SYMMETRY AND DUALITY IN MONOIDAL CATEGORIES

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Abstract. These are lecture notes for a minicourse given at the Categorification in Representation Theory Workshop at the University of Sydney, February 6–10, 2023. We examine how the concepts of symmetry and duality can be formulated in the context of monoidal categories. We begin by introducing the technique of string diagrams. Then we examine the possible types of dualities and symmetries that can arise in monoidal categories, using intuitive diagrammatic arguments to deduce some of their basic properties. All of this will lead us naturally to oriented and unoriented Brauer categories, which are intimately related to the representation theory of Lie groups. We conclude with a brief discussion of further directions into the realm of supergroups.

Contents

1. Introduction

The goal of these notes is to introduce the reader to the notions of symmetry and duality in the context of monoidal categories, with an aim towards the study of representations of Lie groups and Lie algebras. Throughout, we will use the language of string diagrams for monoidal categories. (For a more detailed exposition of this formalism, we refer the reader to [TV17, Sel11].) We will motivate the definitions of the oriented and unoriented Brauer categories by examining how different types of duality interact with symmetry. This will allow us to give efficient presentations of categories with nice universality properties. We then explain the connection between these categories and the representation theory of the classical Lie groups.

Throughout these notes, $\Bbbk$ denotes a field, and $\mathbb{N} = \mathbb{Z}_{\geq 0}$ is the set of nonnegative integers.

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2. Strict monoidal categories and string diagrams

In this section we briefly review the definition of strict linear monoidal categories. These will be our main tool for exploring the concepts of symmetry and duality. We then discuss string diagrams, which are, in many ways, the best language to use to work with monoidal categories.

2.1. **Definitions.** Throughout this document, all categories are assumed to be locally small. In other words, we have a *set* of morphisms between any two objects.

A *strict monoidal category* is a category $\mathcal{C}$ equipped with

- a bifunctor (the *tensor product*) $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and
- a *unit object* $\mathbb{1}$

such that, for all objects X, Y, and Z of $\mathcal{C}$, we have

- $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and
- $\mathbb{1} \otimes X = X = X \otimes \mathbb{1}$,

and, for all morphisms f, g, and h of $\mathcal{C}$, we have

- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ and
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

Here, and throughout the document, 1_X denotes the identity endomorphism of an object X.

Remark 2.1. Note that, in a (not necessarily strict) *monoidal category*, the equalities above are replaced by isomorphism, and one imposes certain coherence conditions. For example, let $\mathcal{V}ec_{\Bbbk}$ be the category of finite-dimensional $\Bbbk$-vector spaces. In this category one has isomorphisms $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$, but these isomorphisms are not equalities in general. Similarly, the unit object in this category is the one-dimensional vector space $\Bbbk$, and we have $\Bbbk \otimes V \cong V \cong V \otimes \Bbbk$ for any vector space V.

We will be building monoidal categories "from scratch" via generators and relations. Thus, we are free to require them to be strict. In general, Mac Lane's coherence theorem for monoidal categories asserts that every monoidal category is monoidally equivalent to a strict one. (For a proof of this fact, see [Mac98, §VII.2] or [Kas95, §XI.5].) So, in practice, we do not lose much by assuming monoidal categories are strict. (See also [Sch01].)

A $\Bbbk$-*linear category* is a category $\mathcal{C}$ such that

- for any two objects X and Y of $\mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a $\Bbbk$-module,
- composition of morphisms is bilinear:
$$f \circ (\alpha g + \beta h) = \alpha(f \circ g) + \beta(f \circ h),$$
$$(\alpha f + \beta g) \circ h = \alpha(f \circ h) + \beta(g \circ h),$$
for all $\alpha, \beta \in \Bbbk$ and morphisms f, g, and h such that the above operations are defined.

The category $\mathcal{V}ec_{\Bbbk}$ is an example of a $\Bbbk$-linear category. For any two $\Bbbk$-modules M and N, the space $\mathrm{Hom}_{\Bbbk}(M, N)$ is again a $\Bbbk$-module under the usual pointwise operations. Composition is bilinear with respect to this $\Bbbk$-module structure.

A *strict* $\Bbbk$-*linear monoidal category* is a category that is both strict monoidal and $\Bbbk$-linear, and such that the tensor product of morphisms is $\Bbbk$-bilinear. Before discussing some examples, we mention the important *interchange law*. Suppose

$$X_1 \xrightarrow{f} X_2 \qquad \text{and} \qquad Y_1 \xrightarrow{g} Y_2$$

are morphisms in a strict $\Bbbk$-linear monoidal category $\mathcal{C}$. Then

$$(1_{X_2} \otimes g) \circ (f \otimes 1_{Y_1}) = \otimes((1_{X_2}, g)) \circ \otimes((f, 1_{Y_1})) = \otimes((1_{X_2}, g) \circ (f, 1_{Y_1})) = \otimes((f, g)) = f \otimes g,$$

where the second equality uses that the tensor product is a bifunctor. Similarly,

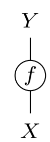
$$(f \otimes 1_{Y_2}) \circ (1_{X_1} \otimes g) = f \otimes g.$$

Thus, the following diagram commutes:

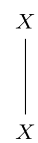
$$\begin{tikzcd}
X_1 \otimes Y_1 \arrow[r, "1\otimes g"] \arrow[d, "f\otimes 1"'] \arrow[dr, "f\otimes g"] & X_1 \otimes Y_2 \arrow[d, "f\otimes 1"] \\
X_2 \otimes Y_1 \arrow[r, "1\otimes g"'] & X_2 \otimes Y_2
\end{tikzcd}$$

2.2. **String diagrams.** Strict monoidal categories are especially well suited to being depicted using the language of *string diagrams*. These diagrams, which are sometimes also called *Penrose diagrams*, have their origins in work of Roger Penrose in physics [Pen71]. Working with strings diagrams helps build intuition. It also often makes certain arguments obvious, whereas the corresponding algebraic proof can be a bit opaque. We give here a brief overview of string diagrams, referring the reader to [TV17, Ch. 2] for a detailed treatment. Throughout this section, $\mathcal{C}$ will denote a strict $\Bbbk$-linear monoidal category.

We will denote a morphism $f\colon X \to Y$ by a strand with a coupon labeled f:

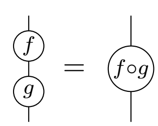


Note that we are adopting the convention that diagrams should be read from bottom to top. The *identity map* $1_X\colon X \to X$ is a string with no coupon:

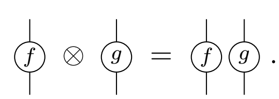


We sometimes omit the object labels (e.g. X and Y above) when they are clear or unimportant. We will also sometimes distinguish identity maps of different objects by some sort of decoration of the string (orientation, dashed versus solid, etc.), rather than by adding object labels.

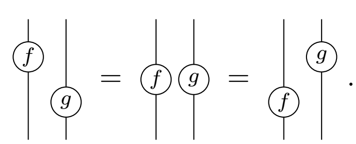
Composition is denoted by *vertical stacking* (recall that we read pictures bottom to top) and tensor product is *horizontal juxtaposition*:



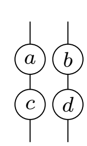
and



The *interchange law* then becomes

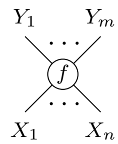


This graphical interpretation of the interchange law is one of the main reasons that the two-dimensional notation of string diagrams works so well for monoidal categories. Much as we may omit parenthesis when multiplying several elements in an associative algebra, string diagrams allow us to draw a single diagram



without specifying if this denotes $(a\otimes b)\circ(c\otimes d)$ or $(a\circ c)\otimes(b\circ d)$, since both expressions are equal.

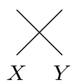
A general morphism $f\colon X_1 \otimes \cdots \otimes X_n \to Y_1 \otimes \cdots \otimes Y_m$ can be depicted as a coupon with n strands emanating from the bottom and m strands emanating from the top:



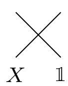
3. Symmetry in monoidal categories

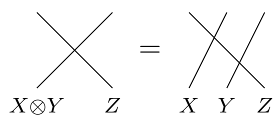
In this section, we discuss how the concept of symmetry can be formalized in the language of monoidal categories. Using string diagrams, the notion becomes quite intuitive.

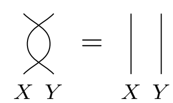
3.1. Symmetric monoidal categories. A *strict symmetric monoidal category* is a strict monoidal category $\mathcal{C}$ equipped with a family of isomorphisms

(3.1)  $: X \otimes Y \to Y \otimes X, \qquad X, Y \in \mathcal{C},$

that are natural in X and Y, and such that

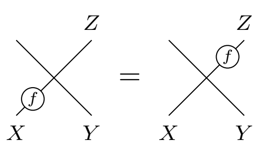
(3.2a)  $= 1_X,$

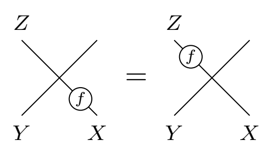
(3.2b)  ,

(3.2c)  ,

for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$. We will refer to the isomorphisms (3.1) as *crossings*.

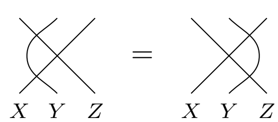
The requirement that the crossings (3.1) are natural in X and Y means that

(3.3a)  ,

(3.3b)  ,

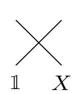
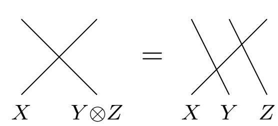
for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$. In fact, it is enough to require (3.3a), since then attaching a crossing to the top and bottom of both sides of the relation, then using (3.2c), gives (3.3b).

Taking f in (3.3a) to be a crossing, we see that

(3.4) 

for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$. This is called the *braid relation*.

Exercise 3.1. Suppose $\mathcal{C}$ is a strict symmetric monoidal category. Show that

 $= 1_X$ and 

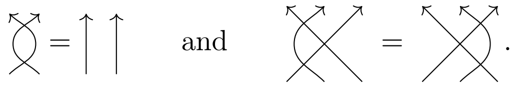
for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

3.2. **A universal symmetric monoidal category.** Define $\mathcal{S}ym$ to be the strict $\Bbbk$-linear monoidal category with:

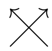
- one generating object $\uparrow$;
- one generating morphism

 $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow;$

- and relations

(3.5) 

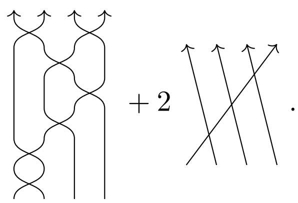
One could write these relations in a more traditional algebraic manner, if so desired. For example, if we let

$s =$  $: \uparrow \otimes \uparrow \ \to \ \uparrow \otimes \uparrow,$

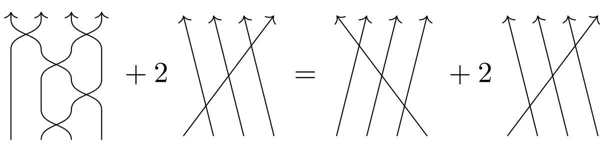
then the two relations (3.5) become

$$s^2 = 1_{\uparrow\otimes\uparrow} \qquad \text{and} \qquad (s \otimes 1_\uparrow) \circ (1_\uparrow \otimes s) \circ (s \otimes 1_\uparrow) = (1_\uparrow \otimes s) \circ (s \otimes 1_\uparrow) \circ (1_\uparrow \otimes s).$$

The objects of $\mathcal{S}ym$ are $\uparrow^{\otimes n}$, $n \in \mathbb{N}$. An example of an endomorphism of $\uparrow^{\otimes 4}$ is



Using the relations, we see that this morphism is equal to



Fix a positive integer n and recall that the group algebra $\Bbbk\mathfrak{S}_n$ of the symmetric group on n letters has a presentation with generators $s_1, s_2, \dots, s_{n-1}$ (the simple transpositions) and relations

$$s_i^2 = 1, \qquad 1 \le i \le n-1, \tag{3.6}$$

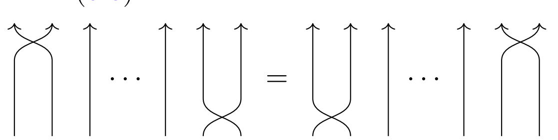
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad 1 \le i \le n-2, \tag{3.7}$$

$$s_i s_j = s_j s_i, \qquad 1 \le i, j \le n-1,\ |i-j| > 1. \tag{3.8}$$

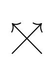
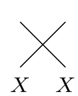
We have an isomorphism of algebras

$$\Bbbk\mathfrak{S}_n \to \mathrm{End}_{\mathcal{S}ym}(\uparrow^{\otimes n}) \tag{3.9}$$

where s_i is sent to the crossing of the i-th and $(i+1)$-st strands, labeled from right to left. Note that the "distant braid relation" (3.8) follows for free from the interchange law:



The category $\mathcal{S}ym$ has a universal property; it is the free $\Bbbk$-linear symmetric monoidal category on one object. More precisely, if X is an object in a symmetric monoidal category $\mathcal{C}$, then there exists a unique monoidal functor

$\mathcal{S}ym \to \mathcal{C}, \qquad \uparrow \mapsto X, \qquad$  $\mapsto$  .

It then follows from (3.9) that we have a homomorphism of algebras

$$\Bbbk\mathfrak{S}_n \to \mathrm{End}_{\mathcal{C}}(X^{\otimes n}).$$

For example, suppose $\mathcal{C} = \mathcal{V}ec_{\Bbbk}$ is the category of finite-dimensional vector spaces over $\Bbbk$. For any two vector spaces V and W, define

$$\mathrm{flip}_{V,W} \colon V \otimes W \to W \otimes V, \qquad v \otimes w \mapsto w \otimes v. \tag{3.10}$$

Then there exists a unique monoidal functor

$\mathcal{S}ym \to \mathcal{V}ec_{\Bbbk}, \qquad \uparrow \mapsto V, \qquad$  $\mapsto \mathrm{flip}_{V,V},$

and we have a homomorphism of algebras

$$\Bbbk\mathfrak{S}_n \to \mathrm{End}_{\Bbbk}(V^{\otimes n}).$$

4. Duality in monoidal categories

We now turn our attention to the concept of duality in monoidal categories. We begin with a general definition of what it means for two objects to be dual. We then examine the interplay between duality and symmetry. We will see that, in categories possessing symmetry and duality, we can define categorical notions of trace and dimension.

4.1. Dual in monoidal categories.

4.1. Duals in monoidal categories. Suppose a strict monoidal category has two objects $\uparrow$ and $\downarrow$. Recalling our convention that we do not draw the identity morphism of the unit object $\mathbb{1}$, a morphism $\downarrow \otimes \uparrow \to \mathbb{1}$ would have string diagram

 $\colon \downarrow \otimes \uparrow \to \mathbb{1},$

where we may decorate the cap with some symbol if we have more than one such morphism. The fact that the top of the diagram is empty space indicates that the codomain of this morphism is the unit object $\mathbb{1}$. Similarly, we can have

 $\colon \mathbb{1} \to \uparrow \otimes \downarrow.$

We say that $\downarrow$ is *left dual* to $\uparrow$ (and $\uparrow$ is *right dual* to $\downarrow$) if we have morphisms

 $\colon \downarrow \otimes \uparrow \to \mathbb{1} \qquad$ and $\qquad$  $\colon \mathbb{1} \to \uparrow \otimes \downarrow$

such that

(4.1) 

The morphisms  and  are called the *unit* and the *counit*, respectively, of the duality. (The relations (4.1) are a generalization of the unit-counit formulation of adjunction of functors.) A monoidal category in which every object has both left and right duals is called a *rigid*, or *autonomous*, category.

If $\uparrow$ and $\downarrow$ are both left and right dual to each other, then, in addition to the above, we also have

 $\colon \uparrow \otimes \downarrow \to \mathbb{1} \qquad$ and $\qquad$  $\colon \mathbb{1} \to \downarrow \to \uparrow$

such that

(4.2) 

To give a concrete example of duality in a monoidal category, consider the category $\mathcal{V}ec_{\Bbbk}$ of finite-dimensional $\Bbbk$-vector spaces. (The category $\mathcal{V}ec_{\Bbbk}$ is not strict, but this will not cause any problems for us. See Remark 2.1.) In this category, the unit object is $\Bbbk$. We claim that, if V is any finite-dimensional $\Bbbk$-vector space, the dual vector space V^* is left and dual to V in the sense defined above.

Define the *evaluation map*

$$\mathrm{ev}_V \colon V^* \otimes V \to \Bbbk, \qquad f \otimes v \mapsto f(v), \tag{4.3}$$

and the *coevaluation map*

$$\mathrm{coev}_V \colon \Bbbk \to V \otimes V^*, \qquad 1 \mapsto \sum_{v \in \mathbf{B}_V} v \otimes \delta_v, \tag{4.4}$$

where $\mathbf{B}_V$ is a basis of V and $\{\delta_v : v \in \mathbf{B}_V\}$ is the dual basis of V^*.

Taking $\uparrow = V$ and $\downarrow = V^*$, we define

$$= \mathrm{ev}_V \qquad \text{and} \qquad = \mathrm{coev}_V\,.$$

Let us check the left-hand relation in (4.1). The left-hand side is the composition

$$V \cong \Bbbk \otimes V \xrightarrow{\ \otimes 1_V} V \otimes V^* \otimes V \xrightarrow{1_V \otimes\ } V \otimes \Bbbk \cong V,$$

$$w \mapsto 1 \otimes w \mapsto \sum_{v \in \mathbf{B}_V} v \otimes \delta_v \otimes w \mapsto \sum_{v \in \mathbf{B}_V} \delta_v(w) \otimes v \mapsto \sum_{v \in \mathbf{B}_V} \delta_v(w) v = w.$$

Thus, this composition is precisely the identity map 1_V, and so the right-hand relation in (4.1) is satisfied. The verification of the left-hand equality in (4.1) is analogous and is left as an exercise for the reader.

In fact, V^* is also *right* dual to V. This can be shown directly by computations analogous to those above, or can be seen as a consequence of the more general result in symmetric monoidal categories (Proposition 4.5).

Exercise 4.1. Show that the coevaluation map coev_V defined in (4.4) is independent of the choice of basis $\mathbf{B}_V$.

From now on, cups and caps in string diagrams will always denote units and counits giving the data of duality between objects.

Exercise 4.2. Units and counits are not unique. For instance, if we fix $\alpha \in \Bbbk^\times$, then $\alpha\,\mathrm{coev}_V$ and $\alpha^{-1}\,\mathrm{ev}_V$ are also units and counits expressing that V^* is left dual to a finite-dimensional vector space V. However, fixing the unit uniquely determines the counit, and vice versa. Indeed, let $\uparrow$ and $\downarrow$ be objects in a monoidal category $\mathcal{C}$.

(a) Suppose that $\colon \downarrow \otimes \uparrow \to \mathbb{1}$ is a morphism in $\mathcal{C}$. Show that there exists at most one morphism $\colon \mathbb{1} \to \uparrow \otimes \downarrow$ satisfying (4.1).

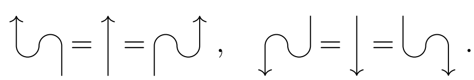
(b) Suppose that $\colon \mathbb{1} \to \uparrow \otimes \downarrow$ is a morphism in $\mathcal{C}$. Show that there exists at most one morphism $\colon \downarrow \otimes \uparrow \to \mathbb{1}$ satisfying (4.1).

4.2. **A universal category with duals.** The *oriented Temperley–Lieb category* $\mathcal{OTL}_\Bbbk$ is the strict $\Bbbk$-linear monoidal category with

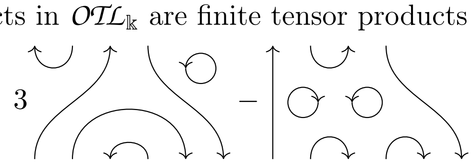
- two generating objects $\uparrow$ and $\downarrow$;
- four generating morphisms,

$$\colon \downarrow \otimes \uparrow \to \mathbb{1}, \quad \colon \mathbb{1} \to \uparrow \otimes \downarrow, \quad \colon \uparrow \otimes \downarrow \to \mathbb{1}, \quad \colon \mathbb{1} \otimes \downarrow \otimes \uparrow;$$

- and relations



Objects in $\mathcal{OTL}_\Bbbk$ are finite tensor products of $\uparrow$ and $\downarrow$. An example of a morphism in $\mathcal{OTL}_\Bbbk$ is



$$\in \mathrm{Hom}_{\mathcal{OTL}_\Bbbk}(\uparrow \otimes \uparrow \otimes \downarrow \otimes \uparrow \otimes \downarrow \otimes \downarrow, \uparrow \otimes \downarrow \otimes \uparrow \otimes \downarrow).$$

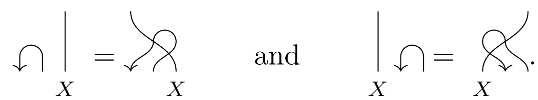
The category $\mathcal{OTL}_\Bbbk$ is the free $\Bbbk$-linear monoidal category on one object with a two-sided dual. If X and Y are two objects in a $\Bbbk$-linear monoidal category $\mathcal{C}$ that are both left and right dual to each other, then there exists a unique monoidal functor

$$\mathcal{OTL}_\Bbbk \to \mathcal{C}, \qquad \uparrow \mapsto X, \qquad \downarrow \mapsto Y,$$

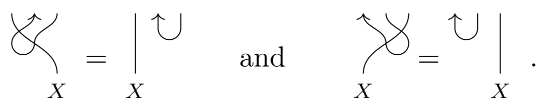
sending , , , and to the units and counits of the dualities between X and Y.

Exercise 4.3. Show that $\mathcal{OTL}_\Bbbk$ is a rigid monoidal category. *Hint*: To show that arbitrary objects have duals, nest cups and caps.

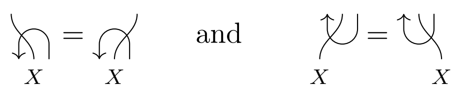
4.3. **Duals in symmetric monoidal categories.** We now examine how duality behaves in symmetric monoidal categories. Throughout this subsection we work in a rigid strict symmetric monoidal category $\mathcal{C}$. Suppose $\uparrow$ is an object of $\mathcal{C}$ with left dual $\downarrow$. For any object X in $\mathcal{C}$, taking f in (3.3) to be gives

(4.5)


Similarly, taking f in (3.3) to be gives

(4.6)


Exercise 4.4. Use (4.5) and (4.6) to show that

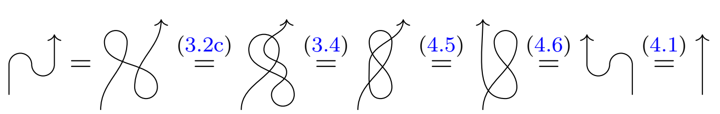
(4.7)


for every object X.

Proposition 4.5. *Suppose $\mathcal{C}$ is a symmetric monoidal category containing an object $\downarrow$ left dual to an object $\uparrow$. Then $\downarrow$ is also right dual to $\uparrow$.*

Proof. Define

We claim that these morphisms satisfy (4.2). Indeed, for the left-hand relation in (4.2), we have



The proof of the right-hand relation in (4.2) is analogous; we simply reflect all our diagrams in the horizontal axis. $\square$

Proposition 4.5 implies that a symmetric monoidal category is rigid if and only if all objects have left duals. From now on, when working in a rigid symmetric monoidal category, for every object $\uparrow$ with dual $\downarrow$, we assume that the duality data is related by

(4.8)


Exercise 4.6. Using (4.8), show that

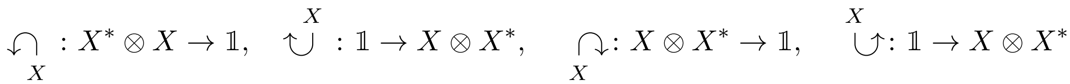
(4.9)


Since we assume that $\mathcal{C}$ is a rigid symmetric monoidal category, it follows from Proposition 4.5 that, for every object X in $\mathcal{C}$, there is an object X^* that is both left and right dual to X.

We will denote the identity endomorphisms of X and X^* by upward and downward strands labeled X:

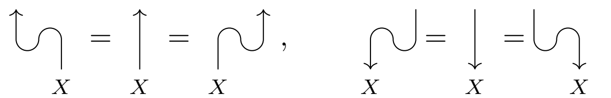
$$1_X = \underset{X}{\uparrow} \qquad \text{and} \qquad 1_{X^*} = \underset{X}{\downarrow}\ .$$

So we have morphisms

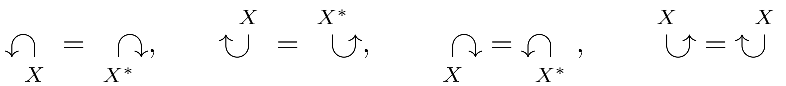


such that

(4.10)

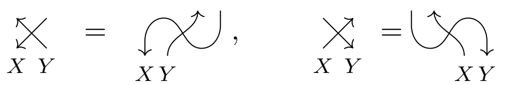


Furthermore, we may choose our cups and caps such that

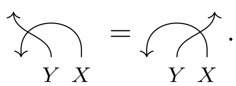


Proposition 4.7. *For all objects X and Y in $\mathcal{C}$, we have*

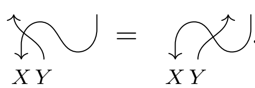
(4.11)



Proof. By the right-hand relation in (4.7), we have



Attaching a left cup to the rightmost bottom endpoint gives



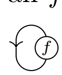
Using (4.10) to straighten the string on the left-hand side then gives the left-hand relation in (4.11). The proof of the right-hand relation in (4.11) is analogous. $\square$

4.4. **Trace and categorical dimension.** Throughout this subsection we work in a strict $\Bbbk$-linear rigid symmetric monoidal category $\mathcal{C}$. Furthermore, we assume that we have an isomorphism of $\Bbbk$-algebras

$$\Bbbk \xrightarrow{\cong} \mathrm{End}_{\mathcal{C}}(\mathbb{1}), \quad \alpha \mapsto \alpha 1_{\mathbb{1}}. \tag{4.12}$$

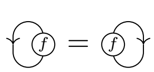
For instance, (4.12) is satisfied in any category of modules over a Hopf algebra, where $\mathbb{1}$ is the one-dimensional trivial module. We use the isomorphism (4.12) to identify $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ with $\Bbbk$.

Suppose $\uparrow$ is an object with dual $\downarrow$. Then, for all $f \in \mathrm{End}_{\mathcal{C}}(\uparrow)$, we may form the closed diagram

(4.13) $\mathrm{tr}(f) :=$  $\in \Bbbk,$

which we call the *trace* of f.

Exercise 4.8. Show that



for all $f \in \mathrm{End}_{\mathcal{C}}(\uparrow)$.

To justify the use of the term *trace*, consider the category $\mathcal{V}ec_{\Bbbk}$ of finite-dimensional vector spaces over $\Bbbk$. If $f \in \operatorname{End}_{\Bbbk}(V)$, then the diagram (4.13) is the composite

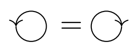
$$\Bbbk \longrightarrow V^* \otimes V \xrightarrow{1_{V^*}\otimes f} V^* \otimes V \longrightarrow \Bbbk,$$
$$1 \mapsto \sum_{v \in \mathbf{B}_V} \delta_v \otimes v \mapsto \sum_{v \in \mathbf{B}_V} \delta_v \otimes f(v) \mapsto \sum_{v \in \mathbf{B}_V} \delta_v(f(v)).$$

The sum $\sum_{v\in\mathbf{B}_V} \delta_v(f(v))$ is the usual trace of a linear operator f. Thus, under the isomorphism (4.12), the diagram (4.13) corresponds to the trace.

In the category $\mathcal{V}ec_{\Bbbk}$, the trace of the identity map of a vector space V is its dimension. Thus, in a general rigid symmetric monoidal category, we define the *categorical dimension* of an object $\uparrow$ to be

$\dim_{\mathcal{C}}(\uparrow) :=$  $\in \Bbbk.$

It follows from Exercise 4.8 that



and so the categorical dimension of an object is equal to the categorical dimension of its dual.

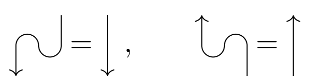
5. The oriented Brauer category

Our goal in this section is to define the free $\Bbbk$-linear rigid symmetric monoidal category on a single object. We will spend some time motivating the definition. Once we have arrived at a concise definition of this category, we will examine applications to representation theory.

5.1. **Motivating the definition.** Let us call the generating object $\uparrow$. Since we want our category to be rigid, this object has a left dual $\downarrow$, together with morphisms

 $: \downarrow \otimes \uparrow \to \mathbb{1}, \qquad$  $: \mathbb{1} \to \uparrow \otimes \downarrow,$

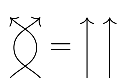
such that

(5.1) 

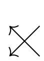
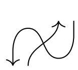
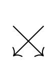
We know from Proposition 4.5 that, as long as our category is symmetric monoidal, $\downarrow$ will also be right dual to $\uparrow$. So we now focus on making our category symmetric monoidal. We begin by adding another generating morphism

 $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow$

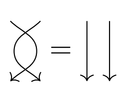
satisfying

(5.2) 

We also need morphisms , , and , subject to similar relations. However, we would like to give as efficient a presentation of our category as possible. Instead of introducing all of these crossings as generating morphisms, Proposition 4.7 suggests that we should *define*

(5.3)  $:=$  and then  $:=$  .

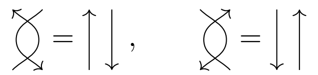
Exercise 5.1. Using the left cup and cap to rotate the relation (5.2), show that



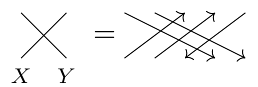
We are still missing a right crossing , which we cannot obtain from as in (4.11), since we do not have right cups and caps. Similarly, we cannot define the right cup and cap as in (4.8) since we do not have a right crossing! So we must introduce another generating morphism

 $: \uparrow \otimes \downarrow \to \downarrow \otimes \uparrow,$

and impose the relations

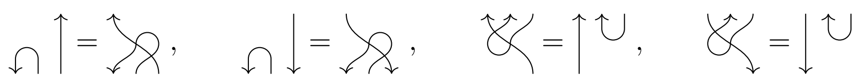
(5.4) 

We now have the relations (3.2c) for all choices of $X, Y \in \{\uparrow, \downarrow\}$. We can then use (3.2a) and (3.2b) to define crossings between arbitrary tensor products of $\uparrow$ and $\downarrow$ (including the empty tensor product, which is $\mathbb{1}$). For instance, for $X = \uparrow \otimes \uparrow \otimes \downarrow$ and $Y = \downarrow \otimes \downarrow$, we have

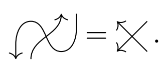


To obtain a symmetric monoidal category, it remains to consider the naturality of the crossings. To do this we need the relation (3.3a) as f ranges over all generating morphisms and $Y \in \{\uparrow, \downarrow\}$. Let us begin with the cups and caps. In this case, naturality already follows from the relations that we have imposed.

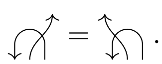
Lemma 5.2. *We have*

(5.5) 

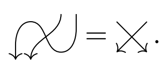
Proof. By definition, we have



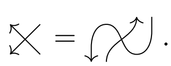
Tensoring on the right with $\uparrow$, then composing on the top with $\uparrow$ and using (5.1), gives



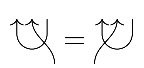
Composing on the bottom with $\uparrow$ and using (5.2) then gives the first relation in (5.5). The proof of the second relation in (5.5) is similar, starting from



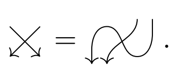
Now start again with



Tensor on the left $\uparrow$, then compose on the bottom with $\uparrow$ and use (5.1), to obtain

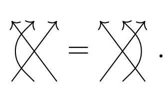


Then compose on the top with $\uparrow$ and use (5.2) to obtain the third relation in (5.5). The proof of the last relation in (5.5) is analogous, starting from



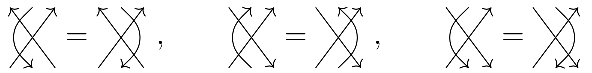
□

It remains to consider the naturality relation (3.3a) when $f \in \{\,,\,\}$ and $Y \in \{\uparrow, \downarrow\}$. We cannot obtain any of these relations from what we have so far. So we impose an additional relation

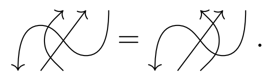
(5.6) 

As the next result shows, this is enough to deduce the rest of the cases.

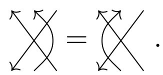
Lemma 5.3. *We have*

(5.7) 

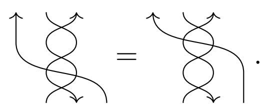
Proof. Starting with (5.6), we tensor on the left and right with $\downarrow$, then compose on the top with $\uparrow\uparrow\downarrow$ and on the bottom with $\downarrow\uparrow\uparrow$, to obtain



Using (5.3) then gives



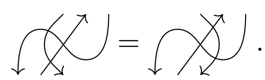
Composing on the top with $\uparrow$ and on the bottom with $\uparrow$ gives



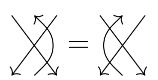
Then using (5.4) yields the first equality in (5.7).

Next, starting with the first equality in (5.7), we compose on the top with $\uparrow$ and on the bottom with $\uparrow$. Using (5.4) then yields the second equality in (5.7).

It remains to prove the third equality in (5.7). Starting with the first equality in (5.7), we tensor on the left and right with $\downarrow$, then compose on the top with $\downarrow\uparrow\downarrow$ and on the bottom with $\downarrow\uparrow\downarrow$ to get

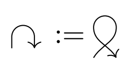
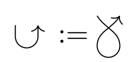


Using (5.3) then gives

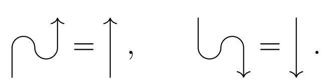


Now compose on the top with $\downarrow$ and on the bottom with $\downarrow$, then use (5.4), to obtain the last equality in (5.7). $\square$

We now have a symmetric monoidal category. Following the discussion in Section 4.3, we can now define

(5.8)  and 

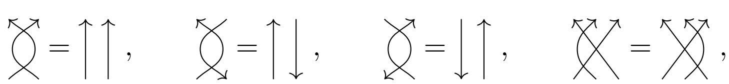
As in the proof Proposition 4.5, we then have



5.2. **Definition.** We now summarize the above discussion with the following definition.

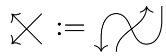
Definition 5.4. The *oriented Brauer category* $\mathcal{OB}_{\Bbbk}$ is the strict $\Bbbk$-linear monoidal category generated by objects $\uparrow$ and $\downarrow$ and morphisms

subject to the relations

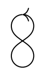
(5.9) 

(5.10) 

In the above, the left crossing is defined by

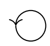
(5.11)  .

For $d \in \Bbbk$, we define $\mathcal{OB}_\Bbbk(d)$ to be the quotient of $\mathcal{OB}_\Bbbk$ by the relation

(5.12)  $= d1_{\mathbb{1}}$.

We call d the *specialization parameter*, and we call $\mathcal{OB}_\Bbbk(d)$ the *specialized oriented Brauer category.*

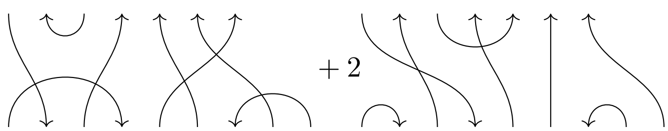
Note that, using (5.8), relation (5.12) becomes

 $= d1_{\mathbb{1}}$,

which means that $\uparrow$ has categorical dimension d.

It follows from our discussion that $\mathcal{OB}_\Bbbk$ is the free $\Bbbk$-linear rigid symmetric monoidal category on one object. Similarly, $\mathcal{OB}_\Bbbk(d)$ is the free $\Bbbk$-linear rigid symmetric monoidal category on an object of categorical dimension d. These categories then have the corresponding universal properties.

Objects in $\mathcal{OB}_\Bbbk(d)$ are finite tensor products of $\uparrow$ and $\downarrow$. An example of a morphism in $\mathcal{OB}_\Bbbk(d)$ is

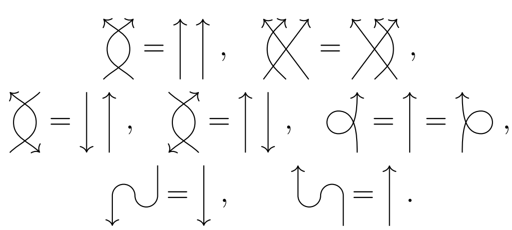


$\in \mathrm{Hom}_{\mathcal{OB}_\Bbbk(d)}(\uparrow\otimes\downarrow\otimes\uparrow\otimes\downarrow\otimes\uparrow\otimes\uparrow\otimes\downarrow\otimes\uparrow\otimes\uparrow, \downarrow\otimes\uparrow\otimes\downarrow\otimes\uparrow\otimes\uparrow\otimes\uparrow\otimes\uparrow)$.

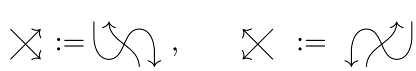
Remark 5.5. In our motivation of the definition of the oriented Brauer category, we first added a left dual $\downarrow$ to our generating object $\uparrow$ and then added morphisms and relations to make it symmetric monoidal. It followed that $\downarrow$ was also right dual to $\uparrow$. Alternatively, we could have first added morphisms and relations ensuring that $\downarrow$ is both left *and* right dual to $\uparrow$, and then made the category symmetric monoidal. This approach leads to an alternative, but equivalent, definition of the oriented Brauer category. Precisely, $\mathcal{OB}_\Bbbk$ is the strict $\Bbbk$-linear monoidal category generated by objects $\uparrow$ and $\downarrow$ and morphisms

 $: \uparrow\otimes\uparrow \to \uparrow\otimes\uparrow$,  $: \downarrow\otimes\uparrow \to \mathbb{1}$,  $: \mathbb{1} \to \uparrow\otimes\downarrow$,  $: \uparrow\otimes\downarrow \to \mathbb{1}$,  $: \mathbb{1} \to \downarrow\otimes\uparrow$,

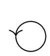
subject to the relations



In the above, the left and right crossings are defined by

(5.13)  .

For $d \in \Bbbk$, we then define $\mathcal{OB}_\Bbbk(d)$ to be the quotient of $\mathcal{OB}_\Bbbk$ by the relation

(5.14)  $= d1_{\mathbb{1}}$.

5.3. **Application to the general linear group.** Let us now use the universal property of the oriented Brauer category to study the representation theory of the general linear group. Let $V = \Bbbk^d$ be the natural module for the general linear group $\mathrm{GL}(d,\Bbbk)$. Since the category $\mathrm{GL}(d,\Bbbk)$-mod of finite-dimensional $\mathrm{GL}(d,\Bbbk)$-modules is a $\Bbbk$-linear rigid symmetric monoidal category, and V is an object of dimension d in this category, we have a monoidal functor

$$\mathbf{F} \colon \mathcal{OB}_\Bbbk(d) \to \mathrm{GL}(d,\Bbbk)\text{-mod} \tag{5.15}$$

determined on objects by

$$\uparrow \mapsto V, \qquad \downarrow \mapsto V^*,$$

and on morphisms by

$$\mathbf{F}\left(\nwarrow\!\!\!\!\!\nearrow\right) = \mathrm{flip}_{V,V}, \quad \mathbf{F}\left(\nwarrow\!\!\!\!\!\searrow\right) = \mathrm{flip}_{V,V^*}, \quad \mathbf{F}\left(\curvearrowleft\right) = \mathrm{ev}_V, \quad \mathbf{F}\left(\uparrow\!\smile\right) = \mathrm{coev}_V, \tag{5.16}$$

where ev_V and coev_V are the evaluation and coevaluation maps from (4.3) and (4.4), and flip is defined in (3.10).

Remark 5.6. In fact, the functor $\mathbf{F}$ is uniquely determined by the first three equalities in (5.16); see Exercise 4.2.

Exercise 5.7. Show that

$$\mathbf{F}\left(\swarrow\!\!\!\!\!\searrow\right) = \mathrm{flip}_{V^*,V^*}, \quad \mathbf{F}\left(\swarrow\!\!\!\!\!\nearrow\right) = \mathrm{flip}_{V^*,V}, \quad \mathbf{F}\left(\curvearrowright\right) = \mathrm{ev}_{V^*}, \quad \mathbf{F}\left(\smile\!\uparrow\right) = \mathrm{coev}_{V^*},$$

where we identify $(V^*)^*$ with V in the usual way, so that

$$\mathrm{ev}_{V^*} \colon V \otimes V^* \to \Bbbk \qquad \text{and} \qquad \mathrm{coev}_{V^*} \colon \Bbbk \to V^* \otimes V.$$

Exercise 5.8. Instead of using the universal property of the oriented Brauer category, show directly that the defining relations (5.9), (5.10), and (5.12) are respected by $\mathbf{F}$.

Theorem 5.9. *If the field $\Bbbk$ is infinite, then the functor $\mathbf{F}$ is full.*

Proof. We need to show that, for all $X, Y \in \mathrm{Ob}(\mathcal{OB}_\Bbbk(d))$, the $\Bbbk$-linear map

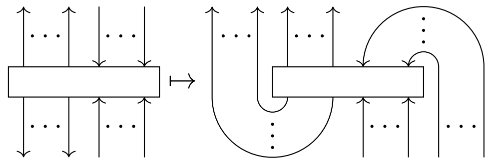
$$\mathbf{F} \colon \mathrm{Hom}_{\mathcal{OB}_\Bbbk(d)}(X,Y) \to \mathrm{Hom}_{\mathrm{GL}(d,\Bbbk)}(F(X),F(Y))$$

is surjective. Suppose that X (respectively, Y) is a tensor product of r_X (respectively, r_Y) copies of $\uparrow$ and s_X (respectively, s_Y) copies of $\downarrow$. Consider the following commutative diagram:

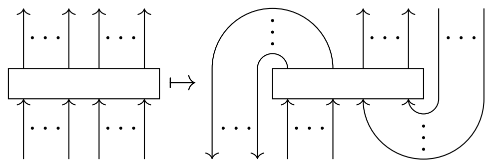
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{OB}_\Bbbk(d)}(X,Y) \arrow[r, "\mathbf{F}"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathrm{GL}(d,\Bbbk)}\big(F(X),F(Y)\big) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{OB}_\Bbbk(d)}(\downarrow^{\otimes s_X} \otimes \uparrow^{\otimes r_X}, \uparrow^{\otimes r_Y} \otimes \downarrow^{\otimes s_Y}) \arrow[r, "\mathbf{F}"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathrm{GL}(d,\Bbbk)}\big((V^*)^{\otimes s_X} \otimes V^{\otimes r_X}, V^{\otimes r_Y} \otimes (V^*)^{\otimes s_Y}\big) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{OB}_\Bbbk(d)}(\uparrow^{\otimes (r_X+s_Y)}, \uparrow^{\otimes (r_Y+s_X)}) \arrow[r, "\mathbf{F}"] & \mathrm{Hom}_{\mathrm{GL}(d,\Bbbk)}\big(V^{\otimes (r_X+s_Y)}, V^{\otimes (r_Y+s_X)}\big)
\end{tikzcd}$$

The top-left vertical map is given by composing on the top and bottom of diagrams with $\swarrow\!\!\!\!\!\nearrow$ to move $\uparrow$'s on the top to the left and $\uparrow$'s on the bottom to the right. The top-right vertical map is given analogously, using $\mathbf{F}(\swarrow\!\!\!\!\!\nearrow) = \mathrm{flip}_{V^*,V}$. The bottom vertical maps are the usual isomorphisms that hold in any rigid monoidal supercategory. In particular, the bottom-left vertical map is the

$\Bbbk$-linear isomorphism given on diagrams by



with inverse



where the rectangle denotes some diagram.

Since all the horizontal maps are isomorphisms, it suffices to show that the bottommost horizontal map has the desired properties. Thus, we must show that the map

$$\mathbf{F}\colon \operatorname{Hom}_{\mathcal{OB}_\Bbbk(d)}(\uparrow^{\otimes r}, \uparrow^{\otimes s}) \to \operatorname{Hom}_{\mathrm{GL}(d,\Bbbk)}(V^{\otimes r}, V^{\otimes s}) \tag{5.17}$$

is surjective for all $r, s \in \mathbb{N}$. For $\alpha \in \Bbbk^\times$, the central element $\alpha I \in \mathrm{GL}(d, \Bbbk)$ acts on $V^{\otimes r}$ as α^r. Thus

$$\operatorname{Hom}_{\mathrm{GL}(d,\Bbbk)}(V^{\otimes r}, V^{\otimes s}) = 0 \qquad \text{for } r \neq s.$$

So it suffices to consider the case $r = s$. In this case, we have an algebra homomorphism

$$\Bbbk\mathfrak{S}_r \to \operatorname{End}_{\mathcal{OB}_\Bbbk}(\uparrow^{\otimes r})$$

sending the simple transposition s_i to the crossing of strings i and $i+1$. Composition with (5.17) corresponds to the action of $\Bbbk\mathfrak{S}_r$ on $V^{\otimes r}$ by permutation of the factors. Then surjectivity of the map (5.17) follows from Schur–Weyl duality. (See [dCP76, Th. 4.1] for a proof that holds in the generality of an infinite ground field. In fact, the result there is more general.) ☐

Remark 5.10. (a) When $d \geq r = s$, the map (5.17) is also injective.

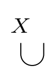
(b) The endomorphism algebra $\operatorname{End}_{\mathcal{OB}_\Bbbk(d)}(\uparrow^{\otimes r} \otimes \downarrow^{\otimes s})$ is sometimes called the *walled Brauer algebra* in the literature.

(c) One can give an explicit basis of each morphism space of $\mathcal{OB}_\Bbbk(d)$ in terms of *oriented Brauer diagrams*.

(d) The semisimplification of the idempotent completion of $\mathcal{OB}_\Bbbk(d)$ is Deligne's interpolating category for the general linear groups.

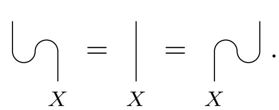
6. Self-duality in monoidal categories

We continue to assume that $\mathcal{C}$ is a rigid strict symmetric monoidal category with our conventions from Section 4. It often occurs in representation theory that a representation is dual to *itself*. In this case, the discussion of Section 4 still applies, but it is no longer useful to orient strings to distinguish between and object and its dual. Instead, we use unoriented strings. In other words, X is self-dual if we have morphisms

 $: X \otimes X \to \mathbb{1}$ and  $: \mathbb{1} \to X \otimes X$

such that

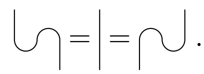
(6.1)



6.1. **A universal category with a self-dual object.** The *Temperley–Lieb category* $\mathcal{TL}$ is the strict $\Bbbk$-linear monoidal category with one generating object I, two generating morphisms

$$\cap \colon \mathsf{I} \otimes \mathsf{I} \to \mathbb{1}, \qquad \cup \colon \mathbb{1} \to \mathsf{I} \otimes \mathsf{I},$$

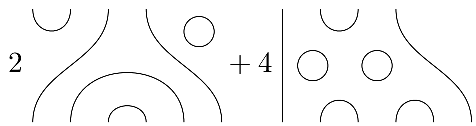
and relations



For $d \in \Bbbk$, we define the *specialized Temperley–Lieb category* $\mathcal{TL}_\Bbbk(d)$ to be the quotient of $\mathcal{TL}_\Bbbk$ by the relation

$$\bigcirc = d1_{\mathbb{1}}.$$

Objects in $\mathcal{TL}_\Bbbk$ and $\mathcal{TL}_\Bbbk(d)$ are of the form $\mathsf{I}^{\otimes n}$, $n \in \mathbb{N}$. For $n \in \mathbb{N}$, the endomorphism algebra $\mathrm{End}_{\mathcal{TL}_\Bbbk(d)}(\mathsf{I}^{\otimes n})$ is the *Temperley–Lieb algebra* $\mathrm{TL}_n(d)$. An example of a morphism in $\mathcal{TL}_\Bbbk$ is



$$\in \mathrm{Hom}_{\mathcal{OTL}_\Bbbk}(\mathsf{I}^{\otimes 6}, \mathsf{I}^{\otimes 4}).$$

The category $\mathcal{TL}_\Bbbk(d)$ is the free $\Bbbk$-linear monoidal category on a self-dual object of categorical dimension d. If X is a self-dual object of categorical dimension d in a $\Bbbk$-linear monoidal category $\mathcal{C}$, then there exists a unique monoidal functor

$$\mathcal{TL}_\Bbbk(d) \to \mathcal{C}, \qquad \mathsf{I} \to X,$$

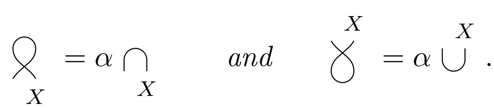
sending $\cap$ and $\cup$ to the unit and counit of the self-duality of X.

6.2. **Symmetry and self-duality.** Now let us examine the interaction between self-duality and symmetry. Suppose that X is a self-dual object with

$$\mathrm{End}_{\mathcal{C}}(X) = \Bbbk 1_X \cong \Bbbk. \tag{6.2}$$

As motivation for this assumption, one should keep in mind the case where $\Bbbk$ is algebraically closed and X is a simple object in some category of modules. Then Schur's lemma implies that $\mathrm{End}(X) \cong \Bbbk$.

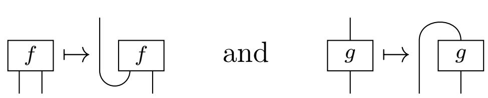
Lemma 6.1. *There exists $\alpha \in \{\pm 1\}$ such that*



Proof. It follows from (6.1) that the $\Bbbk$-linear maps

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes X, \mathbb{1}) \to \mathrm{End}_{\mathcal{C}}(X) \qquad \text{and} \qquad \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{Hom}_{\mathcal{C}}(X \otimes X, \mathbb{1})$$

given by

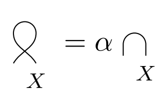


are mutually inverse. Here f and g represent arbitrary morphisms in $\mathrm{Hom}_{\mathcal{C}}(X \otimes X, \mathbb{1})$ and $\mathrm{End}_{\mathcal{C}}(X)$, respectively.

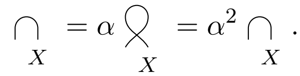
Thus $\mathrm{Hom}_{\mathcal{C}}(X \otimes X, \mathbb{1}) \cong \Bbbk$, with basis

$$\underset{X}{\cap}\,.$$

It follows that

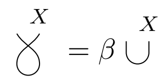


for some $\alpha \in \Bbbk$. Attaching a crossing to the bottom then gives

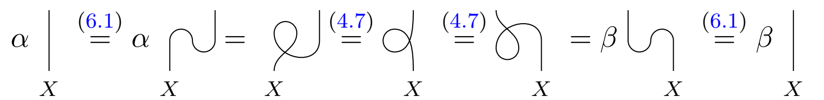


Hence $\alpha^2 = 1$, and so $\alpha \in \{\pm 1\}$.

An analogous argument shows that



for some $\beta \in \{\pm 1\}$. It remains to show that $\alpha = \beta$. To see this, note that



Hence $\alpha = \beta$, as desired. $\square$

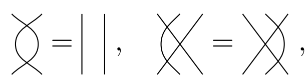
7. The unoriented Brauer category

We now define the analogue of the oriented Brauer category where the generating object is self-dual.

Definition 7.1. The *Brauer category* $\mathcal{B}_{\Bbbk}$ is the strict $\Bbbk$-linear monoidal category generated by an object I and morphisms

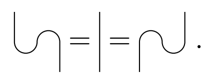
subject to the relations

(7.1)



(7.2)

(7.3)

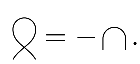


For $d \in \Bbbk$, we define the *specialized Brauer category* $\mathcal{B}_{\Bbbk}(d)$ to be the quotient of $\mathcal{B}_{\Bbbk}$ by the relation

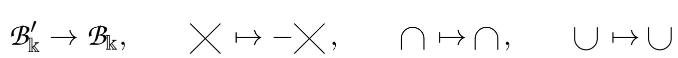
$$\bigcirc = d1_{\mathbb{1}}. \tag{7.4}$$

We call d the *dimension parameter*. When we wish to emphasize the difference from the oriented Brauer category, we will call $\mathcal{B}_{\Bbbk}$ and $\mathcal{B}_{\Bbbk}(d)$ the *unoriented Brauer category* and the *specialized unoriented Brauer category*, respectively.

Remark 7.2. Lemma 6.1 suggests that another natural definition of the Brauer category would come from replacing the first relation in (7.2) by the relation

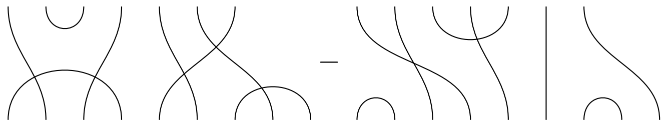


However, the category $\mathcal{B}'_{\Bbbk}$ obtained this way is isomorphic to $\mathcal{B}_{\Bbbk}$ via the monoidal functor defined by



The category $\mathcal{B}_{\Bbbk}$ is the free $\Bbbk$-linear rigid symmetric monoidal category on a symmetrically self-dual object. Similarly, $\mathcal{B}_{\Bbbk}(d)$ is the free $\Bbbk$-linear rigid symmetric monoidal category on a symmetrically self-dual object of categorical dimension d. These categories have the corresponding universal properties.

Objects in $\mathcal{B}_{\Bbbk}(d)$ are of the form $\mathsf{I}^{\otimes n}$, $n \in \mathbb{N}$. An example of a morphism in $\mathcal{B}_{\Bbbk}(d)$ is



$$\in \mathrm{Hom}_{\mathcal{B}_{\Bbbk}(d)}\left(\mathsf{I}^{\otimes 9}, \mathsf{I}^{\otimes 7}\right).$$

The endomorphism algebras $\mathrm{End}_{\mathcal{B}_{\Bbbk}(d)}(\mathsf{I}^{\otimes n})$, $n \in \mathbb{N}$, $d \in \Bbbk$, are called *Brauer algebras.*

7.1. **Application to the orthogonal and symplectic groups.** Let us now use the universal property of the Brauer category to study the representation theory of the orthogonal group. Let $V = \Bbbk^d$ be the natural module for the orthogonal group $\mathrm{O}(d, \Bbbk)$. By definition, $\mathrm{O}(d, \Bbbk)$ is the subgroup of $\mathrm{GL}(d, \Bbbk)$ fixing a nondegenerate symmetric bilinear form

$$\varphi \colon V \otimes V \to \Bbbk.$$

Choose an orthonormal basis $\mathbf{B}_V$ of V, and define

$$\varphi' \colon \Bbbk \to V \otimes V, \qquad 1 \mapsto \sum_{v \in \mathbf{B}_V} v \otimes v.$$

Exercise 7.3. Show that φ' is independent of the choice of $\mathbf{B}_V$.

The bilinear form φ identifies V with its dual V^*. In other words, V is a self-dual object of the category $\mathrm{O}(d, \Bbbk)$-mod of finite-dimensional $\mathrm{O}(d, \Bbbk)$-modules. Since V has dimension d, we have a monoidal functor

$$\mathcal{B}_{\Bbbk}(d) \to \mathrm{O}(d, \Bbbk)\text{-mod} \tag{7.5}$$

determined on objects by

$$\mathsf{I} \mapsto V$$

and on morphisms by

$$\times \mapsto \mathrm{flip}_{V,V}, \qquad \cap \mapsto \varphi, \qquad \cup \mapsto \varphi'.$$

An analogous construction works for the symplectic groups. Let $V = \Bbbk^{2d}$ be the natural module for the symplectic group $\mathrm{Sp}(2d, \Bbbk)$. By definition, $\mathrm{Sp}(2d, \Bbbk)$ is the subgroup of $\mathrm{GL}(2d, \Bbbk)$ fixing a nondegenerate skew-symmetric bilinear form

$$\omega \colon V \otimes V \to \Bbbk.$$

Choose a basis $\mathbf{B}_V$ of V, and let $\{v^\vee : v \in \mathbf{B}_V\}$ be the left dual basis given by

$$\omega(v^\vee, w) = \delta_{vw}, \qquad v, w \in \mathbf{B}_V.$$

Then define

$$\omega' \colon \Bbbk \to V \otimes V, \qquad 1 \mapsto \sum_{v \in \mathbf{B}_V} v \otimes v^\vee.$$

We have a monoidal functor

$$\mathcal{B}_{\Bbbk}(-2d) \to \mathrm{Sp}(2d, \Bbbk)\text{-mod} \tag{7.6}$$

determined on objects by

$$\mathsf{I} \mapsto V$$

and on morphisms by

$$\times \mapsto -\mathrm{flip}_{V,V}, \qquad \cap \mapsto \omega, \qquad \cup \mapsto \omega'.$$

Exercise 7.4. Show directly that relations (7.1) to (7.3) are respected by the functor (7.6).

Theorem 7.5. *If $\Bbbk$ is an infinite field of characteristic not equal to two, then the functors* (7.5) *and* (7.6) *are full.*

Proof. The proof of this theorem is similar to the proof of Theorem 5.9. One uses the cups and caps to reduce the statement to a classical one about the Brauer algebra; see [DH09]. □

Remark 7.6. In Theorem 7.5, it is crucial that the target is the category of modules for the orthogonal *group* $\mathrm{O}(d,\mathbb{C})$. If we replaced the group by its Lie algebra $\mathfrak{so}(d,\mathbb{C})$, which would be equivalent to replacing $\mathrm{O}(d,\mathbb{C})$ by $\mathrm{SO}(d,\mathbb{C})$, then the functor would *not* be full in general. For example, if $d \geq 3$ is an odd integer, then $\Lambda^d(V)$ is isomorphic to the trivial module as $\mathrm{SO}(d,\mathbb{C})$-modules, but *not* as $\mathrm{O}(d,\mathbb{C})$-modules, since the negative of the identity matrix acts on $\Lambda^d(V)$ as -1. Thus, the isomorphism $\Lambda^d(V) \cong \mathbb{C}$ of $\mathrm{SO}(d,\mathbb{C})$-modules is not in the image of $\mathbf{F}$.

Note that, in the symplectic case, V has *negative* categorical dimension since

$$\mathbf{F}\left(\bigcirc\right) = \sum_{v\in\mathbf{B}_V} \varphi(v,v^\vee)1_{\mathbb{1}} = -\sum_{v\in\mathbf{B}_V} \varphi(v^\vee,v)1_{\mathbb{1}} = -2d1_{\mathbb{1}}. \tag{7.7}$$

This suggests another approach to the symplectic case. If we view V as a purely odd vector *superspace*, then it has *super*dimension $-2d$ and ω is a nondegenerate supersymmetric bilinear form. In fact, the orthogonal and symplectic cases can be unified in this way. If $V = \Bbbk^{m|2n}$ is a vector superspace and φ is nondegenerate supersymmetric bilinear form, then we have a monoidal functor

$$\mathcal{B}_{\Bbbk}(m-2n) \to \mathrm{OSp}(m|2n)\text{-smod},$$
$$\mid\ \mapsto V, \qquad \times \mapsto \mathrm{flip}_{V,V}, \qquad \cap \mapsto \varphi, \qquad \cup \mapsto \varphi',$$

where $\mathrm{OSp}(m|2n)$-smod is the category of finite-dimensional modules over the orthosymplectic group $\mathrm{OSp}(m|2n)$, and flip is now given by

$$\mathrm{flip}_{V,W} \colon V\otimes W \to W \otimes V, \qquad v\otimes w \mapsto (-1)^{\bar v\bar w} w\otimes v, \tag{7.8}$$

for vector superspaces V and W. This functor is full; see [DLZ18, LZ17].

8. Going further

In this final section, we briefly examine generalizations of the theory we have examined so far. In particular, we look at the setting of monoidal supercategories. Another further direction, which we do not discuss here, is to the setting of quantized enveloping (super)algebras. To simplify our discussion, we work over the ground field $\Bbbk = \mathbb{C}$ throughout this section.

8.1. Monoidal supercategories.

As noted in Section 7.1, we have a full functor

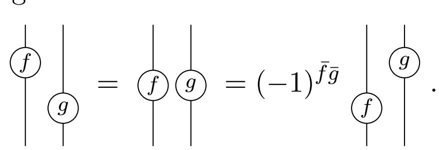
$$\mathcal{B}_{\mathbb{C}}(m-2n) \to \mathrm{OSp}(m|2n,\mathbb{C})\text{-smod}.$$

The functor of (5.15) can also be extended to the super setting, yielding a full functor

$$\mathcal{OB}_{\mathbb{C}}(m-n) \to \mathrm{GL}(m|n,\mathbb{C})\text{-smod}.$$

Once we have moved to the super setting, it makes sense to generalize our discussion of symmetric and duality from the setting of monoidal categories to that of monoidal *super*categories. For a detailed treatment of monoidal supercategories, we refer the reader to [BE17]. The main difference is that, in a monoidal supercategory $\mathcal{C}$, we have that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a vector *super*space for all objects X and Y. The interchange law becomes

(8.1)



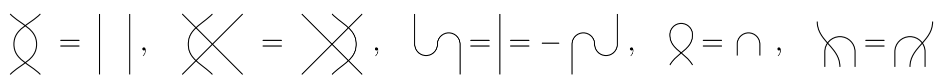
8.2. **The periplectic Brauer supercategory.** Our discussion of symmetric monoidal supercategories is almost identical to that for symmetric monoidal categories. The only difference is that, in our applications to representation theory, the crossing is sent to the super analogue (7.8) of the flip map.

When we discuss duality in monoidal supercategories, some genuinely new possibilities arise. In particular, the unit and counit morphisms could be odd. This happens when an object X is isomorphic to the parity shift of the dual of an object Y. If $X \ncong Y$, then we can replace Y by its parity shift ΠY to return to the situation of even units and counits. However, for *self*-dual objects, this trick does not work. This leads to the following variation of the Brauer category.

Definition 8.1 (Periplectic Brauer category)**.** The *periplectic Brauer category* $\mathcal{PB}_\Bbbk$ is the strict monoidal supercategory generated by one object I and morphisms

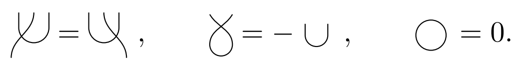
$$\times : \mathsf{I}^{\otimes 2} \to \mathsf{I}^{\otimes 2}, \qquad \cap : \mathsf{I}^{\otimes 2} \to \mathbb{1}, \qquad \cup : \mathbb{1} \to \mathsf{I}^{\otimes 2},$$

subject to the relations

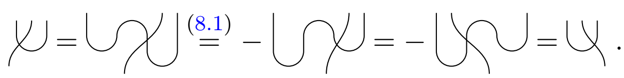
(8.2) 

The morphisms $\cup$ and $\cap$ are both odd, and $\times$ is even.

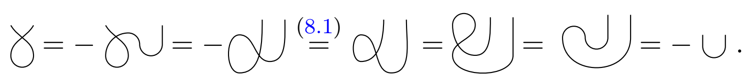
Proposition 8.2. *In $\mathcal{PB}_\Bbbk$, we have*

(8.3) 

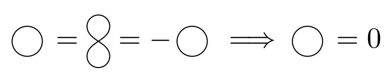
Proof. For the first relation in (8.3), we have



Then, for the second relation in (8.3), we compute



Finally, for the third relation in (8.3), we have



In the first equality above, we used the fifth equality in (8.2) and, in the second equality above, we used the second equality in (8.3). $\square$

The third relation in (8.3) tells us that the generating object I in $\mathcal{PB}_\Bbbk$ has categorical dimension zero. In fact, this should not surprise us. If vector superspace V is isomorphic to the parity shift of its dual, then the dimension of its even part is equal to the dimension of its odd part. Hence, the superdimension of V is zero.

The periplectic Brauer supercategory is the free symmetric monoidal supercategory on a supersymmetrically odd-self-dual object. This leads to the fact that there exists a monoidal superfunctor

$$\mathcal{PB}_\mathbb{C} \to \mathrm{P}(n)\text{-smod}$$

where $\mathrm{P}(n)$ is the *periplectic supergroup*, which is the automorphism supergroup of $\mathbb{C}^{n|n}$ preserving a nondegenerate supersymmetric odd bilinear form. This functor is full. This result was proved in [CE21, Th. 6.2.1], with the key ingredient being [DLZ18, §4.9]. Note that fullness holds for *all* $n \in \mathbb{N}$, with the supercategory $\mathcal{PB}_\mathbb{C}$ not depending on n.

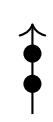
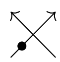
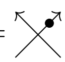
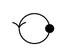
8.3. **The oriented Brauer–Clifford supercategory.** In the monoidal (super)categories $\mathcal{OB}_\Bbbk(d)$, $\mathcal{B}_\Bbbk(d)$, and $\mathcal{PB}_\Bbbk(d)$, the endomorphism algebras of the generating objects consist of scalar multiples of the identity. We even explicitly considered this type of assumption in (6.2). As mentioned in Section 6.2, this assumption is natural if $\Bbbk$ is algebraically closed and we want our generating object to map to a simple object V in some category of modules, since Schur's lemma implies that $\mathrm{End}(V) \cong \Bbbk$.

In the super world, the situation is a bit different. If V is a simple module, then Schur's lemma implies that $\mathrm{End}(V)$ is a complex division superalgebra. There are precisely *two* such complex division superalgebras: $\mathbb{C}$ itself and the Clifford superalgebra Cl generated by an odd element ε satisfying $\varepsilon^2 = -1$. We say that a simple module V is of *type M* if $\mathrm{End}(V) \cong \Bbbk$ and of *type Q* if $\mathrm{End}(V) \cong \mathrm{Cl}$. This leads to the following variation of the oriented Brauer category.

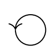
Definition 8.3 (The oriented Brauer–Clifford supercategory)**.** The *oriented Brauer–Clifford supercategory* $\mathcal{OBC}_\mathbb{C}$ is the strict monoidal supercategory obtained from $\mathcal{OB}_\mathbb{C}$ by adjoining an odd morphism

 $: \uparrow \to \uparrow,$

and imposing the additional relations

 $= -\uparrow$,  $=$  ,  $= 0.$

One can show that, in $\mathcal{OBC}_\mathbb{C}$, we have

 $= 0.$

(For details, see [BCK19, (3.16)].) Thus, as for the periplectic Brauer category, the categorical dimension of the generating object $\uparrow$ is zero. Again, this should not surprise us. If V is a simple module of type Q, then the odd element ε of the Clifford superalgebra gives an odd automorphism of V. Hence, the even and odd parts of V have the same dimension, and so the superdimension of V is zero.

The oriented Brauer–Clifford supercategory is the free symmetric monoidal supercategory on an object of type Q. This leads to the fact that there exists a monoidal superfunctor

$$\mathcal{OBC}_\mathbb{C} \to \mathrm{Q}(n)\text{-smod}$$

where $\mathrm{Q}(n)$ is the *isomeric supergroup* (also known as the queer supergroup), which is the automorphism group of a simple module of type Q and dimension $n|n$. This functor is full; see [BCK19, Th. 4.1]. Note that this holds for *all* $n \in \mathbb{N}$, with the supercategory $\mathcal{OBC}_\mathbb{C}$ not depending on n.

One might also expect that there should be an isomeric version of the unoriented Brauer category. However, one can show that such a category collapses to the zero category.

8.4. **More general ground fields.** What happens if we want to use the methods described in these notes to representation theory over more general fields? If we want our generating objects to correspond to simple objects, then Schur's lemma tells us that our strands should carry decorations labelled by elements in a division superalgebra over our ground field $\Bbbk$. When $\Bbbk$ is not algebraically closed, there can be many such division superalgebras. For instance, when $\Bbbk = \mathbb{R}$, there are exactly *three* real division algebras: the real numbers, the complex numbers, and the quaternions. If we move to the super setting, there are *ten* real division superalgebras. This leads to ten versions of the oriented Brauer category. For the unoriented case, our division superalgebra must carry and anti-involution. Exactly four of the ten real division superalgebras admit an anti-involution. So, when $\Bbbk = \mathbb{R}$, we are led to four versions of the Brauer category. These variations of the oriented and unoriented Brauer categories are related to the representation theory of *real* supergroups. We refer the reader to [SSS23] for details.

References

[BCK19] J. Brundan, J. Comes, and J. R. Kujawa. A basis theorem for the degenerate affine oriented Brauer-Clifford supercategory. *Canad. J. Math.*, 71(5):1061–1101, 2019. arXiv:1706.09999, doi:10.4153/cjm-2018-030-8.

[BCNR17] J. Brundan, J. Comes, D. Nash, and A. Reynolds. A basis theorem for the affine oriented Brauer category and its cyclotomic quotients. *Quantum Topol.*, 8(1):75–112, 2017. arXiv:1404.6574, doi:10.4171/QT/87.

[BE17] J. Brundan and A. P. Ellis. Monoidal supercategories. *Comm. Math. Phys.*, 351(3):1045–1089, 2017. arXiv:1603.05928, doi:10.1007/s00220-017-2850-9.

[CE21] K. Coulembier and M. Ehrig. The periplectic Brauer algebra III: The Deligne category. *Algebr. Represent. Theory*, 24(4):993–1027, 2021. arXiv:1704.07547, doi:10.1007/s10468-020-09976-8.

[dCP76] C. de Concini and C. Procesi. A characteristic free approach to invariant theory. *Advances in Math.*, 21(3):330–354, 1976. doi:10.1016/S0001-8708(76)80003-5.

[DH09] S. Doty and J. Hu. Schur-Weyl duality for orthogonal groups. *Proc. Lond. Math. Soc. (3)*, 98(3):679–713, 2009. arXiv:0712.0944, doi:10.1112/plms/pdn044.

[DLZ18] P. Deligne, G. I. Lehrer, and R. B. Zhang. The first fundamental theorem of invariant theory for the orthosymplectic super group. *Adv. Math.*, 327:4–24, 2018. arXiv:1508.04202, doi:10.1016/j.aim.2017.06.009.

[Kas95] C. Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. doi:10.1007/978-1-4612-0783-2.

[LZ17] G. I. Lehrer and R. B. Zhang. The first fundamental theorem of invariant theory for the orthosymplectic supergroup. *Comm. Math. Phys.*, 349(2):661–702, 2017. arXiv:1401.7395, doi:10.1007/s00220-016-2731-7.

[Mac98] S. MacLane. *Categories for the working mathematician*. Springer-Verlag, New York-Berlin, second edition, 1998. Graduate Texts in Mathematics, Vol. 5.

[Pen71] R. Penrose. Applications of negative dimensional tensors. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, pages 221–244. Academic Press, London, 1971.

[Sch01] P. Schauenburg. Turning monoidal categories into strict ones. *New York J. Math.*, 7:257–265, 2001. URL: http://nyjm.albany.edu:8000/j/2001/7_257.html.

[Sel11] P. Selinger. A survey of graphical languages for monoidal categories. In *New structures for physics*, volume 813 of *Lecture Notes in Phys.*, pages 289–355. Springer, Heidelberg, 2011. arXiv:0908.3347, doi:10.1007/978-3-642-12821-9_4.

[SSS23] S. Samchuck-Schnarch and A. Savage. Diagrammatics for real supergroups. 2023. arXiv:2301.01414.

[TV17] V. Turaev and A. Virelizier. *Monoidal categories and topological field theory*, volume 322 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, 2017. doi:10.1007/978-3-319-49834-8.

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