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IRREDUCIBLE FINITE-DIMENSIONAL REPRESENTATIONS OF EQUIVARIANT MAP ALGEBRAS

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Abstract. Suppose a finite group acts on a scheme X and a finite-dimensional Lie algebra $\mathfrak{g}$. The corresponding *equivariant map algebra* is the Lie algebra $\mathfrak{M}$ of equivariant regular maps from X to $\mathfrak{g}$. We classify the irreducible finite-dimensional representations of these algebras. In particular, we show that all such representations are tensor products of evaluation representations and one-dimensional representations, and we establish conditions ensuring that they are all evaluation representations. For example, this is always the case if $\mathfrak{M}$ is perfect.

Our results can be applied to multiloop algebras, current algebras, the Onsager algebra, and the tetrahedron algebra. Doing so, we easily recover the known classifications of irreducible finite-dimensional representations of these algebras. Moreover, we obtain previously unknown classifications of irreducible finite-dimensional representations of other types of equivariant map algebras, such as the generalized Onsager algebra.

Introduction

When studying the category of representations of a (possibly infinite-dimensional) Lie algebra, the irreducible finite-dimensional representations often play an important role. Let X be a scheme and let $\mathfrak{g}$ be a finite-dimensional Lie algebra, both defined over an algebraically closed field k of characteristic zero and both equipped with the action of a finite group Γ by automorphisms. The *equivariant map algebra* $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$ is the Lie algebra of regular maps $X \to \mathfrak{g}$ which are equivariant with respect to the action of Γ. Denoting by A the coordinate ring of X, an equivariant map algebra can also be realized as the fixed point Lie algebra $\mathfrak{M} = (\mathfrak{g} \otimes A)^\Gamma$ with respect to the diagonal action of Γ on $\mathfrak{g} \otimes A$. The purpose of the current paper is to classify the irreducible finite-dimensional representations of such algebras.

One important class of examples of equivariant map algebras are the (twisted) loop algebras which play a crucial role in the theory of affine Lie algebras. The description of the irreducible finite-dimensional representations of loop algebras goes back to the work of Chari and Pressley [Cha86], [CP86], [CP98]. Their work has had a long-lasting impact. Generalizations and more precise descriptions of their work have appeared in many papers, for example in Batra [Bat04], Chari-Fourier-Khandai [CFK], Chari-Fourier-Senesi [CFS08], Chari-Moura [CM04], Feigin-Loktev [FL04], Lau [Lau], Li [Li04], and Rao [Rao93, Rao01]. Other examples of equivariant map algebras whose irreducible finite-dimensional representations have been

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classified are the Onsager algebra [DR00] and the tetrahedron algebra (or three-point $\mathfrak{sl}_2$ loop algebra) [Har07]. In all these papers it was proven, sometimes using complicated combinatorial or algebraic arguments and sometimes without explicitly stating so, that all irreducible finite-dimensional representations are evaluation representations.

In the current paper, we provide a complete classification of the irreducible finite-dimensional representations of an arbitrary equivariant map algebra. This class of Lie algebras includes all the aforementioned examples, and we obtain classification results in these cases with greatly simplified proofs. However, the class of Lie algebras covered by the results of this paper is infinitely larger than this set of examples. To demonstrate this, we work out some previously unknown classifications of the irreducible finite-dimensional representations of other Lie algebras such as the generalized Onsager algebra.

If $\mathfrak{M} = M(X,\mathfrak{g})^\Gamma$ is an equivariant map algebra, $\alpha \in \mathfrak{M}$, and x is a k-rational point of X, the image $\alpha(x)$ is not an arbitrary element of $\mathfrak{g}$, but rather an element of $\mathfrak{g}^x = \{u \in \mathfrak{g} : g \cdot u = u \text{ for all } g \in \Gamma_x\}$ where $\Gamma_x = \{g \in \Gamma : g \cdot x = x\}$. For a finite subset $\mathbf{x}$ of k-rational points of X and finite-dimensional representations $\rho_x : \mathfrak{g}^x \to \operatorname{End}_k V_x$, $x \in \mathbf{x}$, we define the associated *evaluation representation* as the composition

$$\mathfrak{M} \xrightarrow{\mathrm{ev}_{\mathbf{x}}} \textstyle\bigoplus_{x\in\mathbf{x}} \mathfrak{g}^x \xrightarrow{\otimes_{x\in\mathbf{x}}\rho_x} \operatorname{End}_k(\bigotimes_{x\in\mathbf{x}} V_x),$$

where $\mathrm{ev}_{\mathbf{x}} : \alpha \mapsto (\alpha(x))_{x\in\mathbf{x}}$ is evaluation at $\mathbf{x}$. Our definition is slightly more general than the classical definition of evaluation representations, which require the V_x to be representations of $\mathfrak{g}$ instead of $\mathfrak{g}^x$. We believe our definition to be more natural, and it leads to a simplification of the classification of irreducible finite-dimensional representations in certain cases. For instance, the Onsager algebra is an equivariant map algebra where $X = \operatorname{Spec} k[t^{\pm 1}]$, $\mathfrak{g} = \mathfrak{sl}_2(k)$, and $\Gamma = \{1, \sigma\}$, with σ acting on X by $\sigma \cdot x = x^{-1}$ and on $\mathfrak{g}$ by the Chevalley involution. For the fixed points $x = \pm 1 \in X$, the subalgebras $\mathfrak{g}^x$ are one-dimensional, in fact Cartan subalgebras, while $\mathfrak{g}^x = \mathfrak{g}$ for $x \neq \pm 1$. In the classification of the irreducible finite-dimensional representations of the Onsager algebra given in [DR00], not all irreducible finite-dimensional representations are evaluation representations since the more restrictive definition is used. Instead, a discussion of *type* is needed to reduce all irreducible finite-dimensional representations (via an automorphism of the enveloping algebra) to evaluation representations. However, under the more general definition of evaluation representation given in the current paper, all irreducible finite-dimensional representations of the Onsager algebra are evaluation representations and no discussion of type is needed. Additionally, contrary to what has been imposed before (for example in the multiloop case), we allow the representations ρ_x to be non-faithful. This will provide us with greater flexibility. Finally, we note that we use the term *evaluation representation* even when we evaluate at more than one point (i.e. when $|\mathbf{x}| > 1$). Such representations are often called tensor products of evaluation representations in the literature, where evaluation representations are at a single point.

By the above, the main question becomes: When does an irreducible finite-dimensional representation of $\mathfrak{M}$ factor through an evaluation map $\mathrm{ev}_{\mathbf{x}}$? Surprisingly, the answer is not always. In particular, the Lie algebra may have one-dimensional representations that are not evaluation representations. Any one-dimensional representation corresponds to a linear form $\lambda \in \mathfrak{M}^*$ vanishing on

$[\mathfrak{M}, \mathfrak{M}]$. In some cases all such linear forms are evaluation representations, but in other cases this is not true. Our main result (Theorem 5.5) is that *any irreducible finite-dimensional representation of* $\mathfrak{M}$ *is a tensor product of an evaluation representation and a one-dimensional representation.*

Obviously, if $\mathfrak{M} = [\mathfrak{M}, \mathfrak{M}]$ is perfect, then our theorem implies that every irreducible finite-dimensional representation of $\mathfrak{M}$ is an evaluation representation. For example, this is so in the case of a multiloop algebra which is an equivariant map algebra $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$ for $X = \operatorname{Spec} k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, $\mathfrak{g}$ simple, and $\Gamma = \mathbb{Z}/(m_1\mathbb{Z}) \times \cdots \times \mathbb{Z}/(m_n\mathbb{Z})$ with the i-th factor of Γ acting on the i-th coordinate of X by a primitive m_i-th root of unity. Note that in this case Γ acts freely on X and therefore all $\mathfrak{g}^x = \mathfrak{g}$. But perfectness of $\mathfrak{M}$ is by no means necessary for all irreducible finite-dimensional representations to be evaluation representations. In fact, in Section 5 we provide an easy criterion for this to be the case, which in particular can be applied to the Onsager algebra to see that all irreducible finite-dimensional representations are evaluation representations in the more general sense. On the other hand, we also provide conditions under which not all irreducible finite-dimensional representations of an equivariant map algebra are evaluation representations (see Proposition 5.19).

An important feature of our classification is the fact that isomorphism classes of evaluation representations are parameterized in a natural and uniform fashion. Specifically, for a rational point $x \in X_{\mathrm{rat}}$, let $\mathcal{R}_x$ be the set of isomorphism classes of irreducible finite-dimensional representations of $\mathfrak{g}^x$ and set $\mathcal{R}_X = \bigsqcup_{x \in X_{\mathrm{rat}}} \mathcal{R}_x$. Then there is a canonical Γ-action on $\mathcal{R}_X$, and isomorphism classes of evaluation representations are naturally enumerated by finitely-supported Γ-equivariant functions $\Psi : X_{\mathrm{rat}} \to \mathcal{R}_X$ such that $\Psi(x) \in \mathcal{R}_x$. Thus we see that evaluation representations "live on orbits". This point of view gives a natural geometric explanation for the somewhat technical algebraic conditions that appear in previous classifications (such as for the multiloop algebras).

We shortly describe the contents of the paper. After a review of some results in the representation theory of Lie algebras in Section 1, we introduce map algebras ($\Gamma = \{1\}$) and equivariant map algebras (arbitrary Γ) in Sections 2 and 3 and discuss old and new examples. We introduce the formalism of evaluation representations in Section 4 and classify the irreducible finite-dimensional representations of equivariant map algebras in Section 5. We show that in general not all irreducible finite-dimensional representations are evaluation representations, and we derive a sufficient criterion for this to nevertheless be the case, as well as a necessary condition. Finally, in Section 6 we apply our general theorem to some specific cases of equivariant map algebras, recovering previous results as well as obtaining new ones.

Notation. Throughout this paper, k is an algebraically closed field of characteristic zero. For schemes, we use the terminology of [EH00]. In particular, an affine scheme X is the (prime) spectrum of a commutative associative k-algebra A. Note that we do *not* assume that A is finitely generated in general. We say that X is an affine variety if A is finitely generated and reduced, in which case we identify X with the maximal spectrum of A. For an arbitrary scheme X, we set $A = \mathcal{O}_X(X)$, except when the possibility of confusion exists (for instance, when more than one scheme is being considered), in which case we use the notation A_X. We let X_{rat} denote the set of k-rational points of X. Recall ([EH00, p. 45]) that $x \in X$ is a

k-rational point of X if its residue field is k. The symbol $\mathfrak{g}$ will always denote a finite-dimensional Lie algebra (but see Remark 5.3). All tensor products will be over k, unless indicated otherwise. If a group Γ acts on a vector space V we denote by $V^\Gamma = \{v \in V : g \cdot v = v \text{ for all } g \in \Gamma\}$ the subspace of fixed points.

1. Review of some results on representations of Lie algebras

For easier reference we review some mostly known results about representations of Lie algebras. Let L be a Lie algebra, not necessarily of finite dimension. The first items concern representations of the Lie algebra

$$L = L_1 \oplus \cdots \oplus L_n,$$

where $L_1, \ldots, L_n$ are ideals of L. Recall that if V_i, $i = 1, \ldots, n$, are L_i-modules, then $V_1 \otimes \cdots \otimes V_n$ is an L-module with action

$$(u_1, \ldots, u_n) \cdot (v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} (v_1 \otimes \cdots \otimes v_{i-1} \otimes u_i \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n).$$

We will always use this L-module structure on $V = V_1 \otimes \cdots \otimes V_n$. It will sometimes be useful to denote this representation by $\rho_1 \otimes \cdots \otimes \rho_n$, where $\rho_i : L_i \to \mathrm{End}_k(V_i)$ are the individual representations. We call an irreducible L-module V *absolutely irreducible* if $\mathrm{End}_L(V) = k\,\mathrm{Id}$. By Schur's Lemma, an irreducible L-module of countable dimension is absolutely irreducible.

We collect in the following proposition various well-known facts that we will need in the current paper.

Proposition 1.1.

(1) *Let V_i, $1 \le i \le n$, be non-zero L_i-modules and let $V = V_1 \otimes \cdots \otimes V_n$. Then V is completely reducible (respectively absolutely irreducible) if and only if all V_i are completely reducible (respectively absolutely irreducible).* For a proof, see for example [Bou58, §7.4, Theorem 2] and use the fact that the universal enveloping algebra of L is $\mathrm{U}(L) = \mathrm{U}(L_1) \otimes \cdots \otimes \mathrm{U}(L_n)$, or see [Li04, Lemma 2.7].

(2) *Let $n = 2$; thus $L = L_1 \oplus L_2$, and let V be an irreducible L-module. Suppose that V, considered as an L_1-module, contains an absolutely irreducible L_1-submodule V_1. Then there exists an irreducible L_2-module V_2 such that $V \cong V_1 \otimes V_2$. The isomorphism classes of V_1 and V_2 are uniquely determined by V.* For a proof in the case $\dim_k V < \infty$ see, for example, [Bou58, §7.7, Proposition 8] and observe that irreducibility of V_2 follows from (1). For a proof in general see [Li04, Lemma 2.7].

(3) *Suppose $L_2, \ldots, L_n$ are finite-dimensional semisimple Lie algebras (note that there are no assumptions on L_1) and let V be an irreducible finite-dimensional L-module. Then there exist irreducible finite-dimensional L_i-modules V_i, $1 \le i \le n$, such that $V \cong V_1 \otimes \cdots \otimes V_n$. The isomorphism classes of the L_i-modules V_i are uniquely determined by V.* This follows from (2) by induction.

(4) *Let L be a finite-dimensional Lie algebra over k. Then:*

  (a) *L has a finite-dimensional faithful completely reducible representation if and only if L is reductive* ([Bou71, §6.4, Proposition 5]).

(b) *L has a finite-dimensional faithful irreducible representation if and only if L is reductive and its center is at most one-dimensional* ([Bou71, §6, Exercise 20]).

(5) *We denote by $L^* = \operatorname{Hom}_k(L, k)$ the dual space of L and identify $(L/[L,L])^* = \{\lambda \in L^* : \lambda([L,L]) = 0\}$. Any $\lambda \in (L/[L,L])^*$ is a one-dimensional, hence irreducible, representation of the Lie algebra L, and every one-dimensional representation of L is of this form. Two linear forms $\lambda, \mu \in (L/[L,L])^*$ are isomorphic as representations if and only if $\lambda = \mu$.* The proof of these facts is immediate.

We combine some of these facts in the following lemma.

Lemma 1.2. *Let L be a (possibly infinite-dimensional) Lie algebra and let $\rho : L \to \operatorname{End}_k(V)$ be an irreducible finite-dimensional representation. Then there exist unique irreducible finite-dimensional representations $\rho_i : L \to \operatorname{End}_k(V_i)$, $i = 1, 2$, such that $\rho = \rho_1 \otimes \rho_2$ and*

(1) *$\rho_1 = \lambda$ for a unique $\lambda \in (L/[L,L])^*$ as in Proposition* 1.1(5) *(hence $V = V_2$ as k-vector spaces) and*

(2) *$L/\operatorname{Ker}\rho_2$ is finite-dimensional semisimple.*

Proof. The representation ρ factors as

$$\begin{tikzcd} L \arrow[rr, "\rho"] \arrow[dr, "\pi"'] & & \operatorname{End}_k(V) \\ & \bar{L} \arrow[ur, hook, "\bar{\rho}"'] & \end{tikzcd}$$

where $\bar{L} = L/\operatorname{Ker}\rho$, π is the canonical epimorphism and $\bar{\rho}$ is a faithful representation of $\bar{L}$, whence $\bar{L}$ is finite-dimensional. By Proposition 1.1(4), $\bar{L}$ is reductive. If $\bar{L}$ is semisimple, let $\lambda = 0$ and $\rho = \rho_2$. Otherwise, by Proposition 1.1(4) again, $\bar{L}$ has a one-dimensional center $\bar{L}_{\mathfrak{z}}$. Let $\bar{L}_{\mathfrak{s}} = [\bar{L}, \bar{L}]$ be the semisimple part of $\bar{L}$. Since $[\bar{\rho}(\bar{L}_{\mathfrak{z}}), \bar{\rho}(\bar{L}_{\mathfrak{s}})] = 0$, it follows that $\bar{\rho}|_{\bar{L}_{\mathfrak{s}}}$ is irreducible. By Burnside's Theorem (see for example [Jac89, Chapter 4.3]) the associative subalgebra of $\operatorname{End}_k(V)$ generated by $\bar{\rho}(\bar{L}_{\mathfrak{s}})$ is therefore equal to $\operatorname{End}_k(V)$. Consequently, $\bar{\rho}(\bar{L}_{\mathfrak{z}}) \subseteq k \operatorname{Id}_V$. Because $\bar{\rho}$ is faithful, we actually have $\bar{\rho}(\bar{L}_{\mathfrak{z}}) = k \operatorname{Id}_V$. We identify $k \operatorname{Id}_V \equiv k$ and then define $\rho_1(x) = \bar{\rho}(\bar{x}_{\mathfrak{z}})$, where $\bar{x}_{\mathfrak{z}}$ is the $\bar{L}_{\mathfrak{z}}$-component of $\bar{x} = \pi(x) \in \bar{L}$. By Proposition 1.1(5), $\rho_1 = \lambda$ for a unique $0 \neq \lambda \in (L/[L,L])^*$. Finally, we define $\rho_2 : L \to \operatorname{End}_k(V)$ by $\rho_2(x) = \bar{\rho}(\bar{x}_{\mathfrak{s}})$, where $\bar{x}_{\mathfrak{s}}$ is the $\bar{L}_{\mathfrak{s}}$-component of $\bar{x}$. Since $\bar{\rho}|_{\bar{L}_{\mathfrak{s}}}$ is faithful, it follows that $\operatorname{Ker}\rho_2 = \pi^{-1}(\bar{L}_{\mathfrak{z}})$, whence $L/\operatorname{Ker}\rho_2 \cong \bar{L}_{\mathfrak{s}}$ is semisimple. This proves existence of the decomposition. Uniqueness follows from the construction above. □

2. Map algebras

In this and the next section we define our main object of study – the Lie algebra of (equivariant) maps from a scheme to another Lie algebra – and discuss several examples. We remind the reader that X is a scheme defined over k and $\mathfrak{g}$ is a finite-dimensional Lie algebra over k. Then $\mathfrak{g}$ is naturally equipped with the structure of an affine algebraic scheme, namely the affine n-space, where $n = \dim \mathfrak{g}$. Addition and multiplication on $\mathfrak{g}$ give rise to morphisms of schemes $\mathfrak{g} \times_k \mathfrak{g} \to \mathfrak{g}$, and multiplication by a fixed scalar yields a morphism of schemes $\mathfrak{g} \to \mathfrak{g}$.

Definition 2.1 (Map algebras)**.** We denote by $M(X, \mathfrak{g})$ the Lie algebra of regular functions on X with values in $\mathfrak{g}$ (equivalently, the set of morphisms of schemes $X \to \mathfrak{g}$), called the *untwisted map algebra* or the *Lie algebra of currents* ([FL04]). The multiplication in $M(X, \mathfrak{g})$ is defined pointwise. That is, for $\alpha, \beta \in M(X, \mathfrak{g})$, we define $[\alpha, \beta] \in M(X, \mathfrak{g})$ to be the composition

$$X \xrightarrow{(\alpha,\beta)} \mathfrak{g} \times_k \mathfrak{g} \xrightarrow{[\cdot,\cdot]} \mathfrak{g}.$$

The addition and scalar multiplication are defined similarly.

Lemma 2.2. *There is an isomorphism*

$$M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes A$$

of Lie algebras over A and hence also over k. The product on $\mathfrak{g} \otimes A$ is given by $[u \otimes f, v \otimes g] = [u, v] \otimes fg$ for $u, v \in \mathfrak{g}$ and $f, g \in A$.

Because of Lemma 2.2, whose proof is routine, we will sometimes identify $M(X, \mathfrak{g})$ and $\mathfrak{g} \otimes A$ in what follows.

Remark 2.3. If $\phi : X \to Y$ is a morphism of schemes, then $\phi^* : M(Y, \mathfrak{g}) \to M(X, \mathfrak{g})$ given by $\phi^*(\alpha) = \alpha \circ \phi$ is a Lie algebra homomorphism. For example, if $\iota : X \hookrightarrow Y$ is an inclusion of schemes, then $\iota^* : M(Y, \mathfrak{g}) \to M(X, \mathfrak{g})$ is the restriction $\phi^* : \alpha \mapsto \alpha|_X$. The assignments $X \mapsto M(X, \mathfrak{g})$, $\phi \mapsto \phi^*$ are easily seen to define a contravariant functor from the category of schemes to the category of Lie algebras. Analogously, for fixed X, the assignment $\mathfrak{g} \mapsto M(X, \mathfrak{g})$ is a covariant functor from the category of finite-dimensional Lie algebras to the category of Lie algebras.

Example 2.4 (Current algebras)**.** Let X be the n-dimensional affine space. Then $A \cong k[t_1, \dots, t_n]$ is a polynomial algebra in n variables and $M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes k[t_1, \dots, t_n]$ is the so-called *current algebra*.

Example 2.5 (Untwisted multiloop algebras)**.** Let $X = \operatorname{Spec} A$, where $A = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the k-algebra of Laurent polynomials in n variables. Then $M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes k[t_1^{\pm 1}, \dots, t_n^{\pm n}]$ is an *untwisted multiloop algebra*. In the case $n = 1$, it is usually called the *untwisted loop algebra of* $\mathfrak{g}$.

Example 2.6 (Tetrahedron Lie algebra)**.** If X is the variety $k \setminus \{0, 1\}$, then $A \cong k[t, t^{-1}, (t-1)^{-1}]$ and $M(X, \mathfrak{sl}_2) \cong \mathfrak{sl}_2 \otimes k[t, t^{-1}, (t-1)^{-1}]$ is the *three point $\mathfrak{sl}_2$ loop algebra*. Removing any two distinct points of k results in an algebra isomorphic to $M(X, \mathfrak{sl}_2)$, and so there is no loss in generality in assuming the points are 0 and 1. It was shown in [HT07] that $M(X, \mathfrak{sl}_2)$ is isomorphic to the *tetrahedron Lie algebra* and to a direct sum of three copies of the *Onsager algebra* (see Example 3.9). We refer the reader to [HT07] and the references cited therein for further details.

3. Equivariant map algebras

Recall that we assume $\mathfrak{g}$ is a finite-dimensional Lie algebra. We denote the group of Lie algebra automorphisms of $\mathfrak{g}$ by $\operatorname{Aut}_k \mathfrak{g}$. Any Lie algebra automorphism of $\mathfrak{g}$, being a linear map, can also be viewed as an automorphism of $\mathfrak{g}$ considered as a scheme. An action of a group Γ on $\mathfrak{g}$ and on a scheme X will always be assumed to be by Lie algebra automorphisms of $\mathfrak{g}$ and scheme automorphisms of X. Recall that there is an induced Γ-action on A given by

$$g \cdot f = f g^{-1}, \quad f \in A, \quad g \in \Gamma,$$

where on the right hand side we view g^{-1} as the corresponding automorphism of X.

Definition 3.1 (Equivariant map algebras)**.** Let Γ be a group acting on a scheme X and a Lie algebra $\mathfrak{g}$ by automorphisms. Then Γ acts on $M(X, \mathfrak{g})$ by automorphisms:

$$g \cdot \alpha = g \alpha g^{-1}, \quad \alpha \in M(X, \mathfrak{g}), \quad g \in \Gamma, \tag{3.1}$$

where on the right hand side g and g^{-1} are viewed as automorphisms of $\mathfrak{g}$ and X, respectively. We define $M(X, \mathfrak{g})^\Gamma$ to be the set of fixed points under this action. That is,

$$M(X, \mathfrak{g})^\Gamma = \{\alpha \in M(X, \mathfrak{g}) : \alpha g = g \alpha \ \forall \ g \in \Gamma\}$$

is the subalgebra of $M(X, \mathfrak{g})$ consisting of Γ-equivariant maps from X to $\mathfrak{g}$. We call $M(X, \mathfrak{g})^\Gamma$ an *equivariant map algebra.*

Example 3.2 (Discrete spaces)**.** Suppose X is a discrete (hence finite) variety. Let X' be a subset of X obtained by choosing one element from each Γ-orbit of X. Then

$$M(X, \mathfrak{g})^\Gamma \cong \prod_{x \in X'} \mathfrak{g}^{\Gamma_x}, \quad \alpha \mapsto (\alpha(x))_{x \in X'}, \quad \alpha \in M(X, \mathfrak{g})^\Gamma,$$

is an isomorphism of Lie algebras, where $\Gamma_x = \{g \in \Gamma : g \cdot x = x\}$ is the stabilizer subgroup of x.

Lemma 3.3. *Let Γ be a group acting on a scheme X and a Lie algebra $\mathfrak{g}$. Suppose $\tau_1 \in \operatorname{Aut}_k \mathfrak{g}$ and $\tau_2 \in \operatorname{Aut} X$. Then we can define a second action of Γ on $\mathfrak{g}$ and X by declaring $g \in \Gamma$ to act by $\tau_1 g \tau_1^{-1}$ on $\mathfrak{g}$ and by $\tau_2 g \tau_2^{-1}$ on X. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be the equivariant map algebras with respect to these two actions. That is,*

$$\mathfrak{M} = \{\alpha \in M(X, \mathfrak{g}) : \alpha g = g \alpha \ \forall \ g \in \Gamma\},$$
$$\mathfrak{M}' = \{\beta \in M(X, \mathfrak{g}) : \beta \tau_2 g \tau_2^{-1} = \tau_1 g \tau_1^{-1} \beta \ \forall \ g \in \Gamma\}.$$

Then $\mathfrak{M} \cong \mathfrak{M}'$ as Lie algebras.

Proof. One easily checks that $\alpha \mapsto \tau_1 \circ \alpha \circ \tau_2^{-1}$ intertwines the two Γ-actions and thus yields the desired automorphism. □

The group Γ acts naturally on $\mathfrak{g} \otimes A$ by extending the map $g \cdot (u \otimes f) = (g \cdot u) \otimes (g \cdot f)$ by linearity. Define

$$(\mathfrak{g} \otimes A)^\Gamma := \{\alpha \in \mathfrak{g} \otimes A : g \cdot \alpha = \alpha \ \forall \ g \in \Gamma\}$$

to be the subalgebra of $\mathfrak{g} \otimes A$ consisting of elements fixed by Γ. The proof of the following lemma is immediate.

Lemma 3.4. *Let Γ be a group acting on a scheme X and a Lie algebra $\mathfrak{g}$. Then the isomorphism $M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes A$ of Lemma 2.2 is Γ-equivariant. In particular, under this isomorphism $(\mathfrak{g} \otimes A)^\Gamma$ corresponds to $M(X, \mathfrak{g})^\Gamma$.*

Remark 3.5. Let $V = \operatorname{Spec} A$, an affine scheme but not necessarily an affine variety. By assumption, Γ acts on X, hence on A. Thus Γ acts on V by [EH00, I-40]. Since $A_V = A_X = A$, we have $M(X, \mathfrak{g})^\Gamma \cong (\mathfrak{g} \otimes A)^\Gamma \cong M(V, \mathfrak{g})^\Gamma$. Therefore, we lose no generality in assuming that X is an affine scheme, and we will often do so in the sequel.

Lemma 3.6. *If Γ acts trivially on $\mathfrak{g}$, then $M(X,\mathfrak{g})^\Gamma \cong M(\mathrm{Spec}(A^\Gamma),\mathfrak{g})$, and hence $M(X,\mathfrak{g})^\Gamma$ is isomorphic to an (untwisted) map algebra.*

Proof. The proof is straightforward. □

Let Γ be a finite group. Recall that any Γ-module B decomposes uniquely as a direct sum $B = B^\Gamma \oplus B_\Gamma$ of the two Γ-submodules B^Γ and $B_\Gamma = \mathrm{Span}\{g \cdot m - m : g \in \Gamma, m \in B\}$. We let $\sharp : B \to B^\Gamma$ be the canonical Γ-module epimorphism, given by $m \mapsto m^\sharp = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} g \cdot m$. If B is a (possibly nonassociative) algebra and Γ acts by automorphisms, then

$$[B^\Gamma,\, B^\Gamma] \subseteq B^\Gamma, \quad [B^\Gamma, B_\Gamma] \subseteq B_\Gamma, \quad [m_0, m^\sharp] = [m_0, m]^\sharp, \tag{3.2}$$

for $m_0 \in B^\Gamma$ and $m \in B$. Since $\mathfrak{g}\otimes A = (\mathfrak{g}^\Gamma \otimes A^\Gamma)\oplus(\mathfrak{g}^\Gamma\otimes A_\Gamma)\oplus(\mathfrak{g}_\Gamma\otimes A^\Gamma)\oplus(\mathfrak{g}_\Gamma\otimes A_\Gamma)$, we get

$$\mathfrak{M} = (\mathfrak{g}^\Gamma \otimes A^\Gamma) \oplus (\mathfrak{g}_\Gamma \otimes A_\Gamma)^\Gamma$$

and

$$[\mathfrak{g}^\Gamma \otimes A^\Gamma, \mathfrak{g}^\Gamma \otimes A^\Gamma] \subseteq \mathfrak{g}^\Gamma \otimes A^\Gamma, \quad [\mathfrak{g}^\Gamma \otimes A^\Gamma, (\mathfrak{g}_\Gamma \otimes A_\Gamma)^\Gamma] \subseteq (\mathfrak{g}_\Gamma \otimes A_\Gamma)^\Gamma.$$

Lemma 3.7. *We suppose that Γ is a finite group and use the notation above. Then we have:*

(1) *If $[\mathfrak{g}^\Gamma, \mathfrak{g}_\Gamma] = \mathfrak{g}_\Gamma$, then*
$$[\mathfrak{g}^\Gamma, \mathfrak{g}^\Gamma] \otimes A^\Gamma \subseteq [\mathfrak{M},\mathfrak{M}] = \big([\mathfrak{M},\mathfrak{M}] \cap (\mathfrak{g}^\Gamma \otimes A^\Gamma)\big) \oplus (\mathfrak{g}_\Gamma \otimes A_\Gamma)^\Gamma.$$
In particular, if $[\mathfrak{g}^\Gamma, \mathfrak{g}] = \mathfrak{g}$, then $\mathfrak{M}$ is perfect.

(2) *If $\mathfrak{g}$ is simple, $\mathfrak{g}^\Gamma \neq \{0\}$ is perfect and the representation of $\mathfrak{g}^\Gamma$ on $\mathfrak{g}_\Gamma$ does not have a trivial non-zero subrepresentation, then $\mathfrak{g} = [\mathfrak{g}^\Gamma, \mathfrak{g}]$, and hence $\mathfrak{M}$ is perfect.*

Proof.

(1) The inclusion $[\mathfrak{g}^\Gamma, \mathfrak{g}^\Gamma] \otimes A^\Gamma \subseteq [\mathfrak{M},\mathfrak{M}]$ is obvious since $\mathfrak{g}^\Gamma \otimes 1 \subseteq \mathfrak{M}$. For $u_0 \in \mathfrak{g}^\Gamma$ and $v \in \mathfrak{g}_\Gamma$ we have, using (3.2), $([u_0, v] \otimes f)^\sharp = [u_0 \otimes 1, (v \otimes f)^\sharp] \in [\mathfrak{M},\mathfrak{M}]$, whence $(\mathfrak{g}_\Gamma \otimes A_\Gamma)^\Gamma = (\mathfrak{g}_\Gamma \otimes A_\Gamma)^\sharp \subseteq [\mathfrak{M},\mathfrak{M}]$ by linearity of $\sharp$. The same argument also shows that $[\mathfrak{g}^\Gamma, \mathfrak{g}] = \mathfrak{g}$ implies that $\mathfrak{M}$ is perfect.

(2) Since $\mathfrak{g}^\Gamma$ is reductive by [Bou75, VII, §1.5, Proposition 14] and perfect by assumption, $\mathfrak{g}^\Gamma$ is semisimple. Thus the representation of $\mathfrak{g}^\Gamma$ on $\mathfrak{g}_\Gamma$ is a direct sum of irreducible representations. If $U \subseteq \mathfrak{g}_\Gamma$ is an irreducible $\mathfrak{g}^\Gamma$-submodule, then $[\mathfrak{g}^\Gamma, U] \subseteq U$ is a submodule, which is non-zero by assumption on $(\mathrm{ad}\, \mathfrak{g}^\Gamma)|_{\mathfrak{g}_\Gamma}$. Hence $[\mathfrak{g}^\Gamma, U] = U$, and thus $[\mathfrak{g}^\Gamma, \mathfrak{g}_\Gamma] = \mathfrak{g}_\Gamma$. Then $\mathfrak{g} = [\mathfrak{g}^\Gamma, \mathfrak{g}^\Gamma] \oplus [\mathfrak{g}^\Gamma, \mathfrak{g}_\Gamma] = [\mathfrak{g}^\Gamma, \mathfrak{g}]$ follows, so that we can apply (1). □

Example 3.8 (Γ abelian)**.** Since the groups in several of the examples discussed later will be abelian, it is convenient to discuss the case of an arbitrary abelian group. We assume that the action of Γ on $\mathfrak{g}$ and on A is diagonalizable – a condition which is always fulfilled if Γ is finite. Hence, denoting by $\Xi = \Xi(\Gamma)$ the character group of Γ, the action of Γ on $\mathfrak{g}$ induces a Ξ-grading of $\mathfrak{g}$, i.e.,

$$\mathfrak{g} = \textstyle\bigoplus_{\xi\in\Xi} \mathfrak{g}_\xi, \quad [\mathfrak{g}_\xi, \mathfrak{g}_\zeta] \subseteq \mathfrak{g}_{\xi+\zeta},$$

where $\mathfrak{g}_\xi = \{u \in \mathfrak{g} : g \cdot u = \xi(g)u \text{ for all } g \in \Gamma\}$ and $\xi, \zeta \in \Xi$. Thus $\mathfrak{g}_0 = \mathfrak{g}^\Gamma$ and $\bigoplus_{0\neq\xi} \mathfrak{g}_\xi = \mathfrak{g}_\Gamma$ in the notation above. We have a similar decomposition for A. The fixed point subalgebra of the diagonal action of Γ on $\mathfrak{g} \otimes A$ is therefore

$$(\mathfrak{g} \otimes A)^\Gamma = \textstyle\bigoplus_{\xi\in\Xi} \mathfrak{g}_\xi \otimes A_{-\xi},$$

where $-\xi$ corresponds to the representation dual to ξ. The assumption $[\mathfrak{g}^\Gamma, \mathfrak{g}_\Gamma] = \mathfrak{g}^\Gamma$ means $[\mathfrak{g}_0, \mathfrak{g}_\xi] = \mathfrak{g}_\xi$ for all $0 \neq \xi$, and if this is fulfilled we have

$$[\mathfrak{M}, \mathfrak{M}] = \left(\sum_\xi [\mathfrak{g}_\xi, \mathfrak{g}_{-\xi}] \otimes A_\xi A_{-\xi} \right) \oplus \bigoplus_{0 \neq \xi} \mathfrak{g}_\xi \otimes A_{-\xi}.$$

As a special case, let $\Gamma = \{1, \sigma\}$ be a group of order 2. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-grading where $\mathfrak{g}_s = \{u \in \mathfrak{g} : \sigma \cdot u = (-1)^s u\}$ for $s = 0, 1 \in \mathbb{Z}_2$. Thus $\mathfrak{M} = (\mathfrak{g} \otimes A)^\Gamma = (\mathfrak{g}_0 \otimes A_0) \oplus (\mathfrak{g}_1 \otimes A_1)$, where $A = A_0 \oplus A_1$ is the $\mathbb{Z}_2$-grading of A induced by σ. Hence

$$[\mathfrak{M}, \mathfrak{M}] = ([\mathfrak{g}_0, \mathfrak{g}_0] \otimes A_0 + [\mathfrak{g}_1, \mathfrak{g}_1] \otimes A_1^2) \oplus ([\mathfrak{g}_0, \mathfrak{g}_1] \otimes A_1).$$

In particular, if $\mathfrak{g}$ is simple and $\sigma \neq \mathrm{Id}$, then

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = ([\mathfrak{g}_0, \mathfrak{g}_0] + [\mathfrak{g}_1, \mathfrak{g}_1]) \oplus ([\mathfrak{g}_0, \mathfrak{g}_1]).$$

This implies $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1$. Also, since $\mathfrak{g}_1 + [\mathfrak{g}_1, \mathfrak{g}_1]$ is the ideal generated by $\mathfrak{g}_1$, which must be all of $\mathfrak{g}$, we have $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0$. Thus

$$[\mathfrak{M}, \mathfrak{M}] = ([\mathfrak{g}_0, \mathfrak{g}_0] \otimes A_0 + \mathfrak{g}_0 \otimes A_1^2) \oplus (\mathfrak{g}_1 \otimes A_1) \quad (|\Gamma| = 2,\ \mathfrak{g} \text{ simple}). \tag{3.3}$$

Therefore, in this case $\mathfrak{M}$ is perfect as soon as $\mathfrak{g}_0$ is perfect, i.e., semisimple, or $A_0 = A_1^2$, i.e., $A = A_0 \oplus A_1$ is a strong $\mathbb{Z}_2$-grading.

Example 3.9 (Generalized Onsager algebra)**.** Let $X = k^\times = \operatorname{Spec} k[t, t^{-1}]$, $\mathfrak{g}$ be a simple Lie algebra, and $\Gamma = \{1, \sigma\}$ be a group of order 2. We choose a set of Chevalley generators $\{e_i, f_i, h_i\}$ for $\mathfrak{g}$ and let Γ act on $\mathfrak{g}$ by the standard Chevalley involution, i.e.,

$$\sigma(e_i) = -f_i,\ \sigma(f_i) = -e_i,\ \sigma(h_i) = -h_i.$$

Let Γ act on $k[t, t^{-1}]$ by $\sigma \cdot t = t^{-1}$, inducing an action of Γ on X. We define the *generalized Onsager algebra* $\mathcal{O}(\mathfrak{g})$ to be the equivariant map algebra associated to these data:

$$\mathcal{O}(\mathfrak{g}) := M(k^\times, \mathfrak{g})^\Gamma \cong (\mathfrak{g} \otimes k[t, t^{-1}])^\Gamma.$$

These algebras have been considered by G. Benkart and M. Lau. The action of σ on $\mathfrak{g}$ interchanges the positive and negative root spaces, and thus the dimension of the fixed point subalgebra $\mathfrak{g}^\Gamma$ is equal to the number of positive roots. This fact, together with the classification of automorphisms of order two (see, for example, [Hel01, Chapter X, §5, Tables II and III]) determines $\mathfrak{g}_0 = \mathfrak{g}^\Gamma$ as follows:

Type of $\mathfrak{g}$	(Type of) $\mathfrak{g}_0$
A_n	$\mathfrak{so}_{n+1}$
B_n $(n \geq 2)$	$\mathfrak{so}_n \oplus \mathfrak{so}_{n+1}$
C_n $(n \geq 2)$	$\mathfrak{gl}_n = k \oplus \mathfrak{sl}_n$
D_n $(n \geq 4)$	$\mathfrak{so}_n \oplus \mathfrak{so}_n$
E_6	C_4
E_7	A_7
E_8	D_8
F_4	$C_3 \oplus A_1$
G_2	$A_1 \oplus A_1$

Since $\mathfrak{so}_2$ is one-dimensional, we see that $\mathfrak{g}^\Gamma$ is semisimple in all cases, except $\mathfrak{g} = \mathfrak{sl}_2$ (type A_1), $\mathfrak{g} = \mathfrak{so}_5$ (type B_2) and $\mathfrak{g} = \mathfrak{sp}_n$ (type C_n). Hence, using (3.3), we have

$$[\mathcal{O}(\mathfrak{g}),\, \mathcal{O}(\mathfrak{g})] = \begin{cases} (\mathfrak{g}_0 \otimes A_1^2) \oplus (\mathfrak{g}_1 \otimes A_1) & \text{if } \mathfrak{g} = \mathfrak{sl}_2, \\ (\mathfrak{so}_3 \otimes A_0 + \mathfrak{g}_0 \otimes A_1^2) \oplus (\mathfrak{g}_1 \otimes A_1) & \text{if } \mathfrak{g} = \mathfrak{so}_5, \\ (\mathfrak{sl}_n \otimes A_0 + \mathfrak{g}_0 \otimes A_1^2) \oplus (\mathfrak{g}_1 \otimes A_1) & \text{if } \mathfrak{g} = \mathfrak{sp}_n, \text{ and} \\ \mathcal{O}(\mathfrak{g}) & \text{otherwise.} \end{cases} \tag{3.4}$$

Therefore

$$\mathcal{O}(\mathfrak{g})/[\mathcal{O}(\mathfrak{g}),\, \mathcal{O}(\mathfrak{g})] \cong \begin{cases} A_0/A_1^2 & \text{if } \mathfrak{g} = \mathfrak{sl}_2,\, \mathfrak{so}_5, \text{ or } \mathfrak{sp}_n, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Example 3.10. One can consider the following situation, which is even more general than Example 3.9. Namely, we consider the same setup except that we allow σ to act by an *arbitrary* involution of $\mathfrak{g}$. Again, by the classification of automorphisms of order two, it is known that $\mathfrak{g}_0$ is either semisimple or has a one-dimensional center. Thus we have

$$\mathfrak{M}/[\mathfrak{M},\, \mathfrak{M}] \cong \begin{cases} 0 & \text{if } \mathfrak{g}_0 \text{ is semisimple, and} \\ A_0/A_1^2 & \text{otherwise.} \end{cases} \tag{3.6}$$

Remark 3.11. For $k = \mathbb{C}$ it was shown in [Roa91] that $\mathcal{O}(\mathfrak{sl}_2)$ is isomorphic to the usual *Onsager algebra*. This algebra was a key ingredient in Onsager's original solution of the 2D Ising model. The algebra $\mathcal{O}(\mathfrak{sl}_n)$, $k = \mathbb{C}$, was introduced in [UI96], although the definition given there differs slightly from the one given in Example 3.9. For $\mathfrak{g} = \mathfrak{sl}_n$, the Chevalley involution of Example 3.9 is given by $\sigma \cdot u = -u^t$, while the involution used in [UI96] is given by $E_{ij} \mapsto (-1)^{i+j+1} E_{ji}$, where E_{ij} is the standard elementary matrix with the (i,j) entry equal to one and all other entries equal to zero. This involution is equal to $\tau_1 \sigma \tau_1^{-1}$, where $\tau_1(u) = DuD^{-1}$ for $D = \operatorname{diag}(\sqrt{-1}, +1, \sqrt{-1}, \dots)$. Furthermore, the involution of X considered in [UI96] is $x \mapsto (-1)^n x^{-1}$, which is equal to σ if n is even and to $\tau_2 \sigma \tau_2^{-1}$, where $\tau_2 \cdot x = \sqrt{-1}x$, if n is odd. Therefore it follows from Lemma 3.3 that the two versions are isomorphic.

Example 3.12 (Multiloop algebras)**.** Fix positive integers $n, m_1, \dots, m_n$. Let

$$\Gamma = \langle g_1, \dots, g_n : g_i^{m_i} = 1,\ g_i g_j = g_j g_i,\ \forall\ 1 \le i, j \le n \rangle.$$

Then $\Xi = \Xi(\Gamma) \cong \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_n\mathbb{Z} \cong \Gamma$. Suppose Γ acts on a semisimple $\mathfrak{g}$. Note that this is equivalent to specifying commuting automorphisms σ_i, $i = 1, \dots, n$, of $\mathfrak{g}$ such that $\sigma_i^{m_i} = \mathrm{id}$. For $i = 1, \dots, n$, let ξ_i be a primitive m_i-th root of unity. As in Example 3.8 we then see that $\mathfrak{g}$ has a Ξ-grading for which the homogenous subspace of degree $\bar{\mathbf{k}}$, $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$, is given by

$$\mathfrak{g}_{\bar{\mathbf{k}}} = \{u \in \mathfrak{g} : \sigma_i(u) = \xi_i^{k_i} u\ \forall\ i = 1, \dots, n\}.$$

Let $X = (k^\times)^n$ and define an action of Γ on X by

$$g_i \cdot (z_1, \dots, z_n) = (z_1, \dots, z_{i-1}, \xi_i z_i, z_{i+1}, \dots, z_n).$$

Then

$$M(\mathfrak{g}, \sigma_1, \dots, \sigma_n, m_1, \dots, m_n) := M(X, \mathfrak{g})^\Gamma \tag{3.7}$$

is the *multiloop algebra* of $\mathfrak{g}$ relative to $(\sigma_1, \dots, \sigma_n)$ and $(m_1, \dots, m_n)$. If all $\sigma_i = \mathrm{Id}$ we recover the untwisted multiloop algebra of Example 2.5. If not all $\sigma_i = \mathrm{Id}$, the algebra M is therefore sometimes called the *twisted loop algebra*.

With Γ and X as above, the induced action of Γ on $A = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is given by

$$\sigma_i \cdot f(t_1, \dots, t_n) = f(t_1, \dots, t_{i-1}, \xi_i^{-1} t_i, t_{i+1}, \dots, t_n), \quad f \in k[t_1^{\pm 1}, \dots, t_n^{\pm 1}].$$

Hence

$$M(\mathfrak{g}, \sigma_1, \dots, \sigma_n, m_1, \dots, m_n) \cong (\mathfrak{g} \otimes k[t_1^{\pm 1}, \dots, t_n^{\pm 1}])^\Gamma \cong \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} \mathfrak{g}_{\bar{\mathbf{k}}} \otimes k t^{\mathbf{k}}, \tag{3.8}$$

where $t^{\mathbf{k}} = t_1^{k_1} \cdots t_n^{k_n}$.

Loop and multiloop algebras play an important role in the theory of affine Kac-Moody algebras [Kac90], extended affine Lie algebras, and Lie tori. The connection between the last two classes of Lie algebras is the following: The core and the centerless core of an extended affine Lie algebra is a Lie torus or centerless Lie torus respectively, every Lie torus arises in this way, and there is a precise construction of extended affine Lie algebras in terms of centerless Lie tori [Neh04]. Any centerless Lie torus whose grading root system is not of type A can be realized as a multiloop algebra [ABFP09].

While most classification results involving (special cases of) equivariant map algebras in the literature use abelian groups, we will see that the general classification developed in the current paper only assumes that the group Γ is finite. Therefore, for illustrative purposes, we include here an example of an equivariant map algebra where the group Γ is not abelian. We will see in Section 6.4 that the representation theory of this algebra is quite interesting.

Example 3.13 (A non-abelian example)**.** Let $\Gamma = S_3$, the symmetric group on 3 objects, let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and let $\mathfrak{g} = \mathfrak{so}_8$, the simple Lie algebra of type D_4. The symmetry group of the Dynkin diagram of type D_4 is isomorphic to S_3, and so Γ acts naturally on $\mathfrak{g}$ by diagram automorphisms (see Remark 4.12). Now, given points x_i, y_i, $i = 1, 2, 3$, of $\mathbb{P}^1$, there is a unique Möbius transformation of $\mathbb{P}^1$ mapping x_i to y_i, $i = 1, 2, 3$. Thus, for any permutation σ of the points $\{0, 1, \infty\}$, there is a unique Möbius transformation of $\mathbb{P}^1$ which induces σ on the set $\{0, 1, \infty\}$. Hence each permutation σ naturally corresponds to an automorphism of X, which we also denote by σ. Therefore we can form the equivariant map algebra $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$. Now, the subalgebra of $\mathfrak{g}$ fixed by the unique order three subgroup of Γ is the simple Lie algebra of type G_2 (see [Kac90, Proposition 8.3]). One easily checks that this subalgebra is fixed by all of Γ. Therefore $\mathfrak{g}^\Gamma$ is perfect. By [Kac90, Proposition 8.3], the representation of $\mathfrak{g}^\Gamma$ on $\mathfrak{g}_\Gamma$ is a direct sum of two 7-dimensional irreducible representations. Thus, by Lemma 3.7, $\mathfrak{M}$ is perfect.

4. Evaluation representations

From now on we assume that Γ is a *finite* group, acting on an affine scheme X and a (finite-dimensional) Lie algebra $\mathfrak{g}$ by automorphisms. We abbreviate $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$.

Definition 4.1 (Restriction)**.** Let Y be a subscheme of X. Then, as in Remark 2.3, we have the *restriction* Lie algebra homomorphism

$$\mathrm{Res}^X_Y : M(X,\mathfrak{g})^\Gamma \to M(Y,\mathfrak{g}), \quad \mathrm{Res}^X_Y(\alpha) = \alpha|_Y, \quad \alpha \in M(X,\mathfrak{g})^\Gamma.$$

If Y is a Γ-invariant subscheme, the image of Res^X_Y is contained in $M(Y,\mathfrak{g})^\Gamma$.

Definition 4.2 (Evaluation)**.** Given a finite subset $\mathbf{x} \subseteq X_{\mathrm{rat}}$, we define the corresponding *evaluation map*

$$\mathrm{ev}_{\mathbf{x}} : M(X,\mathfrak{g})^\Gamma \to \textstyle\bigoplus_{x\in\mathbf{x}} \mathfrak{g}, \quad \alpha \mapsto (\alpha(x))_{x\in\mathbf{x}}, \quad \alpha \in M(X,\mathfrak{g})^\Gamma.$$

Definition 4.3. For a subset Z of X we define

$$\Gamma_Z = \{g \in \Gamma : g\cdot z = z \;\forall\; z \in Z\} \quad \text{and} \quad \mathfrak{g}^Z = \mathfrak{g}^{\Gamma_Z}.$$

Obviously, Γ_Z is a subgroup of Γ and $\mathfrak{g}^Z$ is a subalgebra of $\mathfrak{g}$. In particular, for any $x \in X$, we put $\Gamma_x = \Gamma_{\{x\}}$ and $\mathfrak{g}^x = \mathfrak{g}^{\{x\}}$.

Lemma 4.4.

(1) *Let Z be a subset of X_{rat}. Then $\alpha(Z) \subseteq \mathfrak{g}^Z$ for all $\alpha \in \mathfrak{M}(X,\mathfrak{g})^\Gamma$. In particular, $\alpha(x) \in \mathfrak{g}^x$ for all $x \in X_{\mathrm{rat}}$.*

(2) *Let Z be a Γ-invariant subscheme of X for which the restriction $A_X \to A_Z$, $f \mapsto f|_Z$, is surjective. Then the restriction map $\mathrm{Res}^X_Z : M(X,\mathfrak{g})^\Gamma \to M(Z,\mathfrak{g})^\Gamma$ is also surjective.*

Proof. Part (1) is immediate from the definitions. In (2), finite-dimensionality of $\mathfrak{g}$ implies that the restriction $M(X,\mathfrak{g}) \to M(Z,\mathfrak{g})$ is surjective. Since Γ acts completely reducibly on $M(X,\mathfrak{g})$ and $M(Z,\mathfrak{g})$, the restriction is then also surjective on the subalgebras of Γ-invariants. □

Definition 4.5. We denote by X_n the set of n-element subsets $\mathbf{x} \subseteq X_{\mathrm{rat}}$ consisting of k-rational points and having the property that $y \notin \Gamma\cdot x$ for distinct $x, y \in \mathbf{x}$.

Corollary 4.6. *For $\mathbf{x} \in X_n$ the image of $\mathrm{ev}_{\mathbf{x}}$ is $\bigoplus_{x\in\mathbf{x}} \mathfrak{g}^x$.*

Proof. Let $Z = \bigcup_{x\in\mathbf{x}} \Gamma\cdot x$. Then Z is a Γ-invariant closed subvariety. Hence $A_X \to A_Z$ is surjective, and therefore $\mathrm{Res}^X_Z : M(X,\mathfrak{g})^\Gamma \to M(Z,\mathfrak{g})^\Gamma$ is also surjective by Lemma 4.4. It is immediate that $\mathrm{ev}_{\mathbf{x}} : M(Z,\mathfrak{g})^\Gamma \to \bigoplus_{x\in\mathbf{x}} \mathfrak{g}^x$ is surjective. □

Definition 4.7 (Evaluation representation)**.** Fix a finite subset $\mathbf{x} \subseteq X_{\mathrm{rat}}$ and let $\rho_x : \mathfrak{g}^x \to \mathrm{End}_k V_x$, $x \in \mathbf{x}$, be representations of $\mathfrak{g}^x$ on the vector spaces V_x. Then define $\mathrm{ev}_{\mathbf{x}}(\rho_x)_{x\in\mathbf{x}}$ to be the composition

$$M(X,\mathfrak{g})^\Gamma \xrightarrow{\mathrm{ev}_{\mathbf{x}}} \textstyle\bigoplus_{x\in\mathbf{x}} \mathfrak{g}^x \xrightarrow{\otimes_{x\in\mathbf{x}}\rho_x} \mathrm{End}_k(\bigotimes_{x\in\mathbf{x}} V_x).$$

This defines a representation of $M(X,\mathfrak{g})^\Gamma$ on $\bigotimes_{x\in\mathbf{x}} V_x$ called a *(twisted) evaluation representation*.

Remark 4.8. We note some important distinctions between Definition 4.7 and other uses of the term *evaluation representation* in the literature. First of all, some authors reserve the term *evaluation representation* for the case $n = 1$ and would refer to the more general case as a tensor product of evaluation representations. Furthermore, traditionally the ρ_x are representations of $\mathfrak{g}$ instead of $\mathfrak{g}^x$ and are required to be faithful. In the case that $\mathfrak{g}^x = \mathfrak{g}$ for all $x \in X$ (for instance, if Γ acts freely on X), this of course makes no difference. However, we will see in the sequel that the more general definition of evaluation representation given above allows for a more uniform classification of irreducible finite-dimensional representations.

Proposition 4.9. *Let $\mathbf{x} \in X_n$, and for $x \in \mathbf{x}$ let $\rho_x : \mathfrak{g}^x \to \operatorname{End}_k V_x$ be an irreducible finite-dimensional representation of $\mathfrak{g}^x$. Then the evaluation representation $\operatorname{ev}_{\mathbf{x}}(\rho_x)_{x\in\mathbf{x}}$ is an irreducible finite-dimensional representation of $M(X,\mathfrak{g})^\Gamma$.*

Proof. Since $\operatorname{ev}_{\mathbf{x}}$ is surjective, this follows from Proposition 1.1(1). □

Corollary 4.10. *If $\mathbf{x}$ is in X^n (but not necessarily in X_n), $\mathfrak{g}^x$ is semisimple for all $x \in \mathbf{x}$, and ρ_x is an arbitrary finite-dimensional representation of $\mathfrak{g}^x$ for each $x \in \mathbf{x}$, then the evaluation representation $\operatorname{ev}_{\mathbf{x}}(\rho_x)_{x\in\mathbf{x}}$ is completely reducible.*

Proof. This follows from Proposition 4.9 and complete reducibility of finite-dimensional representations of each $\mathfrak{g}^x$. □

By abuse of notation, we will sometimes denote a representation of $\mathfrak{g}$ by the underlying vector space V. Then $\operatorname{ev}_x V$, $x \in X$, will denote the corresponding evaluation representation of $M(X,\mathfrak{g})^\Gamma$. Note that, with the notation of Definition 4.7, we have

$$\operatorname{ev}_{\mathbf{x}}(\rho_x)_{x\in\mathbf{x}} \cong \bigotimes_{x\in\mathbf{x}} \operatorname{ev}_x V_x. \tag{4.1}$$

Note that X_{rat} is Γ-invariant and that $X_{\mathrm{rat}} = X$ if X is an affine variety. Let $x \in X_{\mathrm{rat}}$ and $g \in \Gamma$. Since $\Gamma_{g\cdot x} = g\Gamma_x g^{-1}$ we see that $\mathfrak{g}^{g\cdot x} = g\cdot\mathfrak{g}^x$. Hence if ρ is a representation of $\mathfrak{g}^x$, then $\rho\circ g^{-1}$ is a representation of $\mathfrak{g}^{g\cdot x}$. Let $\mathcal{R}_x$ denote the set of isomorphism classes of irreducible finite-dimensional representations of $\mathfrak{g}^x$, and put $\mathcal{R}_X = \bigsqcup_{x\in X_{\mathrm{rat}}} \mathcal{R}_x$. Then Γ acts on $\mathcal{R}_X$ by

$$\Gamma\times\mathcal{R}_X \to \mathcal{R}_X, \quad (g,[\rho]) \mapsto g\cdot[\rho] := [\rho\circ g^{-1}] \in \mathcal{R}_{g\cdot x},$$

where $[\rho]\in\mathcal{R}_x$ denotes the isomorphism class of a representation ρ of $\mathfrak{g}^x$.

Definition 4.11. Let $\mathcal{E}$ denote the set of finitely supported Γ-equivariant functions $\Psi : X_{\mathrm{rat}} \to \mathcal{R}_X$ such that $\Psi(x)\in\mathcal{R}_x$. Here the support $\operatorname{supp}\Psi$ of $\Psi\in\mathcal{E}$ is the set of all $x\in X_{\mathrm{rat}}$ for which $\Psi(x)\neq 0$, where 0 denotes the isomorphism class of the trivial representation.

For isomorphic representations ρ and ρ' of $\mathfrak{g}^x$, the evaluation representations $\operatorname{ev}_x\rho$ and $\operatorname{ev}_x\rho'$ are isomorphic. Therefore, for $[\rho]\in\mathcal{R}_x$, we can define $\operatorname{ev}_x[\rho]$ to be the isomorphism class of $\operatorname{ev}_x\rho$, and this is independent of the representative ρ. Similarly, for a finite subset $\mathbf{x}\subseteq X_{\mathrm{rat}}$ and representations ρ_x of $\mathfrak{g}^x$ for $x\in\mathbf{x}$, we define $\operatorname{ev}_{\mathbf{x}}([\rho_x])_{x\in\mathbf{x}}$ to be the isomorphism class of $\operatorname{ev}_{\mathbf{x}}(\rho_x)_{x\in\mathbf{x}}$.

Remark 4.12. Suppose $\mathfrak{g}$ is a semisimple Lie algebra. In the case that Γ is cyclic and acts on $\mathfrak{g}$ by admissible diagram automorphisms (no edge joins two vertices in the same orbit), there exists a simple description of $\mathcal{E}$ which can be seen as follows. Let I be the set of vertices of the Dynkin diagram of $\mathfrak{g}$. An action of Γ on this Dynkin diagram gives rise to an action of Γ on $\mathfrak{g}$ via

$$g\cdot h_i = h_{g\cdot i}, \quad g\cdot e_i = e_{g\cdot i}, \quad g\cdot f_i = f_{g\cdot i}, \quad g\in\Gamma,$$

where $\{h_i,e_i,f_i\}_{i\in I}$ is a set of Chevalley generators of $\mathfrak{g}$. We then have a natural action of Γ on the weight lattice P of $\mathfrak{g}$ given by

$$g\cdot\omega_i = \omega_{g\cdot i}, \quad g\in\Gamma,$$

where $\{\omega_i\}_{i\in I}$ is the set of fundamental weights of $\mathfrak{g}$. Now, isomorphism classes of irreducible finite-dimensional representations of $\mathfrak{g}$ are naturally enumerated by the set of dominant weights P^+ by associating to $\lambda \in P^+$ the isomorphism class of the irreducible highest weight representation of highest weight λ. Let $\tilde{\mathcal{E}}$ denote the set of Γ-equivariant functions $X_{\mathrm{rat}} \to P^+$ with finite support. It follows that for $\Psi \in \tilde{\mathcal{E}}$ and $x \in X_{\mathrm{rat}}$ we have $\Psi(x) \in (P^+)^{\Gamma_x}$, where $(P^+)^{\Gamma_x}$ denotes the set of Γ_x-invariant elements of P^+. There is a canonical bijection between $(P^+)^{\Gamma_x}$ and the positive weight lattice of $\mathfrak{g}^x$ (see [Lus93, Proposition 14.1.2]), and so we can associate to $\Psi(x)$ the isomorphism class of the corresponding representation of $\mathfrak{g}^x$. Thus, we have a natural bijection between $\tilde{\mathcal{E}}$ and $\mathcal{E}$. Therefore, in the case that Γ acts on $\mathfrak{g}$ by admissible diagram automorphisms, the evaluation representations are naturally enumerated by Γ-equivariant maps from X_{rat} to the positive weight lattice P^+ of $\mathfrak{g}$. In the case that Γ acts freely on X, we can drop the assumption that the diagram automorphisms be admissible.

Lemma 4.13. *Suppose $\Psi \in \mathcal{E}$ and $x \in \operatorname{supp} \Psi$. Then for all $g \in \Gamma$,*

$$\operatorname{ev}_x \Psi(x) = \operatorname{ev}_{g\cdot x} (g \cdot \Psi(x)) = \operatorname{ev}_{g\cdot x} \Psi(g \cdot x).$$

Proof. For any $g \in \Gamma$ and representation ρ of $\mathfrak{g}^x$, the following diagram commutes:

$$\begin{tikzcd} & \mathfrak{g}^x \arrow[dd, "g"] \arrow[dr, "\rho"] & \\ M(X,\mathfrak{g})^\Gamma \arrow[ur, "\mathrm{ev}_x"] \arrow[dr, "\mathrm{ev}_{g\cdot x}"'] & & \operatorname{End}_k V \\ & \mathfrak{g}^{g\cdot x} \arrow[ur, "\rho\circ g^{-1}"'] & \end{tikzcd}$$

Thus, $\operatorname{ev}_x \rho = \operatorname{ev}_{g\cdot x}(\rho \circ g^{-1})$ and the result follows. □

Definition 4.14. For $\Psi \in \mathcal{E}$, we define $\operatorname{ev}_\Psi = \operatorname{ev}_{\mathbf{x}}(\Psi(x))_{x\in\mathbf{x}}$, where $\mathbf{x} \in X_n$ contains one element of each Γ-orbit in $\operatorname{supp}\Psi$. By Lemma 4.13, ev_Ψ is independent of the choice of $\mathbf{x}$. If Ψ is the map that is identically 0 on X, we define ev_Ψ to be the isomorphism class of the trivial representation of $\mathfrak{M}$. Thus $\Psi \mapsto \operatorname{ev}_\Psi$ defines a map $\mathcal{E} \to \mathcal{S}$, where $\mathcal{S}$ denotes the set of isomorphism classes of irreducible finite-dimensional representations of $\mathfrak{M}$.

Proposition 4.15. *The map $\mathcal{E} \to \mathcal{S}$, $\Psi \mapsto \operatorname{ev}_\Psi$, is injective.*

Proof. Suppose $\Psi \neq \Psi' \in \mathcal{E}$. Then there exists $x \in X_{\mathrm{rat}}$ such that $\Psi(x) \neq \Psi'(x)$. Without loss of generality, we may assume $\Psi(x) \neq 0$. Let

$$m = (\dim \operatorname{ev}_\Psi)/(\dim \Psi(x)), \qquad m' = (\dim \operatorname{ev}_{\Psi'})/(\dim \Psi'(x)),$$

where the dimension of an isomorphism class of representations is simply the dimension of any representative of that class and $m' = \dim \operatorname{ev}_{\Psi'}$ if $\Psi'(x) = 0$. By (4.1), m and m' are positive integers. By Corollary 4.6, there exists a subalgebra $\mathfrak{a}$ of $\mathfrak{M}$ such that $\operatorname{ev}_x(\mathfrak{a}) = \mathfrak{g}^x$ and $\operatorname{ev}_{x'}(\mathfrak{a}) = 0$ for all $x' \in (\operatorname{supp}\Psi \cup \operatorname{supp}\Psi') \setminus \{\Gamma \cdot x\}$. Then

$$\operatorname{ev}_\Psi|_{\mathfrak{a}} = \Psi(x)^{\oplus m}, \quad \operatorname{ev}_{\Psi'}|_{\mathfrak{a}} = \Psi'(x)^{\oplus m'}.$$

Since $\Psi(x) \neq \Psi'(x)$, we have $\operatorname{ev}_\Psi \neq \operatorname{ev}_{\Psi'}$. In the above, we have used the convention that the restriction of an isomorphism class is the isomorphism class of the restriction of any representative and that a direct sum of isomorphism classes is the isomorphism class of the corresponding direct sum of representatives. □

Lemma 4.16. *Let $\mathfrak{K}$ be an ideal of $\mathfrak{M}$. If $\mathfrak{M}/\mathfrak{K}$ does not contain non-zero solvable ideals, e.g. if $\mathfrak{M}/\mathfrak{K}$ is finite-dimensional semisimple, then $\mathfrak{K}$ is an A^Γ-submodule of $\mathfrak{M}$.*

Proof. Let $f \in A^\Gamma$. Since A^Γ is contained in the centroid of $\mathfrak{M}$, the k-subspace $f\mathfrak{K}$ is an ideal of $\mathfrak{M}$: $[\mathfrak{M}, f\mathfrak{K}] = f[\mathfrak{M}, \mathfrak{K}] \subseteq f\mathfrak{K}$. Let $\pi : \mathfrak{M} \to \mathfrak{M}/\mathfrak{K}$ be the canonical epimorphism. Since $[f\mathfrak{K} + \mathfrak{K}, f\mathfrak{K} + \mathfrak{K}] \subseteq [\mathfrak{K} + f\mathfrak{K} + f^2\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K}$, the ideal $\pi(f\mathfrak{K})$ of $\mathfrak{M}/\mathfrak{K}$ is abelian, whence $f\mathfrak{K} \subseteq \mathfrak{K}$. □

Proposition 4.17. *Suppose I is an ideal of $\mathfrak{M}$ such that $\mathfrak{M}/I = N_1 \oplus \cdots \oplus N_s$ with N_i a finite-dimensional simple Lie algebra for $i = 1, \ldots, s$. Let $\pi : \mathfrak{M} \to \mathfrak{M}/I$ denote the canonical projection and for $i = 1, \ldots, s$, and let $\pi_i : \mathfrak{M} \to N_i$ denote the map π followed by the projection from $\mathfrak{M}/I$ to N_i. Then there exist $x_1, \ldots, x_s \in X_{\mathrm{rat}}$ such that*

$$\pi(f\alpha) = (f(x_1)\pi_1(\alpha), \ldots, f(x_s)\pi_s(\alpha)) \ \forall\ f \in A^\Gamma,\ \alpha \in \mathfrak{M}. \tag{4.2}$$

Proof. It suffices to show that for $i = 1, \ldots, s$, $f \in A^\Gamma$ and $\alpha \in \mathfrak{M}$ we have $\pi_i(f\alpha) = f(x_i)\pi_i(\alpha)$ for some $x_i \in X$. Since N_i is simple, the action of $\mathfrak{M}$ on N_i induced by the adjoint action is irreducible. By Lemma 4.16, N_i is an A^Γ-module and π_i is an A^Γ-module homomorphism. Since the action of A^Γ commutes with the action of $\mathfrak{M}$, we have that A^Γ must act by scalars and thus as a character $\chi : A^\Gamma \to k$. This character corresponds to evaluation at a point $\tilde{x}_i \in X/\!/\Gamma := \operatorname{Spec} A^\Gamma$. Choosing any x_i in the preimage of $\tilde{x}_i$ under the canonical projection $X \to X/\!/\Gamma$, the result follows. □

Remark 4.18. If $s = 1$, then in the graded setting these types of maps have been studied extensively in [ABFP08]. There ρ is a character of the full centroid of $\mathfrak{M}$, while in the above A^Γ is a priori only a subalgebra of the centroid of $\mathfrak{M}$.

Proposition 4.19. *If* $\operatorname{supp} \Psi \subseteq \{x \in X : \mathfrak{g}^x = \mathfrak{g}\}$*, then any evaluation representation in the isomorphism class* ev_Ψ *of $M(X, \mathfrak{g})^\Gamma$ is obtained by restriction from an evaluation representation of the untwisted map algebra $M(X, \mathfrak{g})$. In particular, the restriction map from the set of isomorphism classes of evaluation representations of $M(X, \mathfrak{g})$ to the set of isomorphism classes of evaluation representations of $M(X, \mathfrak{g})^\Gamma$ is surjective if $\mathfrak{g}^x = \mathfrak{g}$ for all $x \in X$.*

Proof. The proof is immediate. □

5. Classification of irreducible finite-dimensional representations

We consider a finite group Γ, acting on a finite-dimensional Lie algebra $\mathfrak{g}$ and an affine scheme X. We put $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$.

Lemma 5.1 ([Bou70, II, Corollaire 2 de la Proposition 6, §3.6])**.** *Let S be a subring of a ring R and I an ideal of S. Then $R \otimes_S (S/I) \cong R/(RI)$.*

Note that A^Γ is the coordinate ring of the quotient $X/\!/\Gamma$. For a point $[x] \in X/\!/\Gamma$, let $\mathfrak{m}_{[x]}$ denote the corresponding maximal ideal of A^Γ and define $A_{[x]} = A \otimes_{A^\Gamma} (A^\Gamma/\mathfrak{m}_{[x]})$. We will sometimes view $[x]$ as an orbit in X.

Proposition 5.2. *Suppose $\mathfrak{K}$ is an ideal of $\mathfrak{M}$ such that the quotient algebra $\mathfrak{M}/\mathfrak{K}$ is finite-dimensional and simple. Then there exists a point $x \in X_{\mathrm{rat}}$ such that the canonical epimorphism $\pi : \mathfrak{M} \to \mathfrak{M}/\mathfrak{K}$ factors through the evaluation map $\mathrm{ev}_x : \mathfrak{M} \to \mathfrak{g}^x$.*

Proof. We have shown in Proposition 4.17 that there exists a character $\chi : A^\Gamma \to k$ such that $\pi(\alpha f) = \pi(\alpha)\chi(f)$ holds for all $\alpha \in \mathfrak{M}$ and $f \in A^\Gamma$. Let $\mathfrak{n} = \operatorname{Ker} \chi \in \operatorname{Spec}(A^\Gamma)$ be the corresponding k-rational point. Temporarily viewing $\mathfrak{M}$ as a A^Γ-module, it follows that $\mathfrak{n}$ annihilates $\mathfrak{M}/\mathfrak{K}$, whence $\mathfrak{n}\mathfrak{M} \subseteq \mathfrak{K}$, and we can factor π through obvious maps:

$$\begin{tikzcd} \mathfrak{M} \arrow[r, two heads, "\pi"] \arrow[d, two heads] & \mathfrak{M}/\mathfrak{K} \\ \mathfrak{M}/\mathfrak{n}\mathfrak{M} \arrow[ur, two heads] & \end{tikzcd}$$

Observe

$$\mathfrak{M}/\mathfrak{n}\mathfrak{M} \cong \mathfrak{M} \otimes_{A^\Gamma} (A^\Gamma/\mathfrak{n}) = (\mathfrak{g} \otimes_k A)^\Gamma \otimes_{A^\Gamma} (A^\Gamma/\mathfrak{n}) = \big(\mathfrak{g} \otimes_k A \otimes_{A^\Gamma} (A^\Gamma/\mathfrak{n})\big)^\Gamma, \tag{5.1}$$

where in the last equality we used that Γ acts trivially on $A^\Gamma/\mathfrak{n}$. Also note, by Lemma 5.1,

$$A \otimes_{A^\Gamma} (A^\Gamma/\mathfrak{n}) \cong A/I \quad \text{for } I = A\mathfrak{n}. \tag{5.2}$$

Hence, putting these canonical isomorphisms together, we get a new factorization:

$$\begin{tikzcd} \mathfrak{M} \arrow[r, two heads, "\pi"] \arrow[d, two heads] & \mathfrak{M}/\mathfrak{K} \\ \mathfrak{M}/\mathfrak{n}\mathfrak{M} \arrow[r, "\cong"] & \big(\mathfrak{g} \otimes_k (A/I)\big)^\Gamma \arrow[u, two heads, "\psi"'] \end{tikzcd}$$

It now remains to show that ψ factors through ev_x for an appropriate $x \in X$. To this end, let $[\mathfrak{n}] = \{\mathfrak{m} \in \operatorname{Spec} A : \mathfrak{m} \cap A^\Gamma = \mathfrak{n}\}$. One knows ([Bou85, V, §2]) that $[\mathfrak{n}]$ is a non-empty set of k-rational points on which Γ acts transitively. Let $[\mathfrak{n}] = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_s\}$. We claim

$$\sqrt{I} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_s. \tag{5.3}$$

Indeed, any $\mathfrak{p} \in V(I) \subseteq \operatorname{Spec} A$ satisfies $\mathfrak{p} \cap A^\Gamma = \mathfrak{n}$ so that $V(I) = [\mathfrak{n}]$. Now (5.3) follows from [Bou61, II, §2.6, Corollaire de la Proposition 13].

We have an exact sequence of algebras and Γ-modules

$$0 \to \mathfrak{g} \otimes_k \big((\textstyle\bigcap_i \mathfrak{m}_i)/I\big) \to \mathfrak{g} \otimes_k (A/I) \to \mathfrak{g} \otimes_k (A/\textstyle\bigcap_i \mathfrak{m}_i) \to 0, \tag{5.4}$$

where

$$\mathfrak{g} \otimes_k (A/\textstyle\bigcap_i \mathfrak{m}_i) \cong \textstyle\bigoplus_i \big(\mathfrak{g} \otimes_k (A/\mathfrak{m}_i)\big) \tag{5.5}$$

since $A/\bigcap_i \mathfrak{m}_i \cong \bigoplus_i A/\mathfrak{m}_i$. Now observe that the summands on the right hand side of (5.5) are permuted by the action of Γ. Thus, if we fix $x = \mathfrak{m} \in [\mathfrak{n}]$ we have

$$\big(\mathfrak{g} \otimes_k (A/\textstyle\bigcap_i \mathfrak{m}_i)\big)^\Gamma \cong (\mathfrak{g} \otimes_k (A/\mathfrak{m}))^{\Gamma_x} = \mathfrak{g}^x.$$

Therefore, taking Γ-invariants in (5.4) we get an epimorphism

$$\zeta : \big(\mathfrak{g} \otimes_k (A/I)\big)^\Gamma \twoheadrightarrow \mathfrak{g}^x$$

with kernel $\big(\mathfrak{g} \otimes_k (\bigcap_i \mathfrak{m}_i)/I\big)^\Gamma$.

Any $\alpha \in \operatorname{Ker} \zeta$ is a finite sum $\alpha = \sum_j u_j \otimes \bar{f}_j$, where every $\bar{f}_j \in (\bigcap_i \mathfrak{m}_i)/I$ is nilpotent by (5.3). The ideal J of $\big(\mathfrak{g} \otimes_k (A/I)\big)^\Gamma$ generated by α is therefore nilpotent. Since $\mathfrak{M}/\mathfrak{K}$ is simple, it does not contain a non-zero nilpotent ideal. Thus

$\psi(J) = 0$. Therefore ψ factors through ζ and we get the following commutative diagram:

$$\begin{tikzcd} \mathfrak{M} \arrow[rr, "\pi"] \arrow[d, "p"'] & & \mathfrak{M}/\mathfrak{K} \\ \left(\mathfrak{g} \otimes_k (A/I)\right)^\Gamma \arrow[urr, "\psi"] \arrow[rr, two heads, "\zeta"'] & & \mathfrak{g}^x \arrow[u] \end{tikzcd}$$

Since $\zeta \circ p = \mathrm{ev}_x$, the result follows. □

Remark 5.3. Proposition 5.2 remains true for k an arbitrary algebraically closed field whose characteristic does not divide the order of Γ and for $\mathfrak{g}$ an arbitrary finite-dimensional (not necessarily Lie or associative) algebra.

Corollary 5.4.

(1) *Suppose* $\varphi : \mathfrak{M} \to \mathrm{End}_k V$ *is an irreducible finite-dimensional representation such that* $\mathfrak{M}/\ker\varphi$ *is a semisimple Lie algebra. Then* φ *is an irreducible finite-dimensional evaluation representation.*

(2) *The irreducible finite-dimensional representations of* $\mathfrak{M}$ *are precisely the representations of the form* $\lambda \otimes \varphi$ *where* $\lambda \in (\mathfrak{M}/[\mathfrak{M},\mathfrak{M}])^*$*, and* φ *is an irreducible finite-dimensional evaluation representation with* $\mathfrak{M}/\mathrm{Ker}\,\varphi$ *semisimple. The factors* λ *and* φ *are uniquely determined.*

Proof. Part (1) follows from Proposition 5.2 and the results of Section 1. Part (2) is then a simple application of Lemma 1.2. □

We now state our main theorem which gives a classification of the irreducible finite-dimensional representations of an arbitrary equivariant map algebra. Recall the definitions of $\mathcal{E}$ (Definition 4.11) and $\mathcal{S}$ (Definition 4.14).

Theorem 5.5. *Suppose* Γ *is a finite group acting on an affine scheme* X *and a finite-dimensional Lie algebra* $\mathfrak{g}$*. Then the map*

$$(\mathfrak{M}/[\mathfrak{M},\mathfrak{M}])^* \times \mathcal{E} \to \mathcal{S}, \quad (\lambda, \Psi) \mapsto \lambda \otimes \mathrm{ev}_\Psi, \qquad \lambda \in (\mathfrak{M}/[\mathfrak{M},\mathfrak{M}])^*, \quad \Psi \in \mathcal{E},$$

is surjective. In particular, all irreducible finite-dimensional representations of $\mathfrak{M}$ *are tensor products of an evaluation representation and a one-dimensional representation.*

Furthermore, we have that $\lambda \otimes \mathrm{ev}_\Psi = \lambda' \otimes \mathrm{ev}_{\Psi'}$ *if and only if there exists* $\Phi \in \mathcal{E}$ *such that* $\dim \mathrm{ev}_\Phi = 1$*,* $\lambda' = \lambda - \mathrm{ev}_\Phi$ *and* $\mathrm{ev}_{\Psi'} = \mathrm{ev}_{\Psi\otimes\Phi}$*. Here* $\Psi \otimes \Phi \in \mathcal{E}$ *is given by* $(\Psi \otimes \Phi)(x) = \Psi(x) \otimes \Phi(x)$*, where* $\Psi(x)$ *(respectively* $\Phi(x)$*) is the one-dimensional trivial representation if* $x \notin \mathrm{supp}\,\Psi$ *(respectively* $x \notin \mathrm{supp}\,\Phi$*). In particular, the restriction of the map* $(\lambda, \Psi) \mapsto \lambda \otimes \mathrm{ev}_\Psi$ *to either factor (times the zero element of the other) is injective.*

Proof. This follows from Corollary 5.4 and Lemma 1.2. □

Remark 5.6. Since $\lambda, \mu \in (\mathfrak{M}/[\mathfrak{M},\mathfrak{M}])^*$ are isomorphic as representations if and only if they are equal as linear functions, in Theorem 5.5 we have identified elements of $(\mathfrak{M}/[\mathfrak{M},\mathfrak{M}])^*$ with isomorphism classes of representations.

Remark 5.7. Note that the evaluation representations of $M(\mathrm{Spec}\,A, \mathfrak{g})^\Gamma$ are the same as the evaluation representations of $M(\mathrm{Spec}(A/\,\mathrm{rad}\,A), \mathfrak{g})^\Gamma$ since one evaluates at rational points. However, the one-dimensional representations of these two Lie algebras can be different in general. Thus, we do not assume that the scheme X is reduced.

Corollary 5.8. *If $\mathfrak{M}$ is perfect, then the map $\Psi \mapsto \mathrm{ev}_\Psi$ is a bijection between $\mathcal{E}$ and $\mathcal{S}$. In particular, this is true if any one of the following conditions holds:*

1. $[\mathfrak{g}^\Gamma, \mathfrak{g}] = \mathfrak{g}$,
2. $\mathfrak{g}$ *is simple,* $\mathfrak{g}^\Gamma \neq \{0\}$ *is perfect and acts on* $\mathfrak{g}_\Gamma$ *without a trivial non-zero submodule, or*
3. Γ *acts on* $\mathfrak{g}$ *by diagram automorphisms.*

Proof. If $\mathfrak{M}$ is perfect, then $[\mathfrak{M}, \mathfrak{M}] = \mathfrak{M}$, and the first statement follows immediately from Theorem 5.5. Conditions (1) or (2) imply that $\mathfrak{M}$ is perfect by Lemma 3.7. It remains to show that condition (3) implies that $\mathfrak{g}^\Gamma$ is perfect. It suffices to consider the case where $\mathfrak{g}$ is simple. If Γ acts on $\mathfrak{g}$ by diagram automorphisms, then there are two possibilities: either Γ is a cyclic group generated by a single diagram automorphism or $\mathfrak{g}$ is of type D_4 and $\Gamma \cong S_3$. If Γ is generated by a single diagram automorphism, it is well known that $\mathfrak{g}^\Gamma$ is a simple Lie algebra and hence perfect (see [Kac90, §8.2]). The case $\Gamma \cong S_3$ was described in Example 3.13, where it was shown that $\mathfrak{g}^\Gamma$ is simple as well. □

Remark 5.9. Note that the three conditions in Corollary 5.8 depend only on the action of Γ on $\mathfrak{g}$ and not on the scheme X or its Γ-action.

Remark 5.10 (Untwisted map algebras). If Γ is trivial (or, more generally, acts trivially on $\mathfrak{g}$), we have $\mathfrak{g}^\Gamma = \mathfrak{g}$. Thus the map $\mathcal{E} \to \mathcal{S}$, $\Psi \mapsto \mathrm{ev}_\Psi$, is a bijection if and only if $\mathfrak{g}$ is perfect. In the case when Γ is trivial, $\mathfrak{g}$ is a finite-dimensional simple Lie algebra and A is finitely generated, a similar statement has recently been made in [CFK].

Corollary 5.11. *Suppose that all irreducible finite-dimensional representations of $\mathfrak{M}$ are evaluation representations (e.g. $\mathfrak{M}$ is perfect) and that all $\mathfrak{g}^x$, $x \in X_{\mathrm{rat}}$, are semisimple. Then a finite-dimensional $\mathfrak{M}$-module V is completely reducible if and only if there exists $\mathbf{x} \in X_n$ for some $n \in \mathbb{N}$, $n > 0$, such that* $\operatorname{Ker} \mathrm{ev}_{\mathbf{x}} \subseteq \operatorname{Ann}_{\mathfrak{M}} V := \{\alpha \in \mathfrak{M} : \alpha \cdot v = 0 \ \forall\ v \in V\}$.

For the case of the current algebra $\mathfrak{M} = \mathfrak{g} \otimes \mathbb{C}[t]$, this corollary is proven in [CG05, Prop. 3.9(iii)].

Proof. Let V be a completely reducible $\mathfrak{M}$-module, hence a finite direct sum of irreducible finite-dimensional representations $V^{(i)}$. By assumption, every $V^{(i)}$ is an evaluation representation, given by some $\mathbf{x}^{(i)} \in X_{n_i}$. Then $\operatorname{Ann}_{\mathfrak{M}} V = \bigcap_i \operatorname{Ann}_{\mathfrak{M}} V^{(i)} \supseteq \bigcap_i \operatorname{Ker} \mathrm{ev}_{\mathbf{x}^{(i)}} = \operatorname{Ker} \mathrm{ev}_{\mathbf{y}}$ for $\mathbf{y} = \bigcup_i \mathbf{x}^{(i)}$. Since $\operatorname{Ker} \mathrm{ev}_x = \operatorname{Ker} \mathrm{ev}_{g \cdot x}$, we can replace $\mathbf{y}$ by some $\mathbf{x} \in X_n$ satisfying $\operatorname{Ker} \mathrm{ev}_{\mathbf{y}} = \operatorname{Ker} \mathrm{ev}_{\mathbf{x}}$.

Conversely, if $\operatorname{Ker} \mathrm{ev}_{\mathbf{x}} \subseteq \operatorname{Ann}_{\mathfrak{M}}(V)$, then the representation of $\mathfrak{M}$ on V factors through the semisimple Lie algebra $\bigoplus_{x \in \mathbf{x}} \mathfrak{g}^x$ and is therefore completely reducible. □

While Theorem 5.5 classifies all the irreducible finite-dimensional representations of an arbitrary equivariant map algebra, in case $\mathfrak{M}$ is not perfect it leaves open the possibility that not all irreducible finite-dimensional representations are evaluation representations. We see that $\mathfrak{M}$ has irreducible finite-dimensional representations that are not evaluation representations precisely when it has one-dimensional representations that are not evaluation representations. We therefore turn our attention to one-dimensional evaluation representations. Let

$$\tilde{X} = \{x \in X_{\mathrm{rat}} : [\mathfrak{g}^x, \mathfrak{g}^x] \neq \mathfrak{g}^x\}.$$

Note that $\tilde{X}$ is a Γ-invariant subset of X (i.e. $\tilde{X}$ is a union of Γ-orbits).

Lemma 5.12. *If* ev_Ψ *is (the isomorphism class of) a one-dimensional representation, then* $\operatorname{supp}\Psi \subseteq \tilde{X}$.

Proof. This follows easily from the fact that for $x \in X \setminus \tilde{X}$ we have that $\mathfrak{g}^x$ is perfect and thus the one-dimensional representations of $\mathfrak{g}^x$ are trivial. $\square$

Let

$$\mathfrak{M}^d = \{\alpha \in \mathfrak{M} : \alpha(x) \in [\mathfrak{g}^x, \mathfrak{g}^x] \ \forall\, x \in X_{\mathrm{rat}}\} = \{\alpha \in \mathfrak{M} : \alpha(x) \in [\mathfrak{g}^x, \mathfrak{g}^x] \ \forall\, x \in \tilde{X}\}.$$

Then it is easy to see that $[\mathfrak{M}, \mathfrak{M}] \subseteq \mathfrak{M}^d$. The proof of the following lemma is straightforward.

Lemma 5.13. *The Lie algebra* $\mathfrak{M}$ *is perfect if and only if* $\mathfrak{M}^d = [\mathfrak{M}, \mathfrak{M}]$ *and* $\tilde{X} = \emptyset$.

Now assume that $|\tilde{X}| < \infty$. Let $\mathbf{x}$ be a set of representatives of the Γ-orbits comprising $\tilde{X}$ and consider the composition

$$\mathfrak{M} \xrightarrow{\mathrm{ev}_{\mathbf{x}}} \bigoplus_{x \in \mathbf{x}} \mathfrak{g}^x \xrightarrow{\pi} \bigoplus_{x \in \mathbf{x}} \mathfrak{z}^x, \quad \mathfrak{z}^x := \mathfrak{g}^x/[\mathfrak{g}^x, \mathfrak{g}^x], \tag{5.6}$$

where the x-component of π is the canonical projection $\mathfrak{g}^x \to \mathfrak{g}^x/[\mathfrak{g}^x, \mathfrak{g}^x]$. If $\mathfrak{g}$ is reductive, then so is every $\mathfrak{g}^x$, and we can identify $\mathfrak{z}^x$ with the center $Z(\mathfrak{g}^x)$ of $\mathfrak{g}^x$. However, we will not assume that $\mathfrak{g}$ is reductive. The kernel of (5.6) is precisely $\mathfrak{M}^d$, and thus the composition factors through $\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}]$, yielding the following commutative diagram:

$$\begin{tikzcd} \mathfrak{M} \arrow[r, two heads, "\mathrm{ev}_{\mathbf{x}}"] \arrow[dr, two heads] & \bigoplus_{x \in \mathbf{x}} \mathfrak{g}^x \arrow[r, two heads, "\pi"] & \bigoplus_{x \in \mathbf{x}} \mathfrak{z}^x \\ & \mathfrak{M}/[\mathfrak{M}, \mathfrak{M}] \arrow[ur, two heads, "\gamma"'] & \end{tikzcd} \tag{5.7}$$

We then have an isomorphism of vector spaces,

$$(\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}])^* \cong (\ker\gamma)^* \oplus \left(\bigoplus_{x \in \mathbf{x}} \mathfrak{z}^x\right)^*. \tag{5.8}$$

Proposition 5.14. *If* $|\tilde{X}| < \infty$ *and* $\mathbf{x}$ *is a set of representatives of the* Γ*-orbits comprising* $\tilde{X}$*, then there is a natural identification*

$$\left(\bigoplus_{x \in \mathbf{x}} \mathfrak{z}^x\right)^* \cong \{\mathrm{ev}_\Psi : \Psi \in \mathcal{E},\ \dim \mathrm{ev}_\Psi = 1\}.$$

Proof. Choose $\lambda \in \left(\bigoplus_{x \in \mathbf{x}} \mathfrak{z}^x\right)^*$. To λ we associate the evaluation representation

$$\mathfrak{M} \xrightarrow{\mathrm{ev}_{\mathbf{x}}} \bigoplus_{x \in \mathbf{x}} \mathfrak{g}^x \xrightarrow{\pi} \bigoplus_{x \in \mathbf{x}} \mathfrak{z}^x \xrightarrow{\lambda} k.$$

By Lemma 5.12, this gives the desired bijective correspondence. $\square$

We can now refine Theorem 5.5 as follows.

Theorem 5.15. *Suppose Γ is a finite group acting on an affine scheme X and a finite-dimensional Lie algebra $\mathfrak{g}$, and assume that $|\tilde{X}| < \infty$. If γ is defined as in* (5.7)*, then the map*

$$(\lambda, \Psi) \mapsto \lambda \otimes \mathrm{ev}_\Psi, \quad \lambda \in (\ker \gamma)^*, \quad \Psi \in \mathcal{E}$$

is a bijection between $(\ker \gamma)^* \times \mathcal{E}$ and $\mathcal{S}$.

Proof. This follows from Theorem 5.5, (5.8) and Proposition 5.14. □

Corollary 5.16. *Assume $|\tilde{X}| < \infty$. Then $[\mathfrak{M}, \mathfrak{M}] = \mathfrak{M}^d$ if and only if all irreducible finite-dimensional representations are evaluation representations.*

Proof. By Theorem 5.15, all irreducible finite-dimensional representations are evaluation representations if and only if γ is injective (and hence an isomorphism, since it is surjective). Then the result follows from the commutative diagram (5.7) since the kernel of (5.6) is $\mathfrak{M}^d$. □

Remark 5.17. Note that if $\mathfrak{g}$ is perfect and Γ acts on X in such a way that there are only a finite number of points of X that have a non-trivial stabilizer, then $|\tilde{X}| < \infty$, and so the hypotheses of Theorem 5.15 are satisfied.

Remark 5.18. In Section 6.3 we will see that the Onsager algebra is an equivariant map algebra which is not perfect but for which γ is injective, and thus all irreducible finite-dimensional representations are nonetheless evaluation representations.

Having considered the case when all irreducible finite-dimensional representations are evaluation representations, we now examine the opposite situation: Equivariant map algebras for which there exist irreducible finite-dimensional representations that are not evaluation representations.

Proposition 5.19. *Suppose X is a Noetherian affine scheme and $\tilde{X}$ is infinite. Then $\mathfrak{M}$ has a one-dimensional representation that is not an evaluation representation.*

Proof. We first set up some notation for one-dimensional evaluation representations. Let $\mathbf{x} \in X_n$ and let $\rho_x : \mathfrak{g}^x \to \mathrm{End}_k(V_x)$, $x \in \mathbf{x}$, be representations such that $\mathrm{ev}_\mathbf{x}(\rho_x)_{x\in\mathbf{x}}$ is a one-dimensional representation. Necessarily $\dim V_x = 1$, say $V_x = kv_x$, so $V = \bigotimes_{x\in\mathbf{x}} V_x = k\mathbf{v}$ for $\mathbf{v} = \bigotimes_{x\in\mathbf{x}} v_x$. We can assume that all $\rho_x \neq 0$, whence $\mathbf{x} \subseteq \tilde{X}$. Let $\tilde{\rho}_x \in (\mathfrak{g}^x)^*$ be defined by $\rho_x(u)(v_x) = \tilde{\rho}_x(u)v_x$ for $u \in \mathfrak{g}^x$. For $\alpha = \sum_i u_i \otimes f_i \in \mathfrak{M}$ we then have

$$\big((\mathrm{ev}_\mathbf{x}(\rho_x)_{x\in\mathbf{x}})(\alpha)\big)(\mathbf{v}) = \left(\sum_{x\in\mathbf{x},\, i} \tilde{\rho}_x(u_i) f_i(x) \right) \mathbf{v}.$$

Suppose that there exist $\alpha \in \mathfrak{M}$ and $f \in A^\Gamma$ such that
(∗)

$$\{\alpha f^m : m \in \mathbb{N}\} \text{ is linearly independent and } \mathrm{Span}\{\alpha f^m : m \in \mathbb{N}\} \cap [\mathfrak{M}, \mathfrak{M}] = 0.$$

Then there exists $\lambda \in \mathfrak{M}^*$ such that $\lambda([\mathfrak{M}, \mathfrak{M}]) = 0$ and $\lambda(\alpha f^m) \in k$, $m \geq 1$, is a root of an irreducible rational polynomial p_m of degree m, for example $p_m(z) = z^m - 2$. Now suppose that the corresponding one-dimensional representation is an evaluation representation. Writing $\alpha = \sum_i u_i \otimes f_i$ with $u_i \in \mathfrak{g}$ and $f_i \in A$, we get from the equation above

$$\lambda(\alpha f^m) = \sum_{x \in \mathbf{x},\, i} \tilde{\rho}_x(u_i)(f_i f^m)(x) = \sum_{x \in \mathbf{x},\, i} \tilde{\rho}_x(u_i) f_i(x) f(x)^m.$$

But this is a contradiction since the elements $\tilde{\rho}_x(u_i) f_i(x) f(x)^m \in k$ all lie in the $\mathbb{Q}$-subalgebra of k generated by the finitely many elements $\tilde{\rho}_x(u_i), f_i(x), f(x)$ of k, while the elements $\lambda(\alpha f^m)$, $m \in \mathbb{N}$, do not lie in such a subalgebra.

We will now construct $\alpha \in \mathfrak{M}$ and $f \in A^\Gamma$ satisfying $(*)$. For a subgroup H of Γ, let

$$X_H = \{x \in X_{\mathrm{rat}} : \Gamma_x = H\}.$$

Since

$$\tilde{X} = \bigcup_{H \,:\, \mathfrak{g}^H \neq [\mathfrak{g}^H, \mathfrak{g}^H]} X_H,$$

there exists a subgroup H of Γ, with $\mathfrak{g}^H \neq [\mathfrak{g}^H, \mathfrak{g}^H]$, such that X_H is infinite. Observe that $X_H = X^H \setminus \bigcup_{K \supsetneq H} X^K$ is open in the closed subset X^H and that X^H is a Noetherian affine scheme since X is. Because X^H has only finitely many irreducible components, there exists an irreducible component Y of X^H such that $\tilde{Y} = Y \cap X_H$ is infinite.

Let $R : A_X^\Gamma \to A_Y$ be the restriction map. Choose an infinite set $\{y_1, y_2, \dots\}$ of points of Y, no two of which are in the same Γ-orbit. It follows as in the proof of Corollary 4.6 that for all $j \in \mathbb{N}$ there exists $f_j \in A_X^\Gamma$ such that $f_j(y_j) \neq 0$ and $f_j(y_i) = 0$ for $i < j$. Since the set $\{R(f_j) : j \in \mathbb{N}\}$ is linearly independent, the image of R is infinite-dimensional. Because A_X^Γ is finitely generated, so is the image of R. Therefore, by the Noether normalization lemma, this image contains an element f_Y such that the set $\{1, f_Y, f_Y^2, \dots,\}$ is linearly independent. Choose $f \in R^{-1}(f_Y)$. It follows that $\{1, f, f^2, \dots,\}$ is also linearly independent. Since $\tilde{Y}$ is open in the irreducible Y we can choose $\tilde{y} \in \tilde{Y}$ such that $f(\tilde{y}) \neq 0$.

Let $\{u_i\}_{i=1}^l$ be a basis of $[\mathfrak{g}^H, \mathfrak{g}^H]$, and complete it to a basis $\{u_i\}_{i=1}^m$ of $\mathfrak{g}^H$. Then complete this to a basis $\{u_i\}_{i=1}^n$ of $\mathfrak{g}$. Again by Corollary 4.6 there exists $\alpha \in \mathfrak{M}$ such that $\alpha(\tilde{y}) = u_m$. Observe that $\alpha_Y = \alpha|_Y$ can be written in the form $\alpha_Y = \sum_{i=1}^m u_i \otimes f_i$ for some $f_i \in A_Y$. Since $\alpha(\tilde{y}) = u_m$, we have $f_m(\tilde{y}) = 1$. Therefore $f_m(y) \neq 0$ for all y in some dense subset of Y. Now suppose

$$\sum_{j=0}^\infty d_j \alpha f^j \in [\mathfrak{M}, \mathfrak{M}]$$

for some $d_j \in k$ with $d_j = 0$ for all but finitely many j. We then have

$$\left.\left(\sum_{j=0}^\infty d_j \alpha f^j\right)\right|_Y = \alpha_Y \sum_{j=0}^\infty d_j f_Y^j = \left(\sum_{i=1}^m u_i \otimes f_i\right) \sum_{j=0}^\infty d_j f_Y^j = \sum_{i=1}^m u_i \otimes \left(f_i \sum_{j=0}^\infty d_j f_Y^j\right).$$

Since $f_m(y) \neq 0$ for y in a dense subset of Y, we must have $\sum_{j=0}^\infty d_j f_Y^j = 0$. Because $\{1, f_Y, f_Y^2, \dots\}$ is linearly independent, we must have $d_j = 0$ for all j. Therefore

$$\mathrm{Span}\{\alpha f^m : m \in \mathbb{N}\} \cap [\mathfrak{M}, \mathfrak{M}] = 0.$$

Now observe that the preceding argument also shows that $\{\alpha, \alpha f, \alpha f^2, \dots\}$ is linearly independent: Suppose $\sum_{j=1}^\infty c_j \alpha f^j = 0$ for some $c_j \in k$ with $c_j = 0$ for all but finitely many j. Then, since $0 \in [\mathfrak{M}, \mathfrak{M}]$, all $c_j = 0$ by what we have just shown. Thus $(*)$ holds, finishing the proof of the proposition. $\square$

Example 5.20. We give an example in which the assumptions of Proposition 5.19 are fulfilled. Let $\Gamma = \{1, \sigma\}$ be the group of order two acting on $\mathfrak{g} = \mathfrak{sl}_2(k)$ by the Chevalley involution with respect to some $\mathfrak{sl}_2$-triple and let X be the affine space k^2 with σ acting on X by fixing the first coordinate of points in X while multiplying the second by -1. For points $(x_1, x_2) \in X$ with $x_2 \neq 0$, the isotropy subalgebra $\mathfrak{g}^x = \mathfrak{g}$, while for $(x_1, 0) \in X$, the subalgebra $\mathfrak{g}^x$ is the fixed point subalgebra of σ, which is one-dimensional. Therefore $\tilde{X} = \{(x_1, 0) \in X\}$ is infinite.

Proposition 5.19 says that when X is an affine variety, a necessary condition for all irreducible finite-dimensional representations to be evaluation representations is that $\tilde{X}$ be finite. We now show that this condition is not sufficient.

Example 5.21. Let $\mathfrak{g} = \mathfrak{sl}_2(k)$ and

$$X = Z(y^2 - x^3) = \{(y, x) : y^2 = x^3\} \subseteq k^2,$$

an affine variety. Then $A = k[y, x]/(y^2 - x^3)$. Let $\Gamma = \langle \sigma \rangle = \mathbb{Z}_2$ act on k^2 by $\sigma \cdot y = -y$ and $\sigma \cdot x = x$. Since this action fixes $y^2 - x^3$, we have an induced action of Γ on X and the only fixed point is the origin. In particular, $\tilde{X}$ only contains the origin and thus is finite. We let σ act on $\mathfrak{g}$ by a Chevalley involution. We have

$$A_1 = yk[y^2, x]/(y^2 - x^3),$$

and so

$$A_1^2 = y^2 k[y^2, x]/(y^2 - x^3) \cong x^3 k[x],$$

while

$$A_0 = A^\Gamma = k[y^2, x]/(y^2 - x^3) \cong k[x].$$

Recall that $\mathfrak{g}_0$ is one-dimensional. Then

$$[\mathfrak{M}, \mathfrak{M}] = ([\mathfrak{g}_0, \mathfrak{g}_0] \otimes A_0 + \mathfrak{g}_0 \otimes A_1^2) \oplus (\mathfrak{g}_1 \otimes A_1) = \big(\mathfrak{g}_0 \otimes x^3 k[x]\big) \oplus (\mathfrak{g}_1 \otimes A_1),$$

while

$$\mathfrak{M}^d = (\mathfrak{g}_0 \otimes xk[x]) \oplus (\mathfrak{g}_1 \otimes A_1).$$

Therefore

$$\mathfrak{M}^d / [\mathfrak{M}, \mathfrak{M}] \cong xk[x]/(x^3),$$

and by Theorem 5.15, $\mathfrak{M}$ has irreducible finite-dimensional representations that are not evaluation representations.

6. Applications

In this section we use our classification to describe the irreducible finite-dimensional representations of certain equivariant map algebras. The classification of these representations for the multiloop, tetrahedron, or Onsager algebra $\mathcal{O}(\mathfrak{sl}_2)$ by the results obtained in Section 5 provide a simplified and unified interpretation of results previously obtained. For example, the classification of the irreducible finite-dimensional representations of $L^\sigma(\mathfrak{g})$ (as found in [CP01] and [CFS08]) via Drinfeld polynomials requires two distinct treatments for the untwisted ($\sigma = \mathrm{Id}$) and twisted ($\sigma \neq \mathrm{Id}$) cases – for these twisted cases, the twisted loops $L^\sigma(\mathfrak{g})$ with $\mathfrak{g}$ of type A_{2n} require special attention. The classification resulting from our approach, however, is uniform. As we will see, the identification of isomorphism classes of representations with equivariant maps $X_{\mathrm{rat}} \to \mathcal{R}_X$ also provides a simple explanation for many of the technical conditions appearing in previous classifications.

We first note that from Remark 5.10 we immediately obtain the classification of irreducible finite-dimensional representations of the current algebras (Example 2.4), of the untwisted loop and multiloop algebras (Example 2.5), and of the n-point algebras $M(X, \mathfrak{g})$, where $X = \mathbb{P}^1 \setminus \{c_1, \dots, c_n\}$. This includes the tetrahedron algebra, which is isomorphic to the three-point $\mathfrak{sl}_2$ loop algebra $\mathfrak{sl}_2 \otimes \mathbb{C}\left[t, t^{-1}, (t-1)^{-1}\right]$ (Example 2.6), and we recover the classification found in [Har07]. In particular, we easily recover in all of these cases the fact that for $\mathbf{x} = \{x_1, \dots, x_l\} \subseteq X$ and irreducible representations $V_1, \dots, V_l$ of $\mathfrak{g}$, the evaluation representation $\mathrm{ev}_{\mathbf{x}}(\otimes V_i)$ is irreducible if and only if $x_i \neq x_j$ for $i \neq j$.

6.1. **Multiloop algebras.** If $\mathfrak{M} = M(\mathfrak{g}, \sigma_1, \dots, \sigma_n, m_1, \dots, m_n)$ is a multiloop algebra (Example 3.12), then $\mathfrak{M}$ is perfect ($\mathfrak{M}$ is an iterated loop algebra; see [ABP06, Lemma 4.9]). Therefore, by Corollary 5.8 we have the following classification:

Corollary 6.1. *The map $\mathcal{E} \to \mathcal{S}$, $\Psi \mapsto \mathrm{ev}_\Psi$, is a bijection. In particular, all irreducible finite-dimensional representations are evaluation representations.*

The irreducible finite-dimensional representations of an arbitrary multiloop algebra have been discussed in [Bat04] and [Lau]. With Corollary 6.1 we recover the recent results in [Lau], which subsume all previous classifications of irreducible finite-dimensional representations of loop algebras. We note that these previous classifications involved some rather complicated algebraic conditions on points of evaluation (see, for example, [Lau, Theorem 5.7]). However, in the approach of Theorem 5.5, such conditions are not necessary. In fact, we see that the presence of these algebraic conditions arises from the description of evaluation representations in terms of individual points rather than as equivariant maps (i.e. elements of $\mathcal{E}$). For instance, if $\mathfrak{M} = M(\mathfrak{g}, \sigma_1, \dots, \sigma_n, m_1, \dots, m_n)$ is an arbitrary multiloop algebra and $\Psi \in \mathcal{E}$, then ev_Ψ is the isomorphism class $\mathrm{ev}_{\mathbf{x}}(\Psi(x_i))_{i=1}^l$, where $\mathbf{x} = \{x_1, \dots, x_l\} \in X_l$ contains one element from each Γ-orbit in $\operatorname{supp} \Psi$ and this class is independent of the choice of $\mathbf{x}$ (Definition 4.7). It is immediate that such an $\mathbf{x}$ must satisfy the condition that $x_1^m, \dots, x_l^m$ (where $x_i^m = (x_{i1}^{m_1}, \dots, x_{in}^{m_n})$ for $x_i = (x_{i1}, \dots, x_{in})$) are pairwise distinct in $(k^\times)^n$. We therefore recover the conditions on the points x_i found in [Lau, Theorem 5.7] which are necessary and sufficient for $\mathrm{ev}_{x_1,\dots,x_l}(\otimes V_i)$ to be irreducible. The other conditions found there are similarly explained.

6.2. **Connections to Drinfeld polynomials.** In [CP01], [CM04], [CFS08] and [Sen10a], the isomorphism classes of irreducible finite-dimensional representations of loop algebras $L(\mathfrak{g})$, $L^\sigma(\mathfrak{g})$ are parameterized by certain collections of polynomials, sometimes referred to as *Drinfeld polynomials*. Here we explain the relationship between this parametrization and ours. For better comparison with the existing literature, we assume in this subsection that $k = \mathbb{C}$.

Denote by $\mathcal{P}$ the set of of n-tuples of polynomials with constant term 1:

$$\mathcal{P} = \{\boldsymbol{\pi} = (\pi_1(u), \dots, \pi_n(u)) : \pi_i \in \mathbb{C}[u], \ \ \pi_i(0) = 1\}.$$

Then the set $\mathcal{P}$ is in bijective correspondence with the isomorphism classes of irreducible finite-dimensional representations of $L(\mathfrak{g})$ ([CP01, Proposition 2.1]). To the element $\boldsymbol{\pi} \in \mathcal{P}$ we associate an irreducible representation $V(\boldsymbol{\pi})$ (the construction of $V(\boldsymbol{\pi})$ is given in [CP01]). We describe this correspondence.

Fix a simple finite-dimensional Lie algebra $\mathfrak{g}$, denote by n its rank, and fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ with Cartan subalgebra $\mathfrak{h} = \bigoplus_{i=1}^n \mathbb{C}h_i \subseteq \mathfrak{g}$ and weight lattice $P = \bigoplus_{i=1}^n \mathbb{Z}\omega_i$, with fundamental weights $\{\omega_i\}_{i=1}^n$, $\omega_i(h_j) = \delta_{ij}$.

Let $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n) \in \mathcal{P}$ and $\{x_i\}_{i=1}^l = \bigcup_{j=1}^n \left\{z \in \mathbb{C}^\times : \pi_j(z^{-1}) = 0\right\}$. Then each π_j can be written uniquely in the form

$$\pi_j(u) = \prod_{i=1}^{l} (1 - x_i u)^{N_{ij}}, \quad N_{ij} \in \mathbb{N}.$$

Let $\mathbf{x} = \{x_1, \dots, x_l\}$. For $i = 1, \dots, l$, define $\lambda_i \in P^+$ by $\lambda_i(h_j) = N_{ij}$, and let $\rho_i : \mathfrak{g} \to \operatorname{End}_k(V(\lambda_i))$ be the corresponding irreducible finite-dimensional representation of $\mathfrak{g}$. Then $V(\boldsymbol{\pi})$ is isomorphic as an $L(\mathfrak{g})$-module to the evaluation representation

$$\operatorname{ev}_{\mathbf{x}}(\rho_i)_{i=1}^l : L(\mathfrak{g}) \xrightarrow{\operatorname{ev}_{\mathbf{x}}} \mathfrak{g}^{\oplus l} \xrightarrow{\bigotimes_{i=1}^l \rho_i} \operatorname{End}_k \left(\bigotimes_{i=1}^l V_i \right).$$

To produce an element $\boldsymbol{\pi} \in \mathcal{P}$ from an irreducible representation V of $L(\mathfrak{g})$, we first find an evaluation representation $\operatorname{ev}_{\mathbf{x}}(\rho_i)_{i=1}^l : L(\mathfrak{g}) \longrightarrow \operatorname{End}_k(\bigotimes_{i=1}^l V(\lambda_i))$ isomorphic to V ([Rao93, Theorem 2.14] or Corollary 6.1). Next, for $i = 1, \dots, l$, we define elements $\boldsymbol{\pi}_{\lambda_i, x_i} \in \mathcal{P}$ by

$$\boldsymbol{\pi}_{\lambda_i, x_i} = \left((1 - x_i u)^{\lambda_i(h_1)}, \dots, (1 - x_i u)^{\lambda_i(h_n)} \right)$$

and define $\boldsymbol{\pi} = \prod_{i=1}^l \boldsymbol{\pi}_{\lambda_i, x_i}$, where multiplication of n-tuples of polynomials occurs componentwise.

Given an element $\boldsymbol{\pi} \in \mathcal{P}$, we can uniquely decompose $\boldsymbol{\pi} = \prod_{i=1}^l \boldsymbol{\pi}_{\lambda_i, x_i}$, $x_i \neq x_j$, and we define

$$\Psi_{\boldsymbol{\pi}} := \{x_i \mapsto [V(\lambda_i)]\} \in \mathcal{E}.$$

Then $V(\boldsymbol{\pi})$ is a representative of $\operatorname{ev}_{\Psi_{\boldsymbol{\pi}}}$.

In [CFS08], there is a similar parametrization of the irreducible finite-dimensional representations of $L^\sigma(\mathfrak{g})$, where σ is a non-trivial diagram automorphism of $\mathfrak{g}$, but in this case the bijective correspondence is between isomorphism classes of irreducible finite-dimensional $L^\sigma(\mathfrak{g})$-modules and the set $\mathcal{P}^\sigma$ of m-tuples of polynomials $\boldsymbol{\pi}^\sigma = (\pi_1, \dots, \pi_m)$, $\pi_i(0) = 1$, where m is the rank of the fixed-point subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$. One feature of this classification is the fact that every irreducible finite-dimensional $L^\sigma(\mathfrak{g})$-module is the restriction of an irreducible finite-dimensional $L(\mathfrak{g})$-module (see [CFS08, Theorem 2]). This fact follows immediately from Proposition 4.19 once we note that in the setup of multiloop algebras, the action of Γ on $X = \mathbb{C}^\times$ is via multiplication by roots of unity and hence is free. Thus $\mathfrak{g}^x = \mathfrak{g}^{\Gamma_x} = \mathfrak{g}^{\{\mathrm{Id}\}} = \mathfrak{g}$ for all $x \in X$. Of course, the approach of the current paper yields an enumeration by elements of $\mathcal{E}$. The induced identification of $\mathcal{E}$ with $\mathcal{P}^\sigma$ is somewhat technical and will not be described here, but can found in [Sen10b].

6.3. **The generalized Onsager algebra.** Our results also provide a classification of the irreducible finite-dimensional representations of the generalized Onsager algebra $\mathcal{O}(\mathfrak{g})$ introduced in Example 3.9 (in fact, for the more general equivariant map algebra of Example 3.10). For $\mathfrak{g} \neq \mathfrak{sl}_2$ this classification was previously unknown.

Proposition 6.2. *Let $\mathfrak{g}$ be a simple Lie algebra, $X = \operatorname{Spec} k[t^{\pm 1}]$, and $\Gamma = \{1, \sigma\}$ be a group of order two. Suppose σ acts on X by $\sigma \cdot x = x^{-1}$, $x \in X$, and on $\mathfrak{g}$*

by an automorphism of order two. Then the map $\mathcal{E} \to \mathcal{S}$, $\Psi \mapsto \mathrm{ev}_\Psi$, is a bijection. In particular, all irreducible finite-dimensional representations are evaluation representations. In particular, this is true for the generalized Onsager algebra $\mathcal{O}(\mathfrak{g})$.

Proof. Recall that $\mathfrak{g}_0 = \mathfrak{g}^\Gamma$ is either semisimple or has a one-dimensional center. In the case when $\mathfrak{g}_0$ is semisimple, the result follows from Corollary 5.8 and (3.6). We thus assume that $Z(\mathfrak{g}_0) \cong \mathfrak{g}_0/[\mathfrak{g}_0, \mathfrak{g}_0]$ is one-dimensional. By (3.6), we have $\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}] \cong A_0/A_1^2$, a Lie algebra with trivial Lie bracket. Now, $A_0 = k[t + t^{-1}]$ and $A_1 = (t - t^{-1})A_0$. Thus, setting $z = t + t^{-1}$, we have

$$A_0/A_1^2 \cong k[z]/\langle z^2 - 4\rangle,$$

which is a two-dimensional vector space. The points 1 and -1 are each Γ-fixed points, and so we must take $\mathbf{x} = \{\pm 1\}$ in (5.7). Therefore $\bigoplus_{x \in \mathbf{x}} Z(\mathfrak{g}^x)$ is also a two-dimensional vector space, and so the map γ in (5.7) is injective (since it is surjective). The result then follows from Theorem 5.15. $\square$

In the special case $\mathfrak{g} = \mathfrak{sl}_2$, $k = \mathbb{C}$, the irreducible finite-dimensional representations of $\mathcal{O}(\mathfrak{sl}_2)$ were described in [DR00] as follows. Let $\{e, h, f\}$ be an $\mathfrak{sl}_2$-triple and define $X = (e + f) \otimes 1$ and $Y = e \otimes t + f \otimes t^{-1}$. Then $\mathcal{O}(\mathfrak{sl}_2)$ is generated by X, Y. Furthermore, if V is an irreducible finite-dimensional representation of $\mathcal{O}(\mathfrak{sl}_2)$, then X and Y are diagonalizable on V, and there exists an integer $d \geq 0$ and scalars $\gamma, \gamma^* \in k$ such that the set of distinct eigenvalues of X (resp. Y) on V is $\{d - 2i + \gamma : 0 \leq i \leq d\}$ (resp. $\{d - 2i + \gamma^* : 0 \leq i \leq d\}$) [Har07, Corollary 2.7]. The ordered pair (γ, γ^*) is called the *type* of V. Replacing X, Y by $X - \gamma I, Y - \gamma^* I$ (in the universal enveloping algebra $U(\mathcal{O}(\mathfrak{sl}_2))$ of $\mathcal{O}(\mathfrak{sl}_2)$) the type becomes $(0, 0)$.

Let $\mathrm{ev}_{x_1} V_1, \ldots, \mathrm{ev}_{x_n} V_n$ denote a finite sequence of evaluation modules for $\mathcal{O}(\mathfrak{sl}_2)$, and V the evaluation module $\mathrm{ev}_{x_1} V_1 \otimes \cdots \otimes \mathrm{ev}_{x_n} V_n$. Any module that can be obtained from V by permuting the order of the factors and replacing any number of the x_i's with their multiplicative inverses will be called *equivalent* to V. The classification of irreducible finite-dimensional $\mathcal{O}(\mathfrak{sl}_2)$-modules of type $(0, 0)$ is described in [DR00] as follows.

Proposition 6.3.

(1) [DR00, Theorem 6] *Every non-trivial irreducible finite-dimensional $\mathcal{O}(\mathfrak{sl}_2)$-module of type $(0, 0)$ is isomorphic to a tensor product of evaluation modules.*

(2) [DR00, Proposition 5] *Let $\mathrm{ev}_{x_1} V_1, \ldots, \mathrm{ev}_{x_n} V_n$ denote a finite sequence of evaluation modules for $\mathcal{O}(\mathfrak{sl}_2)$, and consider the $\mathcal{O}(\mathfrak{sl}_2)$-module $\mathrm{ev}_{x_1} V_1 \otimes \cdots \otimes \mathrm{ev}_{x_n} V_n$. This module is irreducible if and only if $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$ are pairwise distinct.*

(3) [DR00, Proposition 5] *Let U and V denote tensor products of finitely many evaluation modules for $\mathcal{O}(\mathfrak{sl}_2)$. Assume each of U, V is irreducible as an $\mathcal{O}(\mathfrak{sl}_2)$-module. Then the $\mathcal{O}(\mathfrak{sl}_2)$-modules U and V are isomorphic if and only if they are equivalent.*

These results are immediate consequences of Proposition 6.2. The type $(0, 0)$ representations are precisely the evaluation representations ev_Ψ such that $\{\pm 1\} \cap \operatorname{supp} \Psi = \emptyset$. We note that under the previous definition of evaluation representations appearing in the literature (see Remark 4.8), *only* type $(0, 0)$ representations are evaluation representations. However, using Definition 4.7, *all* irreducible finite-dimensional representations (of arbitrary type) are evaluation representations. The key is that we allow one-dimensional representations of $\mathfrak{g}^x$ for $x \in \{\pm 1\}$, where $\mathfrak{g}^x$ is one-dimensional. Thus our definition allows for a more uniform description of the representations. The condition that the points $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$ be pairwise distinct also follows automatically as in the case of multiloop algebras.

6.4. **A non-abelian example.** Let $\mathfrak{g} = \mathfrak{so}_8$, $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $\Gamma = S_3$ as in Example 3.13, and let $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$. Since $\mathfrak{M}$ is perfect, by Corollary 5.8 all irreducible finite-dimensional representations of $\mathfrak{M}$ are evaluation representations and these are naturally enumerated by $\mathcal{E}$. We identify $\Gamma = S_3$ with the permutations of the set $\{0, 1, \infty\}$ and use the usual cycle notation for permutations. For instance, $(0\,\infty)$ denotes the permutation given by $0 \mapsto \infty$, $\infty \mapsto 0$, $1 \mapsto 1$. A straightforward computation shows that the points with non-trivial stabilizer are listed in the table below (note that $\{0, 1, \infty\} \not\in X$). Each $\mathfrak{g}^x$ is the fixed point algebra of a diagram automorphism of $\mathfrak{g}$ and thus a simple Lie algebra of type B_3 or G_2.

x	Γ_x	Type of $\mathfrak{g}^x$
-1	$\{\mathrm{Id}, (0\,\infty)\} \cong \mathbb{Z}_2$	B_3
2	$\{\mathrm{Id}, (1\,\infty)\} \cong \mathbb{Z}_2$	B_3
$\frac{1}{2}$	$\{\mathrm{Id}, (0\,1)\} \cong \mathbb{Z}_2$	B_3
$e^{\pm\pi i/3}$	$\{\mathrm{Id}, (0\,1\,\infty), (0\,\infty\,1)\} \cong \mathbb{Z}_3$	G_2

Furthermore, the sets $\{-1, 2, \frac{1}{2}\}$ and $\{e^{\pi i/3}, e^{-\pi i/3}\}$ are Γ-orbits. We thus see that the representation theory of $\mathfrak{M}$ is quite rich. Elements of $\mathcal{E}$ can assign to the three-element orbit (the isomorphism class of) any irreducible representation of the simple Lie algebra of type B_3, to the two-element orbit any irreducible representation of the simple Lie algebra of type G_2 and to any of the other (six-element) orbits, any irreducible representation of the simple Lie algebra $\mathfrak{g}$ of type D_4.

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