq 0$ it suffices to show that its evaluation at $q = 1$ is non-zero. The variety (39) consists of all t_x-stable flags $\mathbf{D} = (0 \subset \mathbf{D}_1 \subset \cdots \subset \mathbf{D}_k = D)$ such that $\dim \mathbf{D}_i = \mathbf{d}^i$ and the intersection of $\mathbf{D}_i$ with $\ker t^x$ is a space of dimension $(\mathbf{n}^b)^j = \sum_{i=1}^j \mathbf{n}_i^b$. There is only one choice for $\mathbf{D}_k$, namely D. Assume we have picked $\mathbf{D}_{j+1}$. $\mathbf{D}_j$ can be any subspace of dimension $\mathbf{d}^i$ such that

$$t^x(\mathbf{D}_{j+1}) \subset \mathbf{D}_j \subset \mathbf{D}_{j+1}$$

and

$$\dim(\mathbf{D}_j \cap \ker t^x) = (\mathbf{n}^b)^j.$$

Note that since $\dim(\mathbf{D}_{j+1} \cap \ker t^x) = (\mathbf{n}^b)^{j+1}$ and $\dim \mathbf{D}_{j+1} = \mathbf{d}^{j+1}$, we have that $\dim t^x(\mathbf{D}_{j+1}) = \mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1}$. Also, since $(t^x)^2 = 0$, $t^x(\mathbf{D}_{j+1}) \subset \ker t^x$. Passing to the quotient by $t^x(\mathbf{D}_{j+1})$ and denoting this by a bar, we see that picking a subspace $\mathbf{D}_j$ subject to the above conditions is equivalent to picking a subspace $\overline{\mathbf{D}_j}$ of $\overline{\mathbf{D}_{j+1}}$ of dimension $\mathbf{d}^j - (\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1})$ such that

$$\dim(\overline{\mathbf{D}_j} \cap \overline{\ker t^x}) = (\mathbf{n}^b)^j - (\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1}).$$

Since $\dim \overline{\mathbf{D}_{j+1}} = \mathbf{d}^{j+1} - (\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1}) = (\mathbf{n}^b)^{j+1}$ and $\dim \overline{\mathbf{D}_{j+1}} \cap \overline{\ker t^x} = \dim \overline{\mathbf{D}_{j+1} \cap \ker t^x} = (\mathbf{n}^b)^{j+1} - (\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1}) = 2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1}$ we see by Lemma 3.5.2 that the value of χ_q of the variety of such spaces evaluated at $q = 1$ is

$$\begin{pmatrix} 2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1} \\ (\mathbf{n}^b)^{j+1} + (\mathbf{n}^b)^j - \mathbf{d}^{j+1} \end{pmatrix} \begin{pmatrix} (\mathbf{n}^b)^{j+1} - (2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1}) \\ \mathbf{d}^j - \mathbf{d}^{j+1} + (\mathbf{n}^b)^{j+1} - ((\mathbf{n}^b)^{j+1} + (\mathbf{n}^b)^j - \mathbf{d}^{j+1}) \end{pmatrix}$$

$$= \begin{pmatrix} 2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1} \\ (\mathbf{n}^b)^{j+1} + (\mathbf{n}^b)^j - \mathbf{d}^{j+1} \end{pmatrix} \begin{pmatrix} \mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1} \\ \mathbf{d}^j - (\mathbf{n}^b)^j \end{pmatrix}.$$

This is thus strictly positive provided that

$$2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1} \geqslant 0, \tag{40}$$

$$(\mathbf{n}^b)^{j+1} + (\mathbf{n}^b)^j - \mathbf{d}^{j+1} \geqslant 0, \tag{41}$$

$$\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1} \geqslant 0, \tag{42}$$

$$\mathbf{d}^j - (\mathbf{n}^b)^j \geqslant 0, \tag{43}$$

$$2(\mathbf{n}^b)^{j+1} - \mathbf{d}^{j+1} \geqslant (\mathbf{n}^b)^{j+1} + (\mathbf{n}^b)^j - \mathbf{d}^{j+1}, \tag{44}$$

$$\mathbf{d}^{j+1} - (\mathbf{n}^b)^{j+1} \geqslant \mathbf{d}^j - (\mathbf{n}^b)^j. \tag{45}$$

Now, recall that $\mathbf{n}^b$ is obtained from $\mathbf{a}$ by forcing all unmatched arrows to be oriented down. Also, $\mathbf{d}^j$ is the number of vertices associated to $V_{\mathbf{d}_1}$ through $V_{\mathbf{d}_j}$ while $(\mathbf{n}^b)^j$ is number of these vertices with down arrows. Thus $\mathbf{d}^j - (\mathbf{n}^b)^j$ is the number of these vertices with up arrows. So (42), (43) and (45) are obvious. Eq. (44) follows from the simple fact that $(\mathbf{n}^b)^{j+1} \geqslant (\mathbf{n}^b)^j$. Eqs. (40) and (41) follow from the fact that each up arrow is matched to a down arrow to its left since all unmatched arrows point down and matchings are oriented to the left.

Thus, χ_q of the variety of choices of $\mathbf{D}_j$ given $\mathbf{D}_{j+1}$ is independent of $\mathbf{D}_{j+1}$ (up to isomorphism) and is non-zero. Using the fact that the Euler characteristic of a locally trivial fibered space is equal to the product of the Euler characteristics of the base and the fiber, we see that the evaluation of c_b at 1 is a product of positive numbers and is thus positive. So $c_b \neq 0$. $\square$

3.6. The action of the intertwiners on $\mathscr{T}_c(\mathbf{d})$

We now compute the action of our intertwiners on the space $\mathscr{T}_c(\mathbf{d})$.

Define the coefficients $\kappa_{\mathbf{a}}^{\mathbf{d},\mathbf{w}}$ by

$$\diamondsuit^{\mathbf{d}} v_{\mathbf{w}} = \sum_{\mathbf{a}} \kappa_{\mathbf{a}}^{\mathbf{d},\mathbf{w}} (\otimes^{\mathbf{d}} v_{\mathbf{a}}).$$

For $b \in \mathrm{CM}^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, define $\mathbf{l}^b, \mathbf{m}^b \in (\mathbb{Z}_{\geqslant 0})^k$ such that $\mathbf{l}^b_i$ is equal to the number of left endpoints of lower curves of b in the box corresponding to $V_{\mathbf{d}_i}$ and $\mathbf{m}^b_i$ is equal to the number of endpoints of middle curves of b in the box corresponding to $V_{\mathbf{d}_i}$.

Theorem 3.6.1. *The set $\mathscr{B}_I$ acting on $\mathscr{T}_c(\mathbf{d})$ spans the space of intertwiners $\oplus_{\mu} H^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$. In particular, if $b \in \mathrm{CM}^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$ is such that $b \leqslant M(\mathbf{d},\mathbf{w})$, then*

$$T_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) = \omega \mathbf{1}_{\mathfrak{M}^{|\mathbf{l}^b|}(|\mathbf{w}|,d)},$$

where

$$\omega = \sum_{\mathbf{a}} \left(\kappa_{\mathbf{a}}^{\mathbf{m}^b,\mathbf{w}-\mathbf{l}^b} k_{\mathbf{a}+\mathbf{l}^b,\mathbf{l}^b,\mathbf{l}^b+\mathbf{m}^b} \prod_{i=1}^{k-1} c_{\mathbf{a}_1^i,\mathbf{a}_2^i} \right),$$

$$\mathbf{a}_1^i = ((\mathbf{l}^b)^{(i,k)}, \mathbf{a}^{(i,k)}, (\mathbf{m}^b - \mathbf{a})^{(i,k)}, (\mathbf{d} - \mathbf{m}^b - 2\mathbf{l}^b)^{(i,k)}),$$

$$\mathbf{a}_2^i = (\mathbf{l}^b_i, \mathbf{a}_i, \mathbf{m}^b_i - \mathbf{a}_i,\ \mathbf{d}_i - \mathbf{m}^b_i - \mathbf{l}^b_i)$$

Otherwise, $T_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) = 0$.

Proof. For a crossingless match $b \in \mathrm{CM}^{\mu}_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, $T_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) = p_! R_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}})$ and

$$R_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) = \sum_{S \leqslant M(\mathbf{d},\mathbf{w})} R_{Y_b}(\eta_{\mathbf{r}^S,\mathbf{n}^S})^{-1} (\diamondsuit^{\mathbf{n}^S - \mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S}).$$

This is equal to zero unless $\bar{S} = b$ for some $S \leqslant M(\mathbf{d}, \mathbf{w})$ (that is, the set of lower curves of b is a subset of the set of lower curves of $M(\mathbf{d}, \mathbf{w})$). If this is the case, then

$$R_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) = (\eta_{\mathbf{r}^S, \mathbf{n}^S})^{-1}(\diamond^{\mathbf{n}^S - \mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S})$$

for the particular $S \leqslant M(\mathbf{d}, \mathbf{w})$ such that $\bar{S} = b$. Then $\mathbf{n}^S = \mathbf{l}^b + \mathbf{m}^b$ and $\mathbf{r}^S = \mathbf{l}^b$. So

$$\begin{aligned} R_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}}) &= (\eta_{\mathbf{l}^b, \mathbf{l}^b + \mathbf{m}^b})^{-1}(\diamond^{\mathbf{m}^b} v_{\mathbf{w}-\mathbf{l}^b}) \\ &= (\eta_{\mathbf{l}^b, \mathbf{l}^b + \mathbf{m}^b})^{-1}\left(\sum_{\mathbf{a}} \kappa^{\mathbf{m}^b, \mathbf{w}-\mathbf{l}^b}_{\mathbf{a}} (\otimes^{\mathbf{m}^b} v_{\mathbf{a}})\right) \\ &= \sum_{\mathbf{a}} \kappa^{\mathbf{m}^b, \mathbf{w}-\mathbf{l}^b}_{\mathbf{a}} f_{\mathbf{a}+\mathbf{l}^b, \mathbf{l}^b, \mathbf{l}^b+\mathbf{m}^b} \\ &= \sum_{\mathbf{a}} \kappa^{\mathbf{m}^b, \mathbf{w}-\mathbf{l}^b}_{\mathbf{a}} k_{\mathbf{a}+\mathbf{l}^b, \mathbf{l}^b, \mathbf{l}^b+\mathbf{m}^b} \mathbf{1}_{A_{\mathbf{a}+\mathbf{l}^b, \mathbf{l}^b, \mathbf{l}^b+\mathbf{m}^b}}. \end{aligned}$$

Let $(W, t) \in \mathfrak{M}(d)$. Then if the set of lower curves of b is a subset of the set of lower curves of $M(\mathbf{d}, \mathbf{w})$,

$$\begin{aligned} T_{Y_b}(g^{\mathbf{d}}_{\mathbf{w}})(W, t) &= \sum_{\mathbf{a}} \kappa^{\mathbf{m}^b, \mathbf{w}-\mathbf{l}^b}_{\mathbf{a}} k_{\mathbf{a}+\mathbf{l}^b, \mathbf{l}^b, \mathbf{l}^b+\mathbf{m}^b} p_! \mathbf{1}_{A_{\mathbf{a}+\mathbf{l}^b, \mathbf{l}^b, \mathbf{l}^b+\mathbf{m}^b}}(W, t) \\ &= \sum_{\mathbf{a}} \kappa^{\mathbf{m}^b, \mathbf{w}-\mathbf{l}^b}_{\mathbf{a}} k_{\mathbf{a}+\mathbf{l}^b, \mathbf{l}^b, \mathbf{l}^b+\mathbf{m}^b} \chi_q(\{\mathbf{D} \mid \alpha(D, \mathbf{D}) = \mathbf{d}, t(\mathbf{D}_i) \subset \mathbf{D}_{i-1}, \\ &\qquad \alpha(\operatorname{im} t, \mathbf{D}) = \mathbf{l}^b, \alpha(W, \mathbf{D}) = \mathbf{l}^b + \mathbf{a}, \alpha(\ker t, \mathbf{D}) = \mathbf{l}^b + \mathbf{m}^b\}). \qquad (46) \end{aligned}$$

We see from Proposition 2.6.2 that $\kappa^{\mathbf{m}^b, \mathbf{w}-\mathbf{l}^b}_{\mathbf{a}} = 0$ unless $|\mathbf{a}| = |\mathbf{w} - \mathbf{l}^b| = |\mathbf{w}| - |\mathbf{l}^b|$. Therefore, since $|\alpha(W, \mathbf{D})| = \dim W$, (46) is zero unless $\dim W = |\mathbf{w}|$. Similarly, it is zero unless $\operatorname{rank} t = \dim(\operatorname{im} t) = |\mathbf{l}^b|$. If these conditions are satisfied, (46) is independent of W and t. We can then evaluate ω, the value of the expression in (46), using Lemma 3.5.1 and the fact that the Euler characteristic of a locally trivial fibered space is the product of the Euler characteristics of the base and the fiber. There is only one possible choice for $\mathbf{D}_0$, namely 0. Assume we have picked $\mathbf{D}_{i-1}$. Then $\mathbf{D}_i$ must satisfy the following conditions:

1. $t(\mathbf{D}_i) \subset \mathbf{D}_{i-1}$ or, equivalently, $\mathbf{D}_i \subset t^{-1}(\mathbf{D}_{i-1})$,
2. $\mathbf{D}_i \supset \mathbf{D}_{i-1}$, $\dim \mathbf{D}_i = \mathbf{d}^{(1,i)}$,
3. $\dim(\mathbf{D}_i \cap \operatorname{im} t) = (\mathbf{l}^b)^{(1,i)}$,
4. $\dim(\mathbf{D}_i \cap W) = (\mathbf{l}^b + \mathbf{a})^{(1,i)}$,
5. $\dim(\mathbf{D}_i \cap \ker t) = (\mathbf{l}^b + \mathbf{m}^b)^{(1,i)}$.

Pass to the quotient by $\mathbf{D}_{i-1}$ and denote this by a bar. Let $\mathbf{F}$ be the flag

$$\mathbf{F} = (\mathbf{F}_0 = 0 \subset \mathbf{F}_1 = \overline{\operatorname{im} t} \subset \mathbf{F}_2 = \overline{W} \subset \mathbf{F}_3 = \overline{\ker t}, \mathbf{F}_4 = \overline{t^{-1}(\mathbf{D}_{i-1})}).$$

Then the above conditions are equivalent to picking $\overline{\mathbf{D}_i} \subset \overline{t^{-1}(\mathbf{D}_{i-1})}$ such that

$$\alpha(\mathbf{D}_i, \mathbf{F}) = (\mathbf{l}_i^b, \mathbf{a}_i, \mathbf{m}_i^b - \mathbf{a}_i, \mathbf{d}_i - \mathbf{m}_i^b - \mathbf{l}_i^b).$$

Since

$$\dim \overline{\operatorname{im} t} = (\mathbf{l}^b)^{(i,k)},$$

$$\dim \overline{W} = (\mathbf{l}^b + \mathbf{a})^{(i,k)},$$

$$\dim \overline{\ker t} = (\mathbf{l}^b + \mathbf{m}^b)^{(i,k)}$$

and

$$\begin{aligned}\dim \overline{t^{-1}(\mathbf{D}_{i-1})} &= \dim t^{-1}(\mathbf{D}_{i-1}) - \dim \mathbf{D}_{i-1}\\ &= \dim(\operatorname{im} t \cap D_{i-1}) + \dim(\ker t) - \dim \mathbf{D}_{i-1}\\ &= (\mathbf{l}^b)^{(1,i-1)} + (d - |\mathbf{l}^b|) - \mathbf{d}^{(1,i-1)}\\ &= (\mathbf{d} - \mathbf{l}^b)^{(i,k)},\end{aligned}$$

the form of the action of the elements of $\mathscr{B}_I$ follows.

It remains to show that the set $\mathscr{B}_I$ spans the space of intertwiners $\oplus_\mu H^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$. Since it follows from Theorem 3.5.1 that the cardinality of $\mathscr{B}_I$ is equal to the dimension of $\oplus_\mu H^\mu_{\mathbf{d}_1,\dots,\mathbf{d}_k}$, it suffices to show the linear independence of the set $\mathscr{B}_I$. Assume that, acting on the space $\mathscr{T}_s(\mathbf{d})$,

$$\sum_i a_i T_{Y_{b_i}} = 0, \quad a_i \neq 0 \ \forall\ i. \tag{47}$$

Since the image of $T_{Y_{b_i}}$ is contained in $\mathscr{M}^{|l_{b_i}|}(d)$ by the above results, we may assume that $|l_{b_i}| = |l_{b_j}|$ for all i and j. Fix an i and consider a $\mathbf{w}$ such that $M(\mathbf{d}, \mathbf{w}) = b_i$. All $T_{Y_{b_j}},\ j \neq i$, act by zero on $g^{\mathbf{d}}_{\mathbf{w}}$ by the above (since $|l_{b_i}| = |l_{b_j}|$, we cannot have $b_j \leqslant b_i = M(\mathbf{d}, \mathbf{w})$). Also, $T_{Y_{b_i}} \neq 0$ by the above. Thus $a_i = 0$ which is a contradiction. Thus the theorem is proved. $\square$

3.7. An isomorphism of $\mathcal{T}_c(\mathbf{d})$ with $\mathcal{T}_s(\mathbf{d})$

For $(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})$, let

$$B_{\mathbf{D},W,t} = \{(\mathbf{D}', W', t') \mid W' = W,\ t' = t, \alpha(t, \mathbf{D}') = \alpha(t, \mathbf{D})\}.$$

For $f \in \mathcal{T}(\mathbf{d})$ let

$$\chi_q(f) = \sum_{x \in \mathfrak{T}(\mathbf{d})} f(x).$$

Let $\xi : \mathcal{T}_c(\mathbf{d}) \to \mathcal{T}_s(\mathbf{d})$ be the map given by

$$\xi(f)(\mathbf{D}, W, t) = \chi_q(R_{B_{\mathbf{D},W,t}} f).$$

The fact that the image of ξ is contained in $\mathcal{T}_s(\mathbf{d})$ follows from the fact that, up to isomorphism, $B_{\mathbf{D},W,t}$ depends only on $\alpha(t, \mathbf{D})$ and dim W.

Proposition 3.7.1. *ξ is an $\mathbf{U}_q$-module isomorphism.*

Proof. This follows easily from Theorems 3.5.1 and 3.6.1 since

$$\xi = \sum_b \frac{1}{c_b} (T_{Y_b}|_{Y_b})^{-1} \circ T_{Y_b}. \qquad \square$$

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