A CATEGORY THEORETIC APPROACH TO FORMAL GROUP LAWS

YANANAN WANG

Contents

1. Introduction

A formal group law is a formal power series that behaves in many ways like the product of a Lie group. In some sense, formal group laws are intermediate between Lie groups and Lie algebras. The theory of formal group laws has found a great number of incredible applications in algebraic geometry, number theory, and algebraic topology. We refer the reader to the book [6] for further details on the original applications of formal groups.

More recently, formal group laws have been studied in their connection to algebraic oriented cohomology theory. Through the formal group algebras introduced in [3], it is possible to compute the cohomology rings of complete flag varieties for arbitrary oriented cohomology theories. These ideas have connections to Hecke algebras (see [7]) and the geometry of Steinberg varieties (see [8]).

The purpose of this paper is to serve as an introduction to the subject of formal group laws accessible to undergraduate students. We introduce some of the main definitions and results, assuming minimal background knowledge. We take a category theoretic point of view, introducing the category of formal groups, in addition to discussing how one can view formal group laws themselves as functors (with homomorphisms of formal group laws being natural transformations).

The structure of this document is as follows. In the Section 2 we recall some basic concepts in ring theory, including the important concept of formal power series. In Section 3, we introduce the concept of formal group laws, illustrating the definitions with a number of important examples. Then, in Section 4, we recall the basics of category theory that will be needed in this paper. In Section 5, we discuss the category of formal group laws. We then explain how formal group laws can be viewed as functors and give a treatment of change of base rings. In the majority of the paper, we restrict our attention to one-dimensional formal group laws. However, in Section 6 we discuss how the concepts seen here can be generalized to higher-dimensional formal group laws.

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2. Background

2.1. Rings and modules.

Definition 2.1 (Module over a ring R)**.** Suppose that R is a ring and 1_R is its multiplicative identity. A left R-module M consists of an abelian group $(M, +)$ and an operation $\cdot : R \times M \to M$ such that for all $r, s \in R$ and $x, y \in M$, we have:

$$r \cdot (x + y) = r \cdot x + r \cdot y$$
$$(r + s) \cdot x = r \cdot x + s \cdot x$$
$$(r \cdot s) \cdot x = r \cdot (s \cdot x)$$
$$1_R \cdot x = x$$

A right R-module M or M_R is defined similarly, except that the ring acts on the right.

Definition 2.2 (Submodule over a ring R)**.** Suppose M is a left R-module and N is a subgroup of M. Then N is a submodule (or R-submodule, to be more explicit) if, for any $n \in N$ and any $r \in R$, the product $r \cdot n$ is in N (or $n \cdot r$ for a right module).

Definition 2.3 (R-algebra)**.** Let R be a commutative ring. An associative R-algebra is a ring A that is an R-module (left, say) such that

$$r(ab) = (ra)b = a(rb)$$

for all $r \in R,\ a, b \in A$. Furthermore, A is assumed to be unital, which is to say it contains element 1 such that

$$1x = x = x1$$

for all $x \in A$.

In addition, A is called a commutative R-algebra if it satisfies

$$ab = ba$$

for all $a, b \in A$

We assume that all R-algebras are commutative throughout this paper.

Definition 2.4 (Subalgebra)**.** Let A be an R-algebra. A *subalgebra* of A is an R-submodule of A that is closed under multiplication and contains the identity element $1 \in A$.

Definition 2.5 (Ideal of an R-algebra)**.** A subset I of an commutative R-algebra A is an *ideal* if for every $x, y \in I$, $z \in A$ and $c \in R$, we have the following three statements:

$$\begin{aligned} x + y \in I &\text{ (I is closed under addition)}, \\ cx \in I &\text{ (I is closed under scalar multiplication)}, \\ z \cdot x \in I &\text{ (I is closed under multiplication by arbitrary elements)}. \end{aligned}$$

Definition 2.6 (Homomorphism of R-algebras)**.** A *homomorphism* between two R-algebras, A and B, is a map $F : A \to B$ such that for all $r \in R$ and $x, y \in A$ we have:

$$\begin{aligned} F(rx) &= rF(x), \\ F(x + y) &= F(x) + F(y), \\ F(xy) &= F(x)F(y). \end{aligned}$$

Definition 2.7 (Nilpotent)**.** An element x of a ring is called *nilpotent* if there exists some positive integer n such that $x^n = 0$.

Definition 2.8 (Nilradical)**.** The *nilradical* of a commutative ring A, denoted by $\mathrm{Nil}(A)$, is the set of all nilpotent elements in the ring.

Lemma 2.9. *Let A be a commutative ring. Then* $\mathrm{Nil}(A)$ *is an ideal of A.*

Proof. To prove $\mathrm{Nil}(A)$ is an ideal, we first show it is closed under addition. If $a, b \in \mathrm{Nil}(A)$, then there exist integers n, m such that $a^n = 0$ and $b^m = 0$. We then claim $(a + b)^{n+m} = 0$. To see this, we use the binomial expansion of $(a + b)^{n+m} = \sum_{k=0}^{n+m} a^{n+m-k} b^k$. Note that if $k \geq m$, then $b^k = 0$, so the previous sum is equal to $\sum_{k=0}^{m-1} a^{n+m-k} b^k$. However, for $k < m$, we have $n + m - k = n + (m - k) \geq n$, so $a^{n+m-k} = 0$ when $k < m$. Hence, we have $(a + b)^{n+m} = 0$.

Also, it remains to show that if $a \in \mathrm{Nil}(A)$ and $c \in A$, then $cx = xc \in \mathrm{Nil}(A)$. Suppose $a^n = 0$ for some integer n, then we have $(ca)^n = c^n a^n = c^n \cdot 0 = 0$. Hence, we have $cx \in \mathrm{Nil}(A)$.

Therefore, we have proved that $\mathrm{Nil}(A)$ is an ideal of A. □

Lemma 2.10. *If $f : S_1 \to S_2$ is a homomorphism of R-algebras, then $f(\mathrm{Nil}(S_1)) \subseteq \mathrm{Nil}(S_2)$.*

Proof. If $x \in f(\mathrm{Nil}(S_1))$, we want to show that $x \in \mathrm{Nil}(S_2)$, which means $x^m = 0$ for some integer m.

Since $x \in f(\mathrm{Nil}(S_1))$ and f is a homomorphism, there exists some $y \in \mathrm{Nil}(S_1)$ such that $f(y) = x$. Hence, we have $y \in S_1$.

Since $y \in \mathrm{Nil}(S_1)$, there exists a positive integer m such that $y^m = 0$. Then we have

$$x^m = f^m(y) = f(y^m) = 0$$

Hence, we have showed that $x^m = 0$ for some integer m. Therefore, we have proved the lemma as required. □

2.2. **Formal Power Series.** This section gives the fundamental definition of formal power series and some properties required for the later sections.

Definition 2.11 (The ring of formal power series)**.** The ring of formal power series in t with coefficients in R is denoted by $R[\![t]\!]$, and is defined as follows. The elements of $R[\![t]\!]$ are infinite expressions of the form

$$\sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots$$

in which $a_n \in R$ for all $n \in \mathbb{N}$.

In addition, the ring of formal power series in finitely many variables $t_1, t_2, t_3, \cdots, t_m$ with coefficients in R is denoted by $R[\![t_1, t_2, t_3, \cdots, t_m]\!]$, and is defined as follows. The elements of $R[\![t_1, t_2, t_3, \cdots, t_m]\!]$ are infinite expressions of the forms

$$\sum_{n_1,\ldots,n_m=0}^{\infty} a_{n_1,n_2,n_3,\cdots,n_m} t_1^{n_1} t_2^{n_2} t_3^{n_3} \cdots t_m^{n_m}$$

in which all the coefficients in $\mathbb{R}$.

The addition of two formal power series under $R[\![t]\!]$ is defined as

$$\sum_{i=0}^{\infty} a_i t^i \times \sum_{j=0}^{\infty} b_j t^j = \sum_{i,j=0}^{\infty} (a_i + b_j) t^{i+j}.$$

The multiplication of two formal power series under $R[\![t]\!]$ is defined as

$$\sum_{i=0}^{\infty} a_i t^i \times \sum_{j=0}^{\infty} b_j t^j = \sum_{i,j=0}^{\infty} a_i b_j t^{i+j}.$$

Addition and multiplication of formal power series in several variables is defined similarly.

Proposition 2.12. *Let $f(t) \in R[\![t]\!]$ be a formal series without constant term. If the constant term of the formal power series $f^{'}(t)$ is invertible in R, then there exist a unique power series $g(t) \in R[\![t]\!]$ such that $f(g(t)) = t = g(f(t))$.*

(For the proof of this proposition, we refer the reader to [1, Chapter IV - VII])

Definition 2.13 (Geometric series)**.** As a special form of power series, the geometric series is a series with a constant ratio between successive terms, and can be written as

$$a + at + at^2 + at^3 + \cdots.$$

When we substitute r for t, where r is a real number with $|r| < 1$, the geometric series converges and its sum is $\frac{1}{1-r}$ because

$$1 + r + r^2 + r^3 + \cdots = \lim_{n\to\infty} (1 + r + r^2 + r^3 + \cdots + r^n) = \lim_{n\to\infty} \left(\frac{1 - r^{n+1}}{1 - r} \right).$$

Thus, we will sometimes use the notation $\frac{1}{1-t}$ for the formal power series $1 + t + t^2 + \cdots$.

3. Formal Group Laws

In this present section, we will introduce the fundamental definitions and properties of formal group laws. We will focus on one-dimensional FGLs in this section, and treat the higher dimensional case in Section 6.

Definition 3.1 (One-Dimensional Formal Group Law)**.** A one-dimensional commutative formal group law is a pair (R, F), where R is a commutative ring, called the coefficient ring, and $F = F(u, v) \in R[\![u, v]\!]$ is a power series satisfying the following axioms:

FGL1: $F(u, 0) = F(0, u) = u$. (*Existence of an identity*)
FGL2: $F(u, v) = F(v, u)$. (*Commutativity*)
FGL3: $F(u, F(v, w)) = F(F(u, v), w)$. (*Associativity*)

Remark 3.2. Let $F(u, v) = \sum_{i,j=0} a_{ij}u^iv^j \in R[\![u, v]\!]$ be a formal group law. Since $u = F(u, 0) = a_{00} + a_{10}u + a_{20}u^2 + \cdots$, we have $a_{00} = 0, a_{i0} = 0$ for all $i > 1$. Similarly, we have $a_{0j} = 0$ for all $j > 1$. Therefore, any formal group law is of the form

$$F(u, v) = u + v + \sum_{i,j>1} a_{ij}u^iv^j.$$

The expression $F(F(u, v), w)$ means that in $F(u, v)$ we replace u by $F(u, v)$ and replace v by w, then expand to get an element in $R[\![u, v, w]\!]$. By Remark 3.2, there is no constant term in $F(u, v)$, so the substitution is well-defined.

Remark 3.3. We frequently use the notation $u +_F v := F(u, v)$, so the axioms for formal group law can be expressed as follows:

FGL1: $u +_F 0 = 0 +_F u = u$. (*Existence of an identity*)
FGL2: $u +_F v = v +_F u$. (*Commutativity*)
FGL3: $u +_F (v +_F w) = (u +_F v) +_F w$. (*Associativity*)

Definition 3.4. Let $F(u, v) \in R[\![u, v]\!]$ be a formal group law. The *formal inverse* of F is a power series $G(t) \in R[\![t]\!]$ such that $u +_F G(u) = 0$. We will therefore put $-_F u := G(u)$. Given an integer $m > 1$, we use the notation $m \cdot_F u := \underbrace{u +_F u..... +_F u}_{\text{m times}}$ and $(-m) \cdot_F u := -_F(m \cdot_F u)$, where $-_F u$ denotes the *formal inverse* of u.

Corollary 3.5. *For any formal group law, there exists a unique inverse.*

Proof: First, let us prove the existence of the inverse. Let $F(u, v) \in R[\![u, v]\!]$ be a formal group law. Let $H(u, v) = u - F(u, v)$. As a power series in the indeterminate v, the power series H has no constant term and $(\frac{\partial H(u,v)}{\partial v})_{v=0} = 1$. Then by Proposition 2.12 there exists $G(u, v)$ such that $H(u, G(u, v)) = v$. Therefore, $F(u, G(u, v)) = u - v$ and $F(u, G(u, u)) = 0$, so there exists the inverse $G(u, u)$.

Now, we can prove the uniqueness. Suppose $G_1(u), G_2(u) \in R[\![u, v]\!]$ are both inverses of F and satisfy $u +_F G_1(u) = u +_F G_2(u) = 0$. Then,

$$G_1(u) = G_1(u) +_F 0 = G_1(u) +_F u +_F G_2(u) = 0 +_F G_2(u) = G_2(u).$$

So, the inverse is unique. □

Example 3.6 (Additive formal group law)**.** The additive formal group law is defined as

$$F_A(u, v) = u + v,$$

we need check it satisfies all three axioms.

Firstly, we have $F_A(u, 0) = u+0 = u$, and $F_A(0, u) = 0+u = u$, so $F_A(u, 0) = F_A(0, u) = u$, and the axiom FGL1 is satisfied.

Also, $F_A(u, v) = u + v = v + u = F_A(v, u)$, so the axiom FGL2 is satisfied.

Finally, we have $F_A(u, F_A(v, w)) = u+(v+w) = u+v+w = (u+v)+w = F_A(F_A(u, v), w)$, so the axiom FGL3 is satisfied.

Hence, it is a formal group law.

Example 3.7 (Mulplicative formal group law)**.** The mulplicative formal group law is defined as

$$F_M(u, v) = u + v + \beta uv,\ \ \beta \in R,\ \ \beta \neq 0,$$

and we need to check it satisfies all three axioms.

Firstly, we have $F_M(u, 0) = u + 0 + 0 = u$, and $F_M(0, u) = 0 + u + 0 = u$, so $F_M(u, 0) = F_M(0, u) = u$, and the axiom FGL1 is satisfied.

Also, $F_M(u, v) = u + v + \beta uv = v + u + \beta vu = F_M(v, u)$, so the axiom FGL2 is satisfied.

Finally, we have

$$\begin{aligned} F_M(u, F_M(v, w)) &= F_M(u, (v + w + \beta vw)) \\ &= u + v + w + \beta vw + \beta u(v + w + \beta vw) \\ &= u + v + w + \beta vw + \beta uv + \beta uw + \beta^2 uvw, \end{aligned}$$

and

$$\begin{aligned} F_M(F_M(u, v), w) &= F_M(u + v + \beta uv, w) \\ &= u + v + \beta uv + w + \beta(u + v + \beta uv)w \\ &= u + v + \beta uv + w + \beta uw + \beta vw + \beta^2 uvw \\ &= u + v + w + \beta vw + \beta uv + \beta uw + \beta^2 uvw, \end{aligned}$$

so $F_M(u, F_M(v, w)) = F_M(F_M(u, v), w)$, the axiom FGL3 is satisfied. Hence, it is a formal group law.

Example 3.8 (Lorentz formal group law)**.** The Lorentz formal group law is defined as

$$F_L(u, v) = \frac{u + v}{1 + \beta uv}.$$

By $F_L(u, v) = \frac{u+v}{1+\beta uv}$, we actually mean the power series $F_L(u, v) = (u + v)\sum_{i\geq 0}(-\beta uv)^i$. Recall Definition 2.13 of geometric series, here we have $t = -\beta uv$, so the power series can be written as $(u + v)\sum_{i\geq 0}(-\beta uv)^i = \frac{u+v}{1+\beta uv}$. Now, we want to check that it satisfies the three conditions.

Firstly, we have $F_L(u, 0) = \frac{u+0}{1+0} = u$, and $F_L(0, u) = \frac{0+u}{1+0} = u$. So $F_L(u, 0) = F_L(0, u) = u$, and axiom FGL1 is satisfied.

Also, $F_L(u, v) = \frac{u+v}{1+\beta uv} = \frac{v+u}{1+\beta vu} = F_L(v, u)$, so axiom FGL2 is satisfied.

Finally, we have

$$\begin{aligned} F_L(u, F_L(v,w)) &= F_L\left(u, \frac{v+w}{1+\beta vw}\right) \\ &= \frac{u + \left(\frac{v+w}{1+\beta vw}\right)}{1+\beta u\left(\frac{v+w}{1+\beta vw}\right)} \\ &= \frac{\left(\frac{u+v+w+\beta uvw}{1+\beta vw}\right)}{\left(\frac{1+\beta vw+\beta uv+\beta uw}{1+\beta vw}\right)} \\ &= \frac{u+v+w+\beta uvw}{1+\beta vw+\beta uv+\beta uw}, \end{aligned}$$

and

$$\begin{aligned} F_L(F_L(u,v), w) &= F_L\left(\frac{u+v}{1+\beta uv}, w\right) \\ &= \frac{\left(\frac{u+v}{1+\beta uv}\right)+w}{1+\beta\left(\frac{u+v}{1+\beta uv}\right)w} \\ &= \frac{\left(\frac{u+v+w+\beta uvw}{1+\beta uv}\right)}{\left(\frac{1+\beta uv+\beta uw+\beta vw}{1+\beta uv}\right)} \\ &= \frac{u+v+w+\beta uvw}{1+\beta uv+\beta uw+\beta vw}, \end{aligned}$$

so $F_L(u, F_L(v,w)) = F_L(F_L(u,v),w)$. Therefore, the axiom FGL3 is satisfied. Hence, it is a formal group law.

Definition 3.9 (Lazard ring)**.** We define the *Lazard ring* $\mathbb{L}$ to be the commutative ring with generators $a_{ij}, i, j \in \mathbb{N}$, and the following relations:

$$a_{i0} = a_{0i} = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise} \end{cases}, \tag{3.1}$$

$$a_{ij} = a_{ji}, \tag{3.2}$$

(3.3) the relations forced upon us by the associativity axiom FGL3.

Example 3.10 (Universal formal group law)**.** The formal group law defined over the Lazard ring $\mathbb{L}$ is called the *universal* FGL, and has the form

$$F_U(u,v) = u + v + \sum_{i,j>1} a_{ij}u^i v^j.$$

We want to check that all the axioms are satisfied.

Firstly, we have $F_U(u,0) = u+0+0 = u$, and $F_U(0,u) = 0+u+0 = u$, so $F_U(u,0) = F_L(0,u) = u$.

Also,

$$\begin{aligned} F(u,v) &= u + v + \sum_{i,j\geq 1} a_{ij}u^i v^j \\ &= v + u + \sum_{i,j\geq 1} a_{ji}v^i u^j \\ &= v + u + \sum_{i,j\geq 1} a_{ij}v^i u^j \quad \text{since } a_{ij} = a_{ji} \\ &= F_U(v,u) \end{aligned}$$

Finally, by Definition 3.9 of the Lazard ring, the formal group law satisfies the associativity axiom, so $F_U(u, F_U(v,w)) = F_U(F_U(u,v), w)$.

Hence, it is a formal group law.

Remark 3.11 (Elliptic formal group law)**.** Another typical example of formal group laws is the *elliptic formal group law*. For further details about it, we refer the reader to [2].

4. Category Theory

In this section we recall some basic notions of category theory that will be used in the current paper. Also, we will illustrate the concepts with examples.

Definition 4.1 (Category)**.** A *category* $\mathcal{C}$ consists of :

- A collection of *objects* (which are typically denoted by $A, B, C, \ldots$);
- For each pair of objects A and B, a collection of *morphisms* $f : A \to B$ from A to B;
  - A is the *domain* and B is the *codomain* of $f : A \to B$
  - the morphisms are typically denoted by $f, g, h, \ldots$
- For each object A, there exists an *identity* morphism $\mathrm{Id}_A : A \to A$
- For any two morphisms $f : A \to B$, $g : B \to C$, there exists a morphism $g \circ f : A \to C$, which we will call the *composite* of f with g

In addition, we require the morhpisms to satisfy the following further axioms:

C1: For any $f : A \to B$, $g : B \to C$, $h : C \to D$, we have $(h \circ g) \circ f = h \circ (g \circ f)$. (*Associativity of composition*)

C2: For any $f : A \to B$ we have $f \circ \mathrm{Id}_A = f = \mathrm{Id}_B \circ f$. (*Identity morphisms behave as identities.*)

Definition 4.2 (Isomorphism, monomorphism, epimorphism, endomorphism, automorphism)**.** Let F, G be categories. Then a morphism $f : F \to G$ is called an *isomorphism* if there exists a morphism $g : G \to F$ such that

$$f \circ g = \mathrm{Id}_G \text{ and } g \circ f = \mathrm{Id}_F .$$

Let F, G be categories, then a morphism $f : F \to G$ is called a *monomorphism* if

$$f \circ g_1 = f \circ g_2 \text{ implies } g_1 = g_2$$

for all morphisms $g_1, g_2 : X \to G$.

Dually to monomorphisms, a morphism $f : F \to G$ is called an *epimorphism* if

$$g_1 \circ f = g_2 \circ f \text{ implies } g_1 = g_2$$

for all morphisms $g_1, g_2 : G \to X$.

A morphism $f\colon F \to F$ (that is, a morphism with identical source and target) is an *endomorphism* of F.

Furthermore, an *automorphism* is a morphism that is both an endomorphism and an isomorphism.

Example 4.3 (Category of vector spaces)**.** For a fixed field K, we have the category Vect_K of vector spaces over K. The objects of Vect_K are vector spaces over K, and the morphisms between vector spaces are linear maps.

Example 4.4 (Category of sets)**.** We have the category Set of sets. The objects of Set are sets, and the morphisms between sets are functions.

Example 4.5 (Category of monoids)**.** We have the category Mon of monoids. The objects of Mon are monoids, and the morphisms between monoids are monoid homomorphisms.

Example 4.6 (Category of groups)**.** We have the category Grp of groups. The objects of Grp are groups, and the morphisms between groups are group homomorphisms (preserving group structure).

Example 4.7 (Category of rings)**.** We have the category Rng of rings. The objects of Rng are rings, and the morphisms between rings are ring homomorphisms.

Definition 4.8 (Functor)**.** Given categories $\mathcal{A}$ and $\mathcal{B}$, a *functor* $F : \mathcal{A} \to \mathcal{B}$ consists of the following:

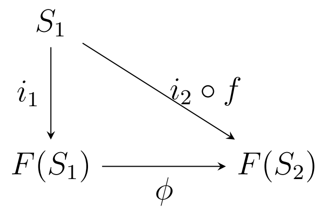
- To each object $X \in \operatorname{Ob}\mathcal{A}$, it associates an object $FX \in \operatorname{Ob}\mathcal{B}$.
- To each morphism $f \in \mathcal{A}(X, Y)$, it associates a morphism $Ff \in \mathcal{B}(FX, FY)$ such that the following propertites hold:
  - For each object $X \in \operatorname{Ob}\mathcal{A}, F\,\mathrm{Id}_X = \mathrm{Id}_{FX}$
  - For a morphism $g \in \mathcal{A}(X, Y)$, and a morphism $f \in \mathcal{A}(Y, Z)$, we have $F(f \circ g) = Ff \circ Fg$.

Example 4.9. The power set functor $P\colon \mathrm{Set} \to \mathrm{Set}$ maps each set to its power set and each function $f\colon X \to Y$ to the map which sends $U \subseteq X$ to its image $F(U) \subseteq Y$.

Example 4.10. The forgetful functor $U\colon \mathrm{Grp} \to \mathrm{Set}$ sends each group $G \in \operatorname{Ob}\mathrm{Grp}$ to its underlying set $U(G) \in \operatorname{Ob}\mathrm{Set}$ and each group homomorphism $f \in \mathrm{Grp}(G_1, G_2)$ to the corresponding set function $Uf \in \mathrm{Set}(U(G_1), U(G_2))$.

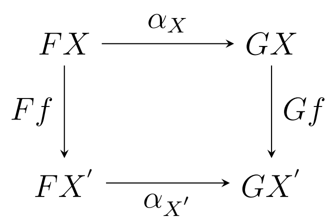
Example 4.11. There is a natural functor $\mathrm{Grp} \to \mathrm{Mon}$ sending a group G to itself as a monoid and each group homomorphism to the corresponding monoid homomorphism.

Example 4.12. There is a free functor $F\colon \mathrm{Set} \to \mathrm{Grp}$ sending each set S to the free group $F(S)$ on S. It sends each set function to a group homomorphism in the following way. Consider the canonical set maps $i_1\colon S_1 \to F(S_1), i_2\colon S_2 \to F(S_2)$. Given a set map $f : S_1 \to S_2$, we have a set map $i_2 \circ f$ from S_1 to $F(S_2)$. Then by the universal property of free groups, there exists a unique group homomorphism $\phi : F(S_1) \to F(S_2)$, such that the following diagram commutes:



We define $Ff = \phi$, then the diagram commutes by construction, so the functor sends each set function f to the group homomorphism Ff. (For more details about free groups and universal properties, we refer the reader to [4, Chapter II])

Definition 4.13 (Natural transformation)**.** Let F, G be functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A natural transformation $\alpha : F \to G$, consists of a collectoin of morphisms $\alpha_X : FX \to GX$, one for each $X \in \operatorname{Ob}\mathcal{C}$, such that for any $X, X' \in \mathcal{C}$ and any morphisms $f \in \mathcal{C}(X, X')$, the diagram

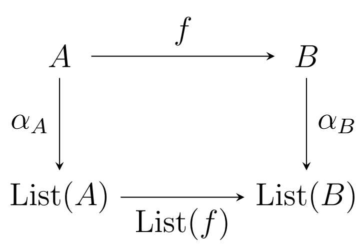


commutes, that is, $\alpha_{X'} \circ Ff = Gf \circ \alpha_X$.

Example 4.14. Consider the functor List: Set $\to$ Set where List sends a set $A \in$ Set to List(A), the set of all finite lists of members of A and List sends a set-function $f : A \to B$, where $B \in$ Set, to the morphism that sends a list $a_0, a_1, a_2, \ldots a_n$ to $fa_0, fa_1, fa_2, \ldots fa_n$.

Claim: there is a natural transformation α : Id $\to$ List, where Id is the identity functor Id : Set $\to$ Set.

Indeed, we need functions α_A which make the following diagram commute for any f : $A \to B$:



For any A, define α_A to be the function which sends an element of A to the length-one list containing just that element, and then we are done.

Note, by the way, that we can think of List as the composite functor GF where F is the free functor from Set to Mon and G is the forgetful functor in the other direction, from Mon to Set.

Example 4.15. Let Vect_K be the category of vector spaces over a field K, with linear maps. Define a functor F: Vect $\to$ Vect that takes a vector space V to its double dual V^{**} and a linear map $\phi: V \to W$ to this double adjoint,

$$F : V \to V^{**} \text{ and } F : \phi \to \phi^{**}.$$

Then, the collection of morphisms $\{\epsilon_V : V \to V^{**} | V \in \mathrm{Vect}\}$ is a natural transformation.

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ {\scriptstyle \epsilon_V}\downarrow & & \downarrow{\scriptstyle \epsilon_W} \\ V^{**} & \xrightarrow[\phi^{**}]{} & W^{**} \end{array} \tag{4.1}$$

To prove this, recall that the dual space V^* of a vector space V is the family of linear functions on V. Hence, the double dual space V^{**} is the family of linear functions on V^*.

For example, if $v \in V$, then the evaluation at v is a map $\overline{v}\colon V^* \to K$ defined by

$$\overline{v}(f) = f(v)$$

for all $f \in V^*$. This map belongs to the double dual V^{**}. Now, let us set

$$\epsilon_V\colon V \to V^{**}, \text{ where } \epsilon_V(v) = \overline{v}.$$

Also, the operator adjoint $\phi^*(f)\colon W^* \to V^*$ of a linear map $\phi\colon V \to W$ is defined by

$$\phi^*(f) = f \circ \phi.$$

Therefore, the second adjoint $\phi^{**}\colon V^{**} \to W^{**}$ is given by

$$\phi^{**}(\alpha) = \alpha \circ \phi^*$$

for $\alpha \in V^{**}$.

Now, we would like to show that, for any linear map $\phi\colon V \to W$, the diagram (4.1) commutes.

We begin by looking at $\epsilon_W \circ \phi$. If $v \in V$, then

$$(\epsilon_W \circ \phi)(v) = \epsilon_W(\phi v) = \overline{\phi v}$$

Now applying $\overline{\phi v}$ to $f \in V$ gives

$$\overline{\phi v}(f) = f(\phi(v)) = \overline{v}(f \circ \phi) = \overline{v}(\phi^*(f)) = (\overline{v} \circ \phi^*)(f),$$

and so $\overline{\phi v} = \overline{v} \circ \phi^*$.

Thus, we have

$$(\epsilon_W \circ \phi)(v) = \overline{\phi v} = \overline{v} \circ \phi^* = \phi^{**}(\overline{v}) = \phi^{**}(\epsilon_V(v)) = (\phi^{**} \circ \epsilon_V)(v),$$

and finally we arrive at

$$\epsilon_W \circ \phi = \phi^{**} \circ \epsilon_V. \tag{4.2}$$

Therefore, the collection of morphisms $\{\epsilon_V : V \to V^{**} \mid V \in \mathrm{Vect}\}$ is a natural transformation.

Definition 4.16 (Initial and terminal object)**.** Let $\mathcal{C}$ be a category. An object $I \in \mathrm{Ob}\,\mathcal{C}$ is an *initial object* if for every object $X \in \mathrm{Ob}\,\mathcal{C}$, there is exactly one morphism $I \to X$. An object $T \in \mathrm{Ob}\,\mathcal{C}$ is a *terminal object* if for every object $X \in \mathrm{Ob}\,\mathcal{C}$, there is exactly one morphism $X \to T$.

Example 4.17. In the category of sets, the empty set is an initial object, and the one-element set is a terminal object.

Example 4.18. In the category of semigroups, the empty semigroup is the unique initial object and any singleton semigroup is a terminal object.

Example 4.19. In the category of non-empty sets, there are no initial objects. The singletons are not initial: while every non-empty set admits a function from a singleton, this function is in general not unique.

Example 4.20. In the category of fields, there are no initial or terminal objects. However, in the subcategory of fields of fixed characteristic, the prime field is an initial object. Recall that a prime field is a finite field of characteristic p where p is prime. In addition, a prime field of characteristic zero is isomorphic to the rational number $\mathbb{Q}$.

As the above examples illustrate, initial and terminal objects do not always exist. However, the following result shows that, if they exist, they are unique up to isomorphism.

Lemma 4.21. *Any two initial objects in a category are isomorphic. Similarly, any two terminal objects in a category are isomorphic.*

Proof. As for initial objects, suppose I_1 and I_2 are both initial objects in category $\mathcal{C}$. By the Definition 4.16 of initial object, there must be unique morphisms $f\colon I_1 \to I_2$ and $g\colon I_2 \to I_1$. Then $g \circ f$ is a morphism from I_1 to itself. In addition, another morphism form I_1 to itself is the identity morphism Id_{I_1}. Since I_1 is an initial object, there can only be one morphism from I_1 to itself, so $g \circ f = \mathrm{Id}_{I_1}$.

Likewise, $f \circ g$ is a morphism from I_2 to itself and Id_{I_2} is another morphism from I_2 to itself. Then since I_2 is an initial object, we have $f \circ g = \mathrm{Id}_{I_2}$. As a result, we have $g \circ f = \mathrm{Id}_{I_1}$ and $f \circ g = \mathrm{Id}_{I_2}$. Hence the unique morphism f has a two-sided inverse and is an isomorphism.

Similarly, we can prove that all terminal objects are isomorphic using the same technique. □

5. Category of Formal Group Laws

In this section, we will discuss how formal group laws naturally form a category.

5.1. Morphisms of Formal Group Laws.

Definition 5.1. Let F, G be formal group laws over a ring R; then a homomorphism $f : F \to G$ over R is a power series with no constant term $f(t) \in R[\![t]\!]$ such that

$$f(F(X,Y)) = G(f(X), f(Y)).$$

Definition 5.2. Let $f(t), g(t) \in R[\![t]\!]$ be two power series without constant terms. Then the composition of f and g is defined to be the formal power series $(f \circ g)(t) = f(g(t))$.

Remark 5.3. If $f(t), g(t) \in R[\![t]\!]$ are two power series with constant terms, then $(f \circ g)(t)$ is not well-defined in general. For instance, take $f(t) = g(t) = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$. Then we have

$$\begin{aligned}(f \circ g)(t) &= f(g(t)) \\ &= 1 + g(t) + (g(t))^2 + (g(t))^3 + \cdots \\ &= 1 + (1 + t + t^2 + t^3 + \cdots) + (1 + t + t^2 + t^3 + \cdots)^2 + (1 + t + t^2 + t^3 + \cdots)^3 + \cdots\end{aligned}$$

Note that when we expand the terms, the constant term from each $g(t)$ term will contribute to the final constant term. Hence, the constant term of this composition is $1 + 1 + 1 + \cdots = \sum_{n=0}^{\infty} 1$, which does not converge.

However, if $f(t), g(t) \in R[\![t]\!]$ are two power serieses without constant terms, then $(f \circ g)(t)$ is well-defined. To see this, take $f(t) = \sum_{n\geq 1} a_n t^n$ and $g(t) = \sum_{n\geq 1} b_n t^n$ without constant term, so we have

$$\begin{aligned}(f \circ g)(t) &= f(g(t)) \\ &= \sum_{n\geq 1}^{\infty} a_n \left(\sum_{m\geq 1} b_m t^m \right)^n \\ &= \sum_{l\geq 1}^{\infty} c_l t^l.\end{aligned}$$

Also, we know that for a fixed k, we have $\sum_{n\geq 1}^{k} a_n \left(\sum_{m\geq 1}^{k} b_m t^m \right)^n = \sum_{l\geq 1}^{k^2} d_l t^l$ for some finite coefficients d_l. We claim that $c_l = d_l$ for any $l \leq k$. Indeed, for coefficients c_l where $l \leq k$, the terms in the infinite sum with $m > k$ or $n > k$ do not contribute since they involve powers of t higher than k. Therefore, each coefficient c_l will be a finite number and therefore the composition is well-defined.

Definition 5.4 (Category of formal group laws)**.** For a fixed ring R, we have the category of formal group laws FGL_R. The objects of FGL_R are formal group laws over R, and the morphisms between formal group laws are the homomorphisms defined in Definition 5.1.

For any morphisms over formal group laws $f\colon B \to C$, $g\colon C \to D$, $h\colon D \to E$, we have

$$\begin{aligned}h \circ (g \circ f)(t) &= h \circ (g(f(t))) \\ &= h(g(f(t)))\end{aligned}$$

and

$$\begin{aligned}(h \circ g)f(t) &= (h(g))(f(t)) \\ &= h(g(f(t))).\end{aligned}$$

Hence, $(h \circ g) \circ f = h \circ (g \circ f)$.

Also, define the identity morphism $\mathrm{Id}\colon F \to F$ as $\mathrm{Id}_F = t$. For any $f\colon B \to C$ we have $f \circ \mathrm{Id}_B = f = \mathrm{Id}_C \circ f$.

Lemma 5.5. *Let R be a field of characteristic zero and assume that β is an invertible element of R. Then the additive and multiplicative formal group laws are isomorphic via the isomorphism $\frac{1}{\beta}(e^t - 1) = \frac{1}{\beta}(t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots)$.*

Proof. Let $F_1(x, y) = x + y + \beta xy$ be the multiplicative formal group law and $F_2(x, y) = x + y$ be the additive formal group law. Let $g(t) = \frac{1}{\beta}(e^t - 1) = \frac{1}{\beta}(t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots)$.

Firstly, we can show that $g\colon F_1(x, y) \to F_2(x, y)$, where $g(t) = \frac{1}{\beta}(e^t - 1)$, is a homomorphism. We have

$$F_1(g(x), g(y)) = F_1 \left(\frac{e^x}{\beta} - \frac{1}{\beta}, \frac{e^y}{\beta} - \frac{1}{\beta} \right)$$

$$
\begin{aligned}
&= \frac{e^x}{\beta} - \frac{1}{\beta} + \frac{e^y}{\beta} - \frac{1}{\beta} + \beta\left(\frac{e^x}{\beta} - \frac{1}{\beta}\right)\left(\frac{e^y}{\beta} - \frac{1}{\beta}\right) \\
&= \frac{e^x}{\beta} - \frac{1}{\beta} + \frac{e^y}{\beta} - \frac{1}{\beta} + \beta\left(\frac{e^{x+y}}{\beta^2} - \frac{e^x}{\beta^2} - \frac{e^y}{\beta^2} + \frac{1}{\beta^2}\right) \\
&= \frac{e^x}{\beta} - \frac{1}{\beta} + \frac{e^y}{\beta} - \frac{1}{\beta} + \frac{e^{x+y}}{\beta} - \frac{e^x}{\beta} - \frac{e^y}{\beta} + \frac{1}{\beta} \\
&= \frac{e^{x+y}}{\beta} - \frac{1}{\beta} \\
&= g(x+y) \\
&= g(F_2(x,y)).
\end{aligned}
$$

Then, we need to prove it is an isomorphism. Define the inverse morphism of g to be $h(t) = (\ln(\beta t+1)) = \beta t - \frac{(\beta t)^2}{2} + \frac{(\beta t)^3}{3} - \frac{(\beta t)^4}{4} + \cdots$, and we need to show h is a homomorphism. Hence, we have

$$
\begin{aligned}
F_2(h(x), h(y)) &= F_2(\ln(\beta x+1), \ln(\beta y+1)) \\
&= \ln(\beta x+1) + \ln(\beta y+1) \\
&= \ln((\beta x+1)(\beta y+1)) \\
&= \ln(\beta^2 xy + \beta x + \beta y + 1) \\
&= \ln(\beta(\beta xy + x + y) + 1) \\
&= h(\beta xy + x + y) \\
&= h(F_1(x,y)).
\end{aligned}
$$

In addition, we also need to show

$$
\begin{aligned}
g(h(t)) &= g(\ln(\beta t+1)) \\
&= \frac{1}{\beta}(e^{(\ln(\beta t+1))} - 1) \\
&= \frac{1}{\beta}(\beta t + 1 - 1) \\
&= \frac{1}{\beta}(\beta t) \\
&= t.
\end{aligned}
$$

and

$$
\begin{aligned}
h(g(t)) &= h(\frac{1}{\beta}(e^t - 1)) \\
&= \ln(\beta \frac{1}{\beta}(e^t - 1) + 1) \\
&= \ln((e^t - 1) + 1) \\
&= \ln(e^t) \\
&= t.
\end{aligned}
$$

Therefore, there is a isomorphism between the additive and multiplicative formal group laws. □

However, over general commutative rings R there is no such homomorphism as defining it requires non-integral rational numbers, and the additive and multiplicative formal groups are usually not isomorphic.

Lemma 5.6. *Let R be a field of characteristic zero, and F a commutative one-dimensional formal group law over R. Then there exists an isomorphism $f\colon F \to F_A$ (the additive formal group law) defined over R.*

(For a proof of Lemma 5.6, we refer the reader to [5, Chapter IV, Section 1])

5.2. **Formal Group Laws as Functors.**

Lemma 5.7. *Let S be a commutative R-algebra and let F be a formal group law over R. Then F induces a binary operation on $\mathrm{Nil}(S)$ and $(\mathrm{Nil}(S), +_F)$ is an abelian group.*

Proof. To show $\mathrm{Nil}(S)$ is a group with group operation $+_F$, we first claim that $\mathrm{Nil}(S)$ is closed under this binary operation. If $x, y \in \mathrm{Nil}(S)$, we have $x +_F y = \sum_{i,j=0}^{\infty} a_{ij}x^iy^j$. Since x and y are nilpotent, $x^{n_1} = 0$ and $y^{n_2} = 0$ for some integer n_1 and n_2. Hence, $x +_F y = \sum_{i,j=0}^{\infty} a_{ij}x^iy^j = \sum_{i,j=0}^{n} a_{ij}x^iy^j + 0 = \sum_{i,j=0}^{n} a_{ij}x^iy^j$ for $n = \min\{n_1, n_2\}$. In addition, $x^i \in \mathrm{Nil}(S)$ and $y^j \in \mathrm{Nil}(S)$ imply $x^iy^j \in \mathrm{Nil}(S)$; $a_{ij} \in R$ and $x^iy^j \in S$ imply $a_{ij}x^iy^j \in \mathrm{Nil}(S)$ since $\mathrm{Nil}(S)$ is an ideal. Hence, $\sum_{i,j=0}^{n} a_{ij}x^iy^j \in \mathrm{Nil}(S)$ and $x +_F y \in \mathrm{Nil}(S)$.

Also, the group $\mathrm{Nil}(S)$ is associative. For $x, y, z \in \mathrm{Nil}(S)$, we have $F(x, F(y, z)) = F(F(x, y), z)$ by the definition of formal group law.

In addition, the group $\mathrm{Nil}(A)$ has an identity 0 since $F(x, 0) = F(0, x) = 0$.

Furthermore, by Corollary 3.5, we know that for each element $x \in \mathrm{Nil}(S)$, there exists an element $y = -_F x \in \mathrm{Nil}(S)$ such that $F(x, y) = 0$.

Therefore, we have proved that F gives a group structure on $\mathrm{Nil}(S)$ with the binary operation $+_F$. □

Let S be a commutative R-algebra. Then any one-dimensional formal group law F over S gives a group structure on the set $\mathrm{Nil}(S)$. Hence, (denote the category of commutative R-algebras by C_R), we want to show that the associative formal group law F gives a functor $\mathbf{F}_F\colon C_R \to \mathrm{Grp}$ from the category of commutative R-algebras to the category of groups.

More precisely, for each object $S \in \mathrm{Ob}\, C_R$, it associates $\mathbf{F}_F(S) = (\mathrm{Nil}(S), +_F) \in \mathrm{Ob}\,\mathrm{Grp}$ defined in the Lemma 5.7.

Also, for each homomorphism $\phi\colon S_1 \to S_2$ where $S_1, S_2 \in \mathrm{Ob}\, C_R$, we know that $\phi(\mathrm{Nil}(S_1)) \subseteq \mathrm{Nil}(S_2)$ by Lemma 2.10. Take $x, y \in \mathrm{Nil}(S_1)$ and let $n \in \mathbb{N}$ such that $x^n = 0 = y^n$. We want to show that $\phi(x +_F y) = \phi(x) +_F \phi(y)$. So we have

$$\begin{aligned}\phi(x +_F y) &= \phi\left(\sum_{i,j=0}^{\infty} a_{ij}x^iy^j\right)\\ &= \phi\left(\sum_{i,j=0}^{n} a_{ij}x^iy^j\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j=0}^{n} \phi\left(a_{ij} x^i y^j\right) \\
&= \sum_{i,j=0}^{n} a_{ij} \phi\left(x^i y^j\right) \\
&= \sum_{i,j=0}^{n} a_{ij} \phi\left(x^i\right) \phi\left(y^j\right) \\
&= \sum_{i,j=0}^{\infty} a_{ij} \phi\left(x^i\right) \phi\left(y^j\right) \\
&= \sum_{i,j=0}^{\infty} a_{ij} \phi(x)^i \phi(y)^j \\
&= \phi(x) +_F \phi(y).
\end{aligned}
$$

Hence, it is proved that $\phi(x +_F y) = \phi(x) +_F \phi(y)$. For a ring homomorphism $\phi\colon S_1 \to S_2$, define $\mathbf{F}_F(\phi)$ as ϕ restricted to $\mathrm{Nil}(S_1)$. Hence, each homomorphism $\phi\colon S_1 \to S_2$ where $S_1, S_2 \in \mathrm{Ob}\, C_R$ is sent by the functor to a homomorphism $\mathbf{F}_F(\phi)\colon \mathbf{F}_F(S_1) \to \mathbf{F}_F(S_2)$.

In addition, it is straightforward to verify that $\mathbf{F}_F(\mathrm{Id}_X) = \mathrm{Id}_{\mathbf{F}_F(X)}$ for each object $X \in \mathrm{Ob}\, C_R$. Also, it is true that for a homomorphism $\phi\colon S_1 \to S_2$ and a homomorphism $\eta\colon S_2 \to S_3$, we have $\mathbf{F}_F(\phi \circ \eta) = \mathbf{F}_F(\phi) \circ \mathbf{F}_F(\eta)$.

Therefore, we have showed that the formal group law F defines a functor $\mathbf{F}_F\colon C_R \to \mathrm{Grp}$ from the category of commutative R-algebras to the category of groups.

Furthermore, we will be talking about the natural transformation given by the morphisms of formal group laws of functors.

Let F and G be formal group laws over R, and consider the associated functors $\mathbf{F}_F$ and $\mathbf{F}_G$ from the category C_R to the category Grp. A natural transformation $f : \mathbf{F}_F \to \mathbf{F}_G$, consists of a collection of morphisms $f_S : \mathbf{F}_F(S) \to \mathbf{F}_G(S)$, one for each $S \in \mathrm{Ob}\, C_R$, such that for any $S_1, S_2 \in C_R$ and any morphisms $\phi\colon S_1 \to S_2$, the diagram

$$
\begin{array}{ccc}
\mathbf{F}_F(S_1) & \xrightarrow{f_{S_1}} & \mathbf{F}_G(S_1) \\
{\scriptstyle \mathbf{F}_F(\phi)}\downarrow & & \downarrow{\scriptstyle \mathbf{F}_G(\phi)} \\
\mathbf{F}_F(S_2) & \xrightarrow[f_{S_2}]{} & \mathbf{F}_G(S_2)
\end{array} \tag{5.1}
$$

commutes; that is, $f_{S_2} \circ \mathbf{F}_F(\phi) = \mathbf{F}_G(\phi) \circ f_{S_1}$.

Let f be a homomorphism between the formal group laws F and G. Thus, f is a power series with no constant term $f(t) \in R[\![t]\!]$ satisfying $f(F(X,Y)) = G(f(X), f(Y))$. For $f\colon F \to G$, define $f_S\colon \mathrm{Nil}(S) \to \mathrm{Nil}(S)$ as $f_S(r) = f(r)$ for $r \in \mathrm{Nil}(S)$, and we want to show the map f_S is well-defined. Take $r \in \mathrm{Nil}(S)$ such that $r^n = 0$ for some integer n, then we have $f_S(r) = \sum_{i=0}^{\infty} a_i r^i = \sum_{i=0}^{n} a_i r^i + 0 = \sum_{i=0}^{n} a_i r^i \in \mathrm{Nil}(S)$. Hence, the map f_S is well-defined.

Next we want to show that f_S is a group homomorphism from the group $\mathbf{F}_F(S) = (\mathrm{Nil}(S), +_F)$ to the group $\mathbf{F}_G(S) = (\mathrm{Nil}(S), +_G)$. To claim it is a homomorphism, we need show that for any $x, y \in \mathrm{Nil}(S)$, it is true that $f(x +_F y) = f(x) +_G f(y)$. By our definition of f, it is already correct that $f(x +_F y) = f(x) +_G f(y)$.

Hence, we have showed that α_S is a group homomorphism from $\mathbf{F}_F(S)$ to $\mathbf{F}_G(S)$.

Proposition 5.8. *If $f\colon F \to G$ is a homomorphism of formal group laws over R, then $(f_S)_{S \in \mathrm{Ob}\, C_R}$ is a natural transformation from $\mathbf{F}_F$ to $\mathbf{F}_G$.*

Proof. To prove the proposition is true, we need to show diagram (5.1) commutes. More specifically, take $r \in \mathbf{F}_F(S_1)$ such that $r^n = 0$, we need to prove that $\mathbf{F}_G(\phi) \circ f_{S_1}(r) = f_{S_2} \circ \mathbf{F}_F(\phi)(r)$. Write f as $f = \sum_{i=0}^{\infty} a_i t^i$. Since f is a ring homomorphism, we have

$$
\begin{aligned}
\mathbf{F}_G(\phi) \circ f_{S_1}(r) &= \mathbf{F}_G(\phi) \circ f(r) \\
&= \phi(f(r)) \\
&= \phi\left(\sum_{i=0}^{\infty} a_i r^i\right) \\
&= \phi\left(\sum_{i=0}^{n} a_i r^i\right) \\
&= \sum_{i=0}^{n} \phi(a_i r^i) \\
&= \sum_{i=0}^{n} a_i \phi(r^i) \\
&= \sum_{i=0}^{\infty} a_i \phi(r^i) \\
&= \sum_{i=0}^{\infty} a_i \phi(r)^i \\
&= f(\phi(r)) \\
&= f_{S_2} \circ \phi(r) \\
&= f_{S_2} \circ \mathbf{F}_F(\phi)(r).
\end{aligned}
$$

Hence, the diagram (5.1) commutes and the result follows. ☐

5.3. **Change of Base Rings.** Suppose $F \in R[\![t]\!]$ is a formal group law over R and $\phi\colon R \to R'$ is a ring homomorphism of commutative rings. Write F as $F(u, v) = \sum_{i,j=0}^{\infty} a_{ij} u^i v^j$ and define $F'(u, v) = \sum_{i,j=0}^{\infty} \phi(a_{ij}) u^i v^j$. Now we want to check that $F' \in R'[\![t]\!]$ is a formal group law over the ring R'.

Firstly, we have

$$F'(u, 0) = \sum_{i,j=0}^{\infty} \phi(a_{ij}) u^i 0^j = \phi(1_R) u + 0 = 1_{R'} u = u$$

and
$$F^{'}(0,u) = 0 + u = u.$$
So the axiom FGL1 is satisfied.

Also, since the formal group law F is commutative and
$$F(u,v) = \sum_{i,j=0}^{\infty} a_{ij}u^i v^j = \sum_{i,j=0}^{\infty} a_{ij}v^i u^j = F(v,u),$$
we have
$$F^{'}(u,v) = \sum_{i,j=0}^{\infty} \phi(a_{ij})u^i v^j = \sum_{i,j=0}^{\infty} \phi(a_{ij})v^i u^j = F^{'}(v,u)$$
because ϕ is a ring homomorphism. Hence, the axiom FGL2 is satisfied.

Finally, since $F(u, F(v,w)) = F(F(u,v), w)$, we have
$$\sum_{i,j=0}^{\infty} a_{ij}u^i(F(v,w))^j = \sum_{i,j=0}^{\infty} a_{ij}u^i \left(\sum_{i,j=0}^{\infty} a_{ij}v^i w^j\right)^j = \sum_{i,j=0}^{\infty} a_{ij}(F(u,v))^i w^j = \sum_{i,j=0}^{\infty} a_{ij} \left(\sum_{i,j=0}^{\infty} a_{ij}u^i v^j\right)^i w^j.$$
By the property of ring homomorphism, we can show that
$$\begin{aligned} F(u, F^{'}(v,w)) &= \sum_{i,j=0}^{\infty} a_{ij}u^i \left(\sum_{i,j=0}^{\infty} \phi(a_{ij})v^i w^j\right)^j \\ &= \sum_{i,j=0}^{\infty} a_{ij} \left(\sum_{i,j=0}^{\infty} \phi(a_{ij})u^i v^j\right)^i w^j \\ &= F(F^{'}(u,v), w). \end{aligned}$$
So, by applying the homomorphism, we get
$$\sum_{i,j=0}^{\infty} \phi(a_{ij})u^i \left(\sum_{i,j=0}^{\infty} \phi(a_{ij})v^i w^j\right)^j = \sum_{i,j=0}^{\infty} \phi(a_{ij}) \left(\sum_{i,j=0}^{\infty} \phi(a_{ij})u^i v^j\right)^i w^j.$$
Hence, we have $F^{'}(u, F^{'}(v,w)) = F^{'}(F^{'}(u,v), w)$ and the axiom FGL3 is satisfied.

Now we define a functor $\mathbf{F}_\phi$ associated with the ring homomorphism $\phi\colon R \to R^{'}$. Let FGL_R be the category of formal group laws over the ring R, and $\mathrm{FGL}_{R^{'}}$ be the category of formal group laws over the ring $R^{'}$. For each object $F = \sum_{i,j=0}^{\infty} a_{ij}u^i v^j \in \operatorname{Ob}\mathrm{FGL}_R$, we have $\mathbf{F}_\phi(F) = \sum_{i,j=0}^{\infty} \phi(a_{ij})u^i v^j \in \operatorname{Ob}\mathrm{FGL}_{R^{'}}$ by applying ϕ to all the coefficients.

Also, for each homomorphism $f\colon F \to G$ where $F, G \in \operatorname{Ob}\mathrm{FGL}_R$, we have $f(F(u,v)) = G(f(u), f(v))$ by Definition 5.1. Now, for each $f \in R[\![x]\!]$, define $\mathbf{F}_\phi(f) \in R^{'}[\![x]\!]$ to be the formal group law obtained by applying ϕ to all the coefficients of f. For each homomorphism $f\colon F \to G$ where $F, G \in \operatorname{Ob}\mathrm{FGL}_R$, we want to show $\mathbf{F}_\phi(f)$ is a homomorphism of formal group laws from $\mathbf{F}_\phi(F)$ to $\mathbf{F}_\phi(G)$ where $\mathbf{F}_\phi(F), \mathbf{F}_\phi(G) \in \operatorname{Ob}\mathrm{FGL}_{R^{'}}$. In other words, we need to prove that $\mathbf{F}_\phi(f)(u +_{\mathbf{F}_\phi(F)} v) = \mathbf{F}_\phi(f)(u) +_{\mathbf{F}_\phi(G)} \mathbf{F}_\phi(f)(v)$. We define $(u +_F v) = \sum_{i,j=0}^{\infty} a_{ij}u^i v^j$, define $u +_G v = \sum_{i,j=0}^{\infty} b_{ij}u^i v^j$ and define $f(t) = \sum_{k=0}^{\infty} c_k t^k$. Also, since f is a homomorphism of formal group laws and we have $f(F(u,v)) = G(f(u), f(v))$, which can be written as

$$\sum_{k=0}^{\infty} c_k \left(\sum_{i,j=0}^{\infty} a_{i,j} u^i v^j \right)^k = \sum_{i,j=0}^{\infty} b_{ij} \left(\sum_{k=0}^{\infty} c_k u^k \right)^i \left(\sum_{k=0}^{\infty} c_k v^k \right)^j . \tag{5.2}$$

Then we have

$$\begin{aligned}
\mathbf{F}_\phi(f)(u +_{\mathbf{F}_\phi(F)} v) &= \mathbf{F}_\phi(f) \left(\sum_{i,j=0}^{\infty} \phi(a_{ij}) u^i v^j \right) \\
&= \sum_{k=0}^{\infty} \phi(c_k) \left(\sum_{i,j=0}^{\infty} \phi(a_{ij}) u^i v^j \right)^k \\
&= \sum_{i,j=0}^{\infty} \phi(b_{ij}) \left(\sum_{k=0}^{\infty} \phi(c_k) u^k \right)^i \left(\sum_{k=0}^{\infty} \phi(c_k) v^k \right)^j \\
&\quad \text{(by the equation (5.2) and the fact that } \phi \text{ is a ring homomorphism)} \\
&= \sum_{i,j=0}^{\infty} \phi(b_{ij}) (\mathbf{F}_\phi(f)(u))^i (\mathbf{F}_\phi(f)(v))^j \\
&= \mathbf{F}_\phi(f)(u) +_{\mathbf{F}_\phi(G)} \mathbf{F}_\phi(f)(v).
\end{aligned}$$

Hence, we proved that $\mathbf{F}_\phi(f)(u +_{\mathbf{F}_\phi(F)} v) = \mathbf{F}_\phi(f)(u) +_{\mathbf{F}_\phi(G)} \mathbf{F}_\phi(f)(v)$. So each homomorphism $f\colon F \to G$ where $F, G \in \operatorname{Ob} \mathrm{FGL}_R$ is sent by the functor $\mathbf{F}_\phi$ to a homomorphism $\mathbf{F}_\phi(f)\colon \mathbf{F}_\phi(F) \to \mathbf{F}_\phi(G)$. In addition, it is straightforward to verify that $\mathbf{F}_\phi(\mathrm{Id}_F) = \mathrm{Id}_{\mathbf{F}_\phi(F)}$ for each object $F \in \operatorname{Ob} \mathrm{FGL}_R$. It remains to show that for a homomorphism $f\colon F \to G$ and a homomorphism $g\colon G \to H$, we have $\mathbf{F}_\phi(f \circ g) = \mathbf{F}_\phi(f) \circ \mathbf{F}_\phi(g)$. Now, define $f(u) = \sum_{i=0}^{\infty} a_i u^i$ and $g(u) = \sum_{i=0}^{\infty} b_i u^i$. Then take $u \in R$, and we have

$$\begin{aligned}
\mathbf{F}_\phi(f \circ g)(u) &= \mathbf{F}_\phi \left(\sum_{i=0}^{\infty} a_i (g(u))^i \right) \\
&= \mathbf{F}_\phi \left(\sum_{i=0}^{\infty} a_i \left(\sum_{i=0}^{\infty} b_j u^j \right)^i \right) \\
&= \sum_{i=0}^{\infty} \phi(a_i) \left(\sum_{i=0}^{\infty} \phi(b_j) u^j \right)^i \qquad \text{because } \phi \text{ is a ring homomorphism} \\
&= \sum_{i=0}^{\infty} \phi(a_i) \left(\mathbf{F}_\phi(g)(u) \right)^i \\
&= \mathbf{F}_\phi(f) \circ \mathbf{F}_\phi(g)(u).
\end{aligned}$$

Hence, we have showed that $\mathbf{F}_\phi$ is a functor from FGL_R to $\mathrm{FGL}_{R'}$.

6. Higher-Dimensional Formal Group Laws

6.1. **Concepts of Higher-Dimensional Formal Group Laws.**

Definition 6.1 (n-Dimensional Formal Group Law)**.** An n-dimensional commutative formal group law is a pair (R, F), where R is a commutative ring, called the coefficient ring, and F is a collection of n power series $F_i(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n) \in R[\![u_1, u_2 \cdots, u_n, v_1, v_2, \cdots, v_n]\!]$ for $1 \leq i \leq n$ in $2n$ variables such that it satisfies the following axioms:

FGL1: $F(u, 0) = F(0, u) = u.$ (*Existence of an identity*)
FGL2: $F(u, v) = F(v, u).$ (*Commutativity*)
FGL3: $F(u, F(v, w)) = F(F(u, v), w).$ (*Associativity*)
where we write F for $(F_1, \cdots, F_n)$, u for $(u_1, \cdots, u_n)$, and so on.

Remark 6.2. We frequently use the notation $u +_F v := F(u, v)$ where F means $(F_1, dotsc, F_n)$ and u means $(u_1, dotsc, u_n)$. So the axioms for a formal group law can be expressed as follows:

FGL1: $u +_F 0 = 0 +_F u = u.$ (*Existence of an identity*)
FGL2: $u +_F v = v +_F u.$ (*Commutativity*)
FGL3: $u +_F (v +_F w) = (u +_F v) +_F w.$ (*Associativity*)

Definition 6.3. Let $F(u, v) \in R[\![u, v]\!]$ be a formal group law. A *formal inverse* of F is a collection of power series $G_i(t) \in R[\![t]\!]$ for $1 \leq i \leq n$ such that $u +_F G(u) = 0$. We will therefore put $-_F u := G(u)$.

Example 6.4 (Additive formal group law)**.** The additive formal group law is defined as

$$F_A(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n) = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n),$$

we need check it satisfies all three axioms.

Firstly, we have

$$F_A(u_1, u_2, \cdots, u_n, 0, 0, \cdots, 0) = (u_1 + 0, u_2 + 0, \cdots, u_n + 0) = (u_1, u_2, \cdots, u_n)$$

and

$$F_A(0, 0, \cdots, 0, u_1, u_2, \cdots, u_n) = (0 + u_1, 0 + u_2, \cdots, 0 + u_n) = (u_1, u_2, \cdots, u_n),$$

So

$$F_A(u_1, u_2, \cdots, u_n, 0, 0, \cdots, 0) = F_A(0, 0, \cdots, 0, u_1, u_2, \cdots, u_n) = (u_1, u_2, \cdots, u_n).$$

and the axiom FGL1 is satisfied.

Also, $F_A(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n) = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n) = (v_1 + u_1, v_2 + u_2, \cdots, v_n + u_n) = F_A(v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n)$, so the axiom FGL2 is satisfied.

Finally, we have

$$\begin{aligned}
&F_A(u_1, u_2, \cdots, u_n, F_A(v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n,)) \\
&= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \cdots, u_n + (v_n + w_n)) \\
&= (u_1 + v_1 + w_1, u_2 + v_2 + w_2, \cdots, u_n + v_n + w_n) \\
&= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \cdots, (u_n + v_n) + w_n) \\
&= F_A(F_A(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n,), w_1, w_2, \cdots, w_n,),
\end{aligned}$$

so the axiom FGL3 is satisfied.

Hence, it is a formal group law.

Example 6.5 (Mulplicative formal group law)**.** The mulplicative formal group law is defined as

$$F_M(u,v) = u + v + \beta uv,\ \beta \in R,\ \beta \neq 0,$$

More precisely, define $F_M(u,v)$ as

$$u + v + \beta uv = (u_1 + v_1 + \beta u_1 v_1, u_2 + v_2 + \beta u_2 v_2, \cdots, u_n + v_n + \beta u_n v_n).$$

We need to check it satisfies all three axioms. The proof is the same as for the one-dimensional case in Example 3.7. Hence, it is a formal group law.

Example 6.6 (Lorentz formal group law)**.** The Lorentz formal group law is defined as

$$F_L(u,v) = \frac{u+v}{1+\beta uv}$$

Thus, we define $u + v = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$, define $1 + \beta uv = (1 + \beta u_1 v_1, 1 + \beta u_2 v_2, \cdots, 1 + \beta u_n v_n)$ and define $\frac{u+v}{1+\beta uv} = (\frac{u_1+v_1}{1+\beta u_1 v_1}, \frac{u_2+v_2}{1+\beta u_2 v_2}, \cdots, \frac{u_n+v_n}{1+\beta u_n v_n})$. By $\frac{u_1+v_1}{1+\beta u_1 v_1}$, we actually mean the power series $(u_1 + v_1)\sum_{i\geq 0}(-\beta u_1 v_1)^i$. Recall Definition 2.13 of geometric series, here we have $r = -\beta u_1 v_1$, so the power series can be written as $(u_1 + v_1)\sum_{i\geq 0}(-\beta u_1 v_1)^i = \frac{u_1+v_1}{1+\beta u_1 v_1}$. Now, we want to check that it satisfies the three conditions. The proof is the same as for the one-dimensional case in Example 3.8. Hence, it is a formal group law.

Definition 6.7 (Homomorphism between higher-dimensional FGLs)**.** A homomorphism from a formal group law F of dimension m to a formal group law G of dimension n is a collection f of n power series in m variables, such that $G(f(u), f(v)) = f(F(u,v))$. More precisely, the homomorphism f needs to satisfy

$$\begin{aligned} &G_i(f_1(u_1, u_2, \cdots, u_m), f_2(u_1, u_2, \cdots, u_m), \cdots, f_n(u_1, u_2, \cdots, u_m), \\ &\qquad f_1(v_1, v_2, \cdots, v_m), f_2(v_1, v_2, \cdots, v_m), \cdots, f_n(v_1, v_2, \cdots, v_m)) \\ &= f_i(F_1(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n), F_2(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n), \cdots, \\ &\qquad F_m(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n)) \end{aligned}$$

where $1 \geq i \geq m$.

Definition 6.8 (Category of higher-dimensional FGLs)**.** For a fixed ring R, we have the category of formal group laws FGL_R. The objects of FGL_R are higher-dimensional formal group laws over R, and the morphisms between formal group laws are the homomorphisms defined in Definition 6.7.

Suppose we have morphisms over formal group laws $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$ where A is of dimension n, B is of dimension m and C is of dimension p. Then we have

$$\begin{aligned} h \circ (g \circ f)(t_1, t_2, \cdots, t_n) &= h \circ (g(f(t_1, t_2, \cdots, t_n))) \\ &= h(g(f(t_1, t_2, \cdots, t_n))) \end{aligned}$$

and

$$\begin{aligned} (h \circ g)f(t_1, t_2, \cdots, t_n) &= (h(g))(f(t_1, t_2, \cdots, t_n)) \\ &= h(g(f(t_1, t_2, \cdots, t_n))). \end{aligned}$$

Hence, $(h \circ g) \circ f = h \circ (g \circ f)$.

Also, define the identity morphism $\mathrm{Id}\colon F \to F$ as $\mathrm{Id}_F = t$. For any $f\colon B \to C$ we have $f \circ \mathrm{Id}_B = f = \mathrm{Id}_C \circ f$.

6.2. **Formal Group Laws as Functors for Higher-Dimensional FGLs.** Let S be a commutative R-algebra. We will show that any n-dimensional formal group law F over S gives a group structure on the set $\mathrm{Nil}(S)^n$. Denote the category of commutative R-algebras by C_R. We want to show that the associative formal group law F gives a functor $\mathbf{F}_F\colon C_R \to \mathrm{Grp}$ from the category of commutative R-algebras to the category of groups.

More precisely, for each object $S \in \mathrm{Ob}\, C_R$, it associates $\mathbf{F}_F(S) = (\mathrm{Nil}(S)^n, +_F) \in \mathrm{Ob}\,\mathrm{Grp}$, which is analogous to the structure defined in the Lemma 5.7.

Also, for each homomorphism $\phi\colon S_1 \to S_2$ where $S_1, S_2 \in \mathrm{Ob}\, C_R$, we know that $\phi(\mathrm{Nil}(S_1)^n) \subseteq \mathrm{Nil}(S_2)^n$ by Lemma 2.10. Take $x, y \in \mathrm{Nil}(S_1)^n$ where x means $(x_1, x_2, \cdots, x_n)$ and y means $(y_1, y_2, \cdots, y_n)$. Let $m \in \mathbb{N}$ such that $x^m = (0, 0, \cdots, 0) = v^m$. We want to show that $\phi(x +_F y) = \phi(x) +_F \phi(y)$. So we have

$$\begin{aligned}
\phi(x +_F y) &= \phi\left(\sum_{i,j=0}^{\infty} a_{ij} x^i y^j\right) \\
&= \phi\left(\sum_{i,j=0}^{m} a_{ij} x^i y^j\right) \\
&= \sum_{i,j=0}^{m} \phi\left(a_{ij} x^i y^j\right) \\
&= \sum_{i,j=0}^{m} a_{ij} \phi\left(x^i y^j\right) \\
&= \sum_{i,j=0}^{m} a_{ij} \phi\left(x^i\right) \phi\left(y^j\right) \\
&= \sum_{i,j=0}^{\infty} a_{ij} \phi\left(x^i\right) \phi\left(y^j\right) \\
&= \sum_{i,j=0}^{\infty} a_{ij} \phi(x)^i \phi(y)^j \\
&= \phi(x) +_F \phi(y).
\end{aligned}$$

Hence, it is proved that $\phi(u +_F v) = \phi(u) +_F \phi(v)$. For a ring homomorphism $\phi\colon S_1 \to S_2$, define $\mathbf{F}_F(\phi)$ as ϕ restricted to $\mathrm{Nil}(S_1)^n$. Hence, each homomorphism $\phi\colon S_1 \to S_2$ where $S_1, S_2 \in \mathrm{Ob}\, C_R$ is sent by the functor to a homomorphism $\mathbf{F}_F(\phi)\colon \mathbf{F}_F(S_1) \to \mathbf{F}_F(S_2)$.

In addition, it is straightforward to verify that $\mathbf{F}_F(\mathrm{Id}_X) = \mathrm{Id}_{\mathbf{F}_F(X)}$ for each object $X \in \mathrm{Ob}\, C_R$. Also, it is true that for a homomorphism $\phi\colon S_1 \to S_2$ and a homomorphism $\eta\colon S_2 \to S_3$, we have $\mathbf{F}_F(\phi \circ \eta) = \mathbf{F}_F(\phi) \circ \mathbf{F}_F(\eta)$.

Therefore, we have showed that the formal group law F defines a functor $\mathbf{F}_F\colon C_R \to \mathrm{Grp}$ from the category of commutative R-algebras to the category of groups.

Furthermore, we will be talking about the natural transformation given by the morphisms of formal group laws of functors.

Let $\mathbf{F}_F, \mathbf{F}_G$ be functors from the category C_R to the category Grp. A natural transformation $f : F \to G$, consists of a collection of morphisms $f_S : \mathbf{F}_F(S) \to \mathbf{F}_G(S)$, one for each $S \in \mathrm{Ob}\, C_R$, such that for any $S_1, S_2 \in C_R$ and any morphisms $\phi\colon S_1 \to S_2$, the diagram

$$\begin{array}{ccc} \mathbf{F}_F(S_1) & \xrightarrow{f_{S_1}} & \mathbf{F}_G(S_1) \\ {\scriptstyle \mathbf{F}_F(\phi)}\downarrow & & \downarrow{\scriptstyle \mathbf{F}_G(\phi)} \\ \mathbf{F}_F(S_2) & \xrightarrow[f_{S_2}]{} & \mathbf{F}_G(S_2) \end{array} \tag{6.1}$$

commutes; that is, $f_{S_2} \circ \mathbf{F}_F(\phi) = \mathbf{F}_G(\phi) \circ f_{S_1}$.

Let f be a homomorphism between the formal group laws F of dimension m and G of dimension n. Thus, $f = (f_1, \ldots, f_n)$ is a collection of power series with no constant term $f_i(t_1, t_2, \cdots, t_m) = \sum_{i_1,i_2,\cdots,i_n=0}^{\infty} a^{(i)}_{i_1,i_2,\cdots,i_n} r_1^{i_1}, r_2^{i_2} \in R[\![t_1, t_2, \cdots, t_m]\!]$ satisfying $f(F(X, Y)) = G(f(X), f(Y))$ where $1 \geq i \geq m$. For $f\colon F \to G$, define $f_S\colon \mathrm{Nil}(S)^m \to \mathrm{Nil}(S)^n$ as $f_S(r) = f(r)$ for $r \in \mathrm{Nil}(S)^n$ where r stands for $(r_1, r_2, \cdots, r_n)$, and we want to show the map f_S is well-defined. Take $r \in \mathrm{Nil}(S)^n$ such that $r^k = 0$ for some integer k, then we have

$$\begin{aligned} f_S(r) = \Bigg(&\sum_{i_1,i_2,\cdots,i_n=0}^{\infty} a^{(1)}_{i_1,i_2,\cdots,i_n} r_1^{i_1}, r_2^{i_2}, \cdots, r_n^{i_n}, \sum_{i_1,i_2,\cdots,i_n=0}^{\infty} a^{(2)}_{i_1,i_2,\cdots,i_n} r_1^{i_1}, r_2^{i_2}, \cdots, r_n^{i_n}, \cdots \\ &\sum_{i_1,i_2,\cdots,i_n=0}^{\infty} a^{(m)}_{i_1,i_2,\cdots,i_n} r_1^{i_1}, r_2^{i_2}, \cdots, r_n^{i_n} \Bigg) = \Bigg(\sum_{i_1,i_2,\cdots,i_n=0}^{k} a^{(1)}_{i_1,i_2,\cdots,i_n} r_1^{i_1}, r_2^{i_2}, \cdots, r_n^{i_n}, \\ &\sum_{i_1,i_2,\cdots,i_n=0}^{k} a^{(2)}_{i_1,i_2,\cdots,i_n} r_1^{i_1}, r_2^{i_2}, \cdots, r_n^{i_n}, \cdots, \sum_{i_1,i_2,\cdots,i_n=0}^{k} a^{(m)}_{i_1,i_2,\cdots,i_n} r_1^{i_1}, r_2^{i_2}, \cdots, r_n^{i_n} \Bigg) \in \mathrm{Nil}(S)^n. \end{aligned}$$

Hence, the map f_S is well-defined.

Next we want to show that f_S is a group homomorphism from the group $\mathbf{F}_F(S) = (\mathrm{Nil}(S)^m, +_F)$ to the group $\mathbf{F}_G(S) = (\mathrm{Nil}(S)^n, +_G)$. To claim it is a homomorphism, we need show that for any $u, v \in \mathrm{Nil}(S)^m$, it is true that $f(x +_F y) = f(x) +_G f(y)$. By our definition of f, it is already correct that $f(x +_F y) = f(x) +_G f(y)$.

Hence, we have showed that f_S is a group homomorphism from $\mathbf{F}_F(S)$ to $\mathbf{F}_G(S)$.

Proposition 6.9. *If $f\colon F \to G$ is a homomorphism of higher-dimensional formal group laws over R, then $(f_S)_{S \in \mathrm{Ob}\, C_R}$ is a natural transformation from $\mathbf{F}_F$ to $\mathbf{F}_G$.*

Proof. To prove the proposition is true, we need to show diagram (6.1) commutes. More specifically, take $r \in \mathbf{F}_F(S_1)$ such that $r^n = 0$, we need to prove that $\mathbf{F}_G(\phi) \circ f_{S_1}(r) = f_{S_2} \circ \mathbf{F}_F(\phi)(r)$. Write f as $f = \sum_{i=0}^{\infty} a_i t^i$. Since ϕ is a ring homomorphism, we have

$$\begin{aligned} \mathbf{F}_G(\phi) \circ f_{S_1}(r) &= \mathbf{F}_G(\phi) \circ f(r) \\ &= \phi(f(r)) \end{aligned}$$

$$
\begin{aligned}
&= \phi\left(\sum_{i=0}^{\infty} a_i r^i\right) \\
&= \phi\left(\sum_{i=0}^{n} a_i r^i\right) \\
&= \sum_{i=0}^{n} \phi(a_i r^i) \\
&= \sum_{i=0}^{n} a_i \phi(r^i) \\
&= \sum_{i=0}^{\infty} a_i \phi(r^i) \\
&= \sum_{i=0}^{\infty} a_i \phi(r)^i \\
&= f(\phi(r)) \\
&= f_{S_2} \circ \phi(r) \\
&= f_{S_2} \circ \mathbf{F}_F(\phi)(r).
\end{aligned}
$$

Hence, the diagram (6.1) commutes and the result follows. □

6.3. **Change of Base Rings with Higher-Dimensional FGLs.** Suppose $F \in R[\![t_1, t_2, \cdots, t_n]\!]$ is a higher-dimensional formal group law over R and $\phi\colon R \to R^{'}$ is a ring homomorphism of commutative rings. Take $u, v \in R$, then write F as

$$
F(u,v) = \left(\sum_{i_1,\cdots,i_n,j_1,\cdots,j_n=0}^{\infty} a^{(1)}_{i_1,\cdots,i_n,i_1,\cdots,i_n} u_1^{i_1}, \cdots, u_n^{i_n}, v_1^{j_1}, \cdots, v_n^{j_n}, \cdots, \right.
\left. \sum_{i_1,\cdots,i_n,j_1,\cdots,j_n=0}^{\infty} a^{(m)}_{i_1,\cdots,i_n,i_1,\cdots,i_n} u_1^{i_1}, \cdots, u_n^{i_n}, v_1^{j_1}, \cdots, v_n^{j_n} \right)
$$

and define

$$
F^{'}(u,v) = \left(\sum_{i_1,\cdots,i_n,j_1,\cdots,j_n=0}^{\infty} \phi(a^{(1)}_{i_1,\cdots,i_n,i_1,\cdots,i_n}) u_1^{i_1}, \cdots, u_n^{i_n}, v_1^{j_1}, \cdots, v_n^{j_n}, \cdots, \right.
\left. \sum_{i_1,\cdots,i_n,j_1,\cdots,j_n=0}^{\infty} \phi(a^{(m)}_{i_1,\cdots,i_n,i_1,\cdots,i_n}) u_1^{i_1}, \cdots, u_n^{i_n}, v_1^{j_1}, \cdots, v_n^{j_n} \right)
$$

Now we want to check that $F^{'} \in R^{'}[\![t_1, t_2, \cdots, t_n]\!]$ is a formal group law over the ring $R^{'}$. The proof for higher-dimensional FGLs is analogous to the one-dimensional case, as proved in Section 5.3.

Finally we can get the conclusion that $\mathbf{F}_\phi$ is a functor from FGL_R to $\mathrm{FGL}_{R'}$.

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