The structure of Frobenius nilHecke algebras

John Stuart

1. Introduction

The purpose of this paper is to supplement [SS20] by Savage and Stuart, which focuses on similar results through the lens of strict monoidal categories. The string diagram approach in [SS20] can be more intuitive for experienced readers, but the different notation often acts as a barrier to those new to the subject. For this reason, we avoid the language of strict monoidal categories, and give a purely algebraic treatment of the subject. Also, several of the omitted proofs in [SS20] can be found in this paper.

One of the earliest appearances of the nilHecke ring was in [KK86] by Kostant and Kumar, and was used to unify different algebraic structures. The name came from its resemblance with the Hecke algebra, differing by a modification to the braid relation so that the generator squares to zero. The nilHecke algebra also shows up as the simplest case of the quiver Hecke algebra categorifying quantum groups. Most readers will recognize the nilHecke algebra as the algebra of divided difference operators on a polynomial ring.

In this paper, we define a Frobenius nilHecke algebra, which is a further modification to the Hecke algebra. As with the original, our algebra depends on a given $n \in \mathbb{N}$, but also a given Frobenius algebra. In the trivial case when the Frobenius algebra is the base field, then our definition reduces to the nilHecke algebra. By taking different choices of Frobenius algebras, we can obtain known algebras in the literature, for example we obtain a superalgebra that is Morita equivalent to the odd nilHecke algebra in [EKL14] by Ellis-Khovanov-Lauda. With other Frobenius algebras, the definition often gives rise to entirely new algebras.

We start with discussing the fundamentals of a general Frobenius superalgebra A in the first chapter. Then for $n \in \mathbb{N}$ we define the *Frobenius nilHecke algebra* $\mathrm{nilHecke}_n(A)$ to be the associative superalgebra generated by u_i for $1 \le i < n$, x_j for $1 \le j \le n$, and $\mathbf{a} \in A^{\otimes n}$ with the following relations:

$$\begin{aligned} u_i^2 &= 0, & x_i x_j &= (-1)^{\varepsilon} x_j x_i, & x_j u_i &= u_i x_j \quad \text{if } j \notin \{i, i+1\},\\ u_i u_j &= u_i u_j \quad \text{if } |i-j| > 1, & \mathbf{a} x_i &= (-1)^{\varepsilon \bar{\mathbf{a}}} x_i \psi_i(\mathbf{a}), & x_{i+1} u_i - u_i x_i &= \tau_{i,i+1},\\ u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1}, & \mathbf{a} u_i &= u_i s_i(\mathbf{a}), & u_i x_{i+1} - x_i u_i &= \tau_{i+1,i}, \end{aligned}$$

where the u_i's are even, the x_i's are the same parity as A's trace map, and any $\mathbf{a}$ has the same parity as $\mathbf{a} \in A^{\otimes n}$. We define $\tau_{i,j}$ in Definition 3.1. Notice that the subalgebra generated by the u_i's is the nilCoxeter algebra, nilCox_n. Next the *Frobenius nilCoxeter algebra* $\mathrm{nilCox}_n(A)$ is the subalgebra generated by the u_i's and $\mathbf{a} \in A^{\otimes n}$, and the *Frobenius polynomial algebra* $\mathrm{Pol}_n(A)$ is the subalgebra generated by the x_i's and $\mathbf{a} \in A^{\otimes n}$. Our algebra $\mathrm{nilHecke}_n(A)$ is similar to the affine wreath product algebra described by Savage in [Sav18]. The main difference between $\mathrm{nilHecke}_n(A)$ and Savage's affine wreath product algebra is that $u_i^2 = 0$ in $\mathrm{nilHecke}_n(A)$, whereas $s_i^2 = 1$ in Savage's paper.

After defining $\mathrm{nilHecke}_n(A)$ and describing some basic results, we show that in the special case of taking A to be the Clifford superalgebra Cl, then we get two Morita equivalences. First, $\mathrm{nilHecke}_n(\mathrm{Cl})$ is Morita equivalent to the odd nilHecke algebra studied in [EKL14] by Ellis-Khovanov-Lauda. Second, we get another Morita equivalence of the nilCoxeter algebra and the odd nilCoxeter ring, also studied in [EKL14]. This first result is analogous to Theorem 4.1 in [Wan09], where Wang establishes an isomorphism between the affine Sergeev algebra and the Clifford algebra tensored with the degenerate spin affine Hecke algebra. The second result is analogous to the well-known Morita equivalence between the Sergeev algebra and the spin symmetric group algebra that can be found in Lemma 13.2.3 of [Kle05] by Kleshchev. Below, we have our results on the left compared with Wang's on the bottom right, and Kleshchev's on the top right:

$$\begin{tikzcd} \mathrm{Cl}^{\otimes n} \otimes \mathrm{ONC}_n \arrow[r, "\cong"] \arrow[d, hook] & \mathrm{nilCox}_n(\mathrm{Cl}) \arrow[d, hook] \\ \mathrm{Cl}^{\otimes n} \otimes \mathrm{ONH}_n \arrow[r, "\cong"] & \mathrm{nilHecke}_n(\mathrm{Cl}) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Cl}^{\otimes n} \otimes \mathbb{C}S_n^- \arrow[r, "\cong"] \arrow[d, hook] & \mathrm{Cl}^{\otimes n} \rtimes \mathbb{C}S_n \arrow[d, hook] \\ \mathrm{Cl}^{\otimes n} \otimes \mathfrak{H}^- \arrow[r, "\cong"] & \mathfrak{H}^{\mathfrak{c}} \end{tikzcd}$$

These isomorphisms give Morita equivalences since Cl is a semi-simple superalgebra.

Finally, we show how the generators u_i of nilCox_n can act on $\mathrm{Pol}_n(A)$ as Demazure operators, and that they satisfy all of the relations in $\mathrm{nilHecke}_n(A)$, where $f \in \mathrm{Pol}_n(A) \subset \mathrm{nilHecke}_n(A)$ act by left multiplication. This allows us to define a faithful action $\mathrm{nilHecke}_n(A) \to \mathrm{End}(\mathrm{Pol}_n(A) \otimes \mathrm{nilCox}_n)$ from which we deduce a basis theorem for $\mathrm{nilHecke}_n(A)$. Afterwards, we also deduce a basis of the subalgebra $\mathrm{nilCox}_n(A)$. In certain special cases, we are able to give explicit equations for the Demazure operator, which is originally defined inductively.

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2. Basics of Frobenius superalgebras

We start by introducing Frobenius superalgebras and proving some basic facts about them.

Definition 2.1 (Superalgebra)**.** A *superalgebra* is an algebra A that has a decomposition $A = A_0 \oplus A_1$ such that $A_i A_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}_2$. When $a \in A_0$, we say that a is even and we write $\bar{a} = 0 \in \mathbb{Z}_2$. Similarily, if $a \in A_1$, we say that a is odd and we write $\bar{a} = 1 \in \mathbb{Z}_2$.

Definition 2.2 (Frobenius superalgebra)**.** A *Frobenius superalgebra* is a pair (A, tr) such that A is a finite dimensional superalgebra over a field $\Bbbk$ and $\mathrm{tr} : A \to \Bbbk$ is a linear map such that for any nonzero $a \in A$, there exists $b \in A$ for which $\mathrm{tr}(ab) \neq 0$. The parity of the trace tr is even ($\varepsilon = 0$) if it is parity preserving (viewing $\Bbbk$ as even) and odd ($\varepsilon = 1$) if it is parity reversing.

Remark 2.3. For every Frobenius superalgebra (A, tr), there exists a *Nakayama* automorphism $\psi \colon A \to A$ such that $\mathrm{tr}(ab) = (-1)^{\bar{a}\bar{b}} \mathrm{tr}(b\psi(a))$. This follows from Proposition 3.1 in [YS12] by A. Skowronski and K. Yamagata. Even more, if we have have two different trace maps tr_1 and tr_2 for the same superalgebra A, then by Proposition 3.4 in [PS16] by Pike and Savage, there is an invertible element $v \in A$ such that

$$\mathrm{tr}_2(a) = \mathrm{tr}_1(av) \quad \forall a \in A.$$

We must have $\bar{v} = \varepsilon_1 + \varepsilon_2$ where $\varepsilon_1, \varepsilon_2$ are the parities of the trace maps. Also, the Nakayama automorphisms satisfy $\psi_2(a) = (-1)^{\bar{v}\bar{a}} v\psi_1(a)v^{-1}$ since

$$\mathrm{tr}_2(ab) = \mathrm{tr}_1(abv) = (-1)^{(\bar{b}+\bar{v})\bar{a}} \mathrm{tr}_1(bv\psi_1(a)) = (-1)^{\bar{b}\bar{a}} \mathrm{tr}_2(b(-1)^{\bar{v}\bar{a}} v\psi_1(a)v^{-1}).$$

For the rest of this text, A refers to an m-dimensional Frobenius superalgebra over a field $\Bbbk$.

Lemma 2.4. *For any basis $\{b_1, \dots, b_m\}$ of A, there exists a unique basis $\{b_1^\vee, \dots, b_m^\vee\}$ of A such that*

$$\mathrm{tr}(b_j^\vee b_i) = \delta_{j,i}.$$

Proof. Fix $k \in \{1, \dots, n\}$ and let $a_1, \dots, a_m \in \Bbbk$. Consider system of equations,

$$\delta_{k,i} = \mathrm{tr}((a_1 b_1 + \cdots + a_m b_m)b_i) = a_1 \mathrm{tr}(b_1 b_i) + \cdots + a_m \mathrm{tr}(b_m b_i),\ \forall i \in \{1, \dots, n\}.$$

Solving this system is equivalent to solving $M\vec{a} = e_k$ where $M \in M_m(\Bbbk)$ with $M_{i,j} = \mathrm{tr}(b_j b_i)$ and $\vec{a} = (a_1, \dots, a_m)$. This matrix is invertible since any nonzero $\vec{v}$ for which $M\vec{v} = \vec{0}$ corresponds to a nonzero element $v \in A$ for which $\mathrm{tr}(vb) = 0\ \forall b \in A$ contradicting the non-degeneracy of the trace map. Therefore take $b_k^\vee = M^{-1} e_k$. □

Proposition 2.5. *For all $a \in A$, we have*

$$\sum_{b \in B} \mathrm{tr}(b^\vee a) b = a = \sum_{b \in B} \mathrm{tr}(ab) b^\vee.$$

Proof. We write $a = a_1 b_1 + \cdots + a_m b_m = c_1 b_1^\vee + \cdots + c_m b_m^\vee$ as decompositions in B and its dual basis. This gives us

$$\sum_{i=1}^m \mathrm{tr}(b_i^\vee a) b_i = \sum_{i=1}^m \mathrm{tr}(b_i^\vee (a_1 b_1 + \cdots + a_m b_m)) b_i = \sum_{i=1}^m (a_1 \mathrm{tr}(b_i^\vee b_1) + \cdots + a_m \mathrm{tr}(b_i^\vee b_m)) b_i = \sum_{i=1}^m a_i b_i = a,$$

$$\sum_{i=1}^m \operatorname{tr}(ab_i)b_i^\vee = \sum_{i=1}^m \operatorname{tr}((c_1b_1^\vee + \cdots + c_mb_m^\vee)b_i)b_i^\vee = \sum_{i=1}^m (c_1 \operatorname{tr}(b_1^\vee b_i) + \cdots + c_m \operatorname{tr}(b_m^\vee b_i))b_i^\vee = \sum_{i=1}^m c_ib_i^\vee = a. \qquad \square$$

3. The Frobenius nilHecke superalgebra

In this section, we'll define our main object of interest: the Frobenius nilHecke superalgebra. We show that although the definition uses both a chosen basis and the trace map of A, the resulting Frobenius nilHecke superalgebra is independent of these choices.

For $a \in A$ and $1 \leq i \leq n$, define:

$$a_i := 1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes(n-i)} \tag{3.1}$$

$$\psi_i := \operatorname{id}_A^{\otimes(i-1)} \otimes \psi \otimes \operatorname{id}_A^{\otimes(n-i)} \tag{3.2}$$

Definition 3.1. Define the *Frobenius polynomial algebra* $\operatorname{Pol}_n(A)$ to be the free product of $A^{\otimes n}$ with the free associative algebra on generators $x_1, \ldots, x_n$ modulo the following relations:

$$x_ix_j = (-1)^\varepsilon x_jx_i$$
$$\mathbf{a}x_i = (-1)^{\varepsilon\bar{\mathbf{a}}} x_i\psi_i(\mathbf{a})$$

where $1 \leq i, j \leq n$, $i \neq j$, $\mathbf{a} \in A^{\otimes n}$, ψ is the Nakayama automorphism, and p is the parity of the trace. The parities of elements is induced from x_i being odd and a_i being the same parity as a in A. Also, define

$$\tau_{i,j} := \sum_{b \in B} (-1)^{\varepsilon\bar{b}} b_ib_j^\vee.$$

Note that multiplication in $A \otimes A$ is defined as follows:

$$(a \otimes b)(c \otimes d) = (-1)^{\bar{b}\bar{c}} ac \otimes bd$$

Proposition 3.2. *The element $\tau_{i,j} \in A^{\otimes n}$ is independent of the basis B.*

Proof. Let $B = \{b^{(1)}, \ldots, b^{(m)}\}$ and $B' = \{c^{(1)}, \ldots, c^{(m)}\}$ be homogeneous bases for A. Then there exist invertible matrices $M, N \in M_m(\Bbbk)$ such that $c^{(i)} = \sum_{j=1}^m M_{ij}b^{(j)}$ and $c^{(i)\vee} = \sum_{j=1}^m N_{ij}b^{(j)\vee}$. We have

$$c^{(i)\vee}c^{(j)} = \left(\sum_{k=1}^m N_{ik}b^{(k)\vee}\right)\left(\sum_{l=1}^m M_{jl}b^{(l)}\right) = \sum_{k,l=1}^m M_{jl}N_{ik}b^{(k)\vee}b^{(l)}.$$

Applying the trace map to both sides, we obtain the following

$$\delta_{ij} = \operatorname{tr}(c^{(i)\vee}c^{(j)}) = \operatorname{tr}\left(\sum_{k,l=1}^m M_{jl}N_{ik}b^{(k)\vee}b^{(l)}\right) = \sum_{k,l=1}^m M_{jl}N_{ik} \operatorname{tr}(b^{(k)\vee}b^{(l)}) = \sum_{k=1}^m M_{jk}N_{ik} = (MN^\top)_{ij},$$
$$\implies MN^\top = I \implies N^\top = M^{-1} \implies N^\top M = I \implies M^\top N = I,$$

where the last step is obtained by transposition. Since the bases are homogeneous, M_{ij} is zero when $c^{(i)}$ and $b^{(j)}$ have different parities. Thus

$$\begin{aligned}\sum_{i=1}^m (-1)^{\varepsilon\overline{c_w^{(i)}}} c_w^{(i)}c_r^{(i)\vee} &= \sum_{i,j,k=1}^m M_{ij}N_{ik}(-1)^{\varepsilon\overline{b^{(j)}}} b_w^{(j)}b_r^{(k)\vee} \\ &= \sum_{j,k=1}^m (M^\top N)_{jk}(-1)^{\varepsilon\overline{b^{(j)}}} b_w{}^{(j)}b_r^{(k)\vee} = \sum_{j=1}^m (-1)^{\varepsilon\overline{b^{(j)}}} b_w^{(j)}b_r^{(j)\vee}. \qquad \square\end{aligned}$$

Definition 3.3 (Frobenius nilHecke algebra)**.** Let the *nilCoxeter algebra* nilCox_n have even generators u_i for $1 \leq i < n$ and the relations

$$u_i^2 = 0, \tag{3.3}$$

$$u_iu_j = u_iu_j \quad \text{if } |i-j| > 1, \tag{3.4}$$

$$u_iu_{i+1}u_i = u_{i+1}u_iu_{i+1}, \tag{3.5}$$

The *Frobenius nilHecke algebra* $\mathrm{nilHecke}_n(A)$ is the quotient of the free product of nilCox_n and $\mathrm{Pol}_n(A)$ by relations

$$\mathbf{a}u_i = u_i s_i(\mathbf{a}), \tag{3.6}$$

$$x_j u_i = u_i x_j \quad \text{if } j \notin \{i, i+1\}, \tag{3.7}$$

$$x_{i+1}u_i - u_i x_i = \tau_{i,i+1}, \tag{3.8}$$

$$u_i x_{i+1} - x_i u_i = \tau_{i+1,i}. \tag{3.9}$$

The parities of elements in $\mathrm{nilHecke}_n(A)$ are induced from those of $\mathrm{Pol}_n(A)$ and nilCox_n. The *Frobenius nilCoxeter algebra* $\mathrm{nilCox}_n(A)$ is the subalgebra of $\mathrm{nilHecke}_n(A)$ generated by the u_i's and all the elements $\mathbf{a} \in A^{\otimes n}$.

Remark 3.4. If A has a $\mathbb{Z}$-grading, then we can endow $\mathrm{nilHecke}_n(A)$ with a $\mathbb{Z}$-grading by defining $|a_i| = |a|$ for $a \in A$, $|x_i| = 1$ and $|u_i| = 0$.

Proposition 3.5. *The superalgebra* $\mathrm{nilHecke}_n(A)$ *is independent of the trace map* tr.

Proof. Suppose we have two Frobenius superalgebras (A, tr_1) and (A, tr_2) with trace map parities ε_1 and ε_2 whose underlying superalgebra is the same. Below, $v \in A$ is the unique element described in Remark 2.3.

$$\mathrm{tr}_1(b_i^{\vee,1} b_j) = \delta_{i,j} = \mathrm{tr}_2(b_i^{\vee,2} b_j) = \mathrm{tr}_1(b_i^{\vee,2} b_j v).$$

Therefore by uniqueness of dual basis elements, we have $(bv)^{\vee,1} = b^{\vee,2}$ where $b^{\vee,i}$ is the dual basis element of b with respect to tr_i. By Proposition 3.2, we have

$$\begin{aligned}
\tau_{i,j}^{(2)} v_i &= \sum_{b\in B} (-1)^{\varepsilon_2 \bar{b}} b_i b_j^{\vee,2} v_i \\
&= \sum_{b\in B} (-1)^{\varepsilon_2 \bar{b} + \bar{v}(\bar{b}+\varepsilon_2)} b_i v_i b_j^{\vee,2} \\
&= \sum_{b\in B} (-1)^{\varepsilon_2 \bar{b} + (\varepsilon_1+\varepsilon_2)(\bar{b}+\varepsilon_2)} (bv)_i (bv)_j^{\vee,1} \\
&= \sum_{b\in B} (-1)^{\varepsilon_1 \bar{b} + \varepsilon_1\varepsilon_2 + \varepsilon_2} (bv)_i (bv)_j^{\vee,1} \\
&= (-1)^{\varepsilon_1+\varepsilon_2} \sum_{b\in B} (-1)^{\varepsilon_1(\bar{b}+\varepsilon_1+\varepsilon_2)} (bv)_i (bv)_j^{\vee,1} \\
&= (-1)^{\varepsilon_1+\varepsilon_2} \sum_{b\in B} (-1)^{\varepsilon_1(\bar{bv})} (bv)_i (bv)_j^{\vee,2} = (-1)^{\varepsilon_1+\varepsilon_2} \tau_{i,j}^{(1)},
\end{aligned}$$

where $\tau_{i,j}^{(k)}$ is $\tau_{i,j}$ defined with respect to tr_k. We will use the result $\tau_{i,i+1}^{(1)} = (-1)^{\varepsilon_1+\varepsilon_2} \tau_{i,i+1}^{(2)} v_i$ to check that the map $\Gamma : \mathrm{nilHecke}_n(A, \mathrm{tr}_1) \to \mathrm{nilHecke}_n(A, \mathrm{tr}_2)$ given by

$$x_i \mapsto (-1)^{\varepsilon_1+\varepsilon_2} x_i v_i, \qquad \mathbf{a} \mapsto \mathbf{a}, \qquad u_i \mapsto u_i,$$

is well defined. Verifications of the relations from Definition 3.3 are straightforward computations left to the reader. Therefore Γ is an algebra homomorphism. Next, Γ is bijective since the inverse $\Gamma^{-1} : \mathrm{nilHecke}_n(A, \mathrm{tr}_2) \to \mathrm{nilHecke}_n(A, \mathrm{tr}_1)$ is defined by

$$x_i \mapsto (-1)^{\varepsilon_1+\varepsilon_2} x_i v_i^{-1}, \qquad \mathbf{a} \mapsto \mathbf{a}, \qquad u_i \mapsto u_i.$$

It is clear that $\Gamma \circ \Gamma^{-1}$ and $\Gamma^{-1} \circ \Gamma$ both act as the identity on generators. ∎

Proposition 3.6. *For* $1 \le i, j < n, i \ne j$ *we have*

$$\psi_j(\tau_{i,j}) = (-1)^{\varepsilon} \tau_{j,i}.$$

Proof. We have

$$\psi_j^{-1}(\tau_{j,i}) = \sum_{b\in B} (-1)^{\varepsilon \bar{b}} \psi^{-1}(b)_j b_i^{\vee} \qquad \text{By Remark 3.1},$$

$$\begin{aligned}
&= \sum_{b\in B}(-1)^{\varepsilon\bar{b}} b_i^\vee (-1)^{\bar{b}+p\bar{b}}\psi^{-1}(b)_j && \text{Since } \overline{b^\vee} = \bar{b}+p,\\
&= (-1)^{\varepsilon}\sum_{b\in B}(-1)^{\varepsilon(\bar{b}+p)} b_i^\vee (b^\vee)_j^\vee && \text{Since } (-1)^{\bar{b}+\bar{b}\varepsilon}\psi^{-1}(b) = (b^\vee)^\vee,\\
&= (-1)^{\varepsilon}\tau_{i,j} && \text{By Proposition 3.2,}
\end{aligned}$$

where the third equality follows from $\mathrm{tr}((-1)^{\bar{b}+\bar{b}\varepsilon}\psi^{-1}(b)^{(j)}b^{(i)\vee}) = \mathrm{tr}(b^{(i)\vee}b^j)$. □

4. The Clifford nilHecke algebra

In this section, we prove that by choosing the Clifford superalgebra ($A = \mathrm{Cl}$), our Frobenius nilHecke algebra is Morita equivalent to the odd nilHecke algebra that can be found in [EKL14] by Ellis-Khovanov-Lauda. We also get a Morita equivalence of subalgebras $\mathrm{nilCox}_n(A)$ and the odd nilCoxeter algebra from [EKL14].

Definition 4.1 (Odd nilHecke algebra)**.** Define ONH_n to be the super algebra generated by $\triangle_i$ for $1 \le i < n$ and y_j for $1 \le j \le n$ with the following relations:

$$\begin{aligned}
\triangle_i\triangle_{i+1}\triangle_i &= \triangle_{i+1}\triangle_i\triangle_{i+1},\\
y_iy_j + y_jy_i &= 0 \quad (i \neq j),\\
y_i\triangle_j + \triangle_j y_i &= 0 \quad (i \neq j, j+1),\\
\triangle_i^2 &= 0,\\
y_{i+1}\triangle_i + \triangle_i y_i &= 1,\\
\triangle_i y_{i+1} + y_i\triangle_i &= 1,\\
\triangle_i\triangle_j + \triangle_j\triangle_i &= 0 \quad (|i-j| > 1),
\end{aligned}$$

where we declare the y_i and $\triangle_i$ to be odd. Define ONC_n to be the subalgebra generated by the $\triangle_i$'s. Note that this definition can also be found in Section 2.2 of [EKL14] by Ellis-Khovanov-Lauda.

Remark 4.2. Define the Clifford superalgebra Cl over $\Bbbk$ with one generator c where $c^2 = 1$. We consider the odd trace map $\mathrm{tr} : \mathrm{Cl} \to \Bbbk$ given by $\mathrm{tr}(a + bc) = b$ for $a, b \in \Bbbk$. This means that we consider x_i to be odd in $\mathrm{nilHecke}_n(\mathrm{Cl})$. The algebra $\mathrm{Cl}^{\otimes n}$ has generators c_i for $1 \le i \le n$ and $c_i^2 = 1$.

Theorem 4.3. *We have a superalgebra isomorphism*

$$\Phi : \mathrm{nilHecke}_n(\mathrm{Cl}) \to \mathrm{Cl}^{\otimes n} \otimes \mathrm{ONH}_n$$

given by

$$c_i \mapsto c_i, \qquad x_i \mapsto y_i, \qquad u_i \mapsto (c_i - c_{i+1})\triangle_i.$$

Proof. A straightforward computation verifies the relations from Definition 3.3. It is equally painless to check the relations from Definition 4.1 for the inverse map $\Phi^{-1} : \mathrm{Cl}^n \otimes \mathrm{ONH}_n \to \mathrm{nilHecke}_n(\mathrm{Cl})$ given by

$$c_i \mapsto c_i, \qquad y_i \mapsto x_i, \qquad \triangle_i \mapsto \frac{1}{2}(c_i - c_{i+1})u_i.$$

Since Φ and its inverse are both homomorphisms, Φ is an isomorphism. □

Corollary 4.4. *As superalgebras,* $\mathrm{nilCox}_n(\mathrm{Cl}) \cong \mathrm{Cl}^{\otimes n} \otimes \mathrm{ONC}_n$*.*

Proof. Restricting Φ from Theorem 4.3 to $\mathrm{nilCox}_n(\mathrm{Cl})$ gives the result. □

Analogous to Section 3.3 in [EKL14] by Ellis-Khovanov-Lauda, we will define the involution σ on $\mathrm{nilHecke}_n(A)$ given by

$$\sigma(u_i) = -u_{n-i}, \qquad \sigma(x_i) = x_{n-i+1}, \qquad \sigma(a_i) = a_{n-i+1}.$$

The relations of $\mathrm{nilHecke}_n(A)$ are straightforward to verify. For example,

$$\sigma(x_{i+1}u_i - u_ix_i - \tau_{i,i+1}) = -x_{n-i}u_{n-i} + u_{n-i}x_{n-i+1} - \tau_{n-i+1,n-i} = 0.$$

When working with string diagrams in a strict monoidal category like in [SS20], σ is the reflection of the diagram in the vertical axis. Next, we'll see an anti-involution obtained by reflecting along the horizontal axis.

When A is symmetric, we define the anti-involution Ψ on $\text{nilHecke}_n(A)$ given by

$$\Psi(u_i) = (-1)^{\varepsilon} u_i, \qquad \Psi(x_i) = x_i, \qquad \Psi(a_i) = a_i.$$

Again, checking the relations is a painless verification:

$$\Psi(x_{i+1}u_i - u_i x_i) = (-1)^{\varepsilon} u_i x_{i+1} - (-1)^{\varepsilon} x_i u_i = (-1)^{\varepsilon} \tau_{i+1,i} \overset{3.6}{=} \Psi(\tau_{i,i+1}).$$

Notice that Ψ acts as the identity on $\text{Pol}_n(A)$ since A is symmetric. Also, we have $\sigma \circ \Psi = \psi \circ \sigma$ since both sides act the same on generators and both are anti-automorphisms.

5. Frobenius Demazure operators

This chapter acts as a stepping stone to the basis theorem in the next chapter. We define the Frobenius Demazure operators and deduce relations that will be used to prove a faithful action. Along the way, we give some explicit formulas for the Frobenius Demazure operators as well.

Definition 5.1. For $1 \le i < n$, define the *Demazure operator* $\partial_i : \text{Pol}_n(A) \to \text{Pol}_n(A)$ inductively

$$\partial_i(x_i) = -\tau_{i,i+1}, \qquad \partial_i(x_{i+1}) = \tau_{i+1,i}, \qquad \partial_i(x_j) = 0 \quad (|i-j| > 1), \qquad \partial_i(\mathbf{a}) = 0 \quad (\mathbf{a} \in A^{\otimes n}),$$

and we extend $\Bbbk$-linearly. Next, we define ∂_i on all of $\text{Pol}_n(A)$ by

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g) \quad f, g \in \text{Pol}_n(A),$$

where the $s_i \in S_n$ act by superpermutations.

Remark 5.2. To see that ∂_i is well-defined, start with the same definition of ∂_i but change the domain to the associative algebra F: the free product of $A^{\otimes n}$ and the non-commuting variables $x_1, ..., x_n$. Clearly this map is well-defined since

$$\begin{aligned}
\partial_i(f(gh)) &= \partial_i(f)gh + s_i(f)\partial_i(gh) \\
&= \partial_i(f)gh + s_i(f)\partial_i(g)h + s_i(f)s_i(g)\partial_i(h) \\
&= \partial_i(fg)h + s_i(fg)\partial_i(h) = \partial_i((fg)h).
\end{aligned}$$

Next, if $r \in F$ is one of the relations from Definition 3.1, we have

$$\partial_i(frg) = \partial_i(f)rg + s_i(f)\partial_i(r)g + s_i(f)s_i(r)\partial_i(g) = 0,$$

since $r = 0 = s_i(r)$ in $\text{Pol}_n(A)$ and $\partial_i(r) = 0$ by simple verifications using Lemmas 3.3.2 and 3.3.4 from [Men20] by Mendonca. Then ∂_i induces a map from $F/J \cong \text{Pol}_n(A)$ where J is the ideal generated by the relations from Definition 3.1.

Proposition 5.3. *In* $\text{End}_{\Bbbk}(\text{Pol}_n(A))$*, we have the following relations:*

$$\partial_i s_i = -s_i \partial_i, \qquad \partial_i s_j = s_j \partial_i \quad (|i-j| > 1), \tag{5.1}$$

$$\partial_i^2 = 0, \qquad \partial_i \partial_j = \partial_j \partial_i \quad (|i-j| > 1). \tag{5.2}$$

Proof. We first show that $\partial_i s_i(f) = -s_i\partial_i(f)$ and $\partial_i s_j(f) = s_j \partial_i(f)$ for $|i-j| > 1$ by induction. Both equations are clearly true on the generators $x_i, x_{i+1}, x_k, \mathbf{a}$. Assume both are true for $f, g \in \text{Pol}_n(A)$.

$$\partial_i(s_i(fg)) = \partial_i s_i(f) s_i(g) + s_i^2(f)\partial_i s_i(g) = -s_i\partial_i(f)s_i(g) - s_i^2(f)s_i\partial_i(g) = -s_i\partial_i(fg),$$

$$\partial_i(s_j(fg)) = \partial_i s_j(f) s_j(g) + s_i s_j(f)\partial_i s_j(g) = s_j\partial_i(f)s_j(g) + s_j s_i(f)s_j\partial_i(g) = s_j\partial_i(fg).$$

This completes the proofs of equations (5.1). Next, we also prove (5.2) by induction. We have

$$\partial_i^2(f) = \partial_i\partial_j(f) = \partial_j\partial_i(f) = 0 \quad (|i-j| > 1),$$

for $f \in \{x_i, x_{i+1}, x_k, \mathbf{a}\}$, so the base cases are satisfied. For induction, we have

$$\partial_i\partial_i(fg) = \partial_i(\partial_i(f)g + s_i(f)\partial_i(g)) = \partial_i^2(f)g + s_i\partial_i(f)\partial_i(g) + \partial_i s_i(f)\partial_i(g) + s_i^2(f)\partial_i^2(g) \overset{(5.1)}{=} 0,$$

$$\partial_i\partial_j(fg) = \partial_i(\partial_j(f)g + s_j(f)\partial_j(g)) = \partial_i\partial_j(f)g + s_i\partial_j(f)\partial_i(g) + \partial_i s_j(f)\partial_j(g) + s_i s_j(f)\partial_i\partial_j(g)$$

$$\overset{(5.1)}{=} \partial_j\partial_i(f)g + \partial_j s_i(f)\partial_i(g) + s_j\partial_i(f)\partial_j(g) + s_j s_i(f)\partial_j\partial_i(g) = \partial_j\partial_i(fg).$$ □

Proposition 5.4. *In* $\mathrm{End}_{\Bbbk}(\mathrm{Pol}_n(A))$, *we have the following relations:*

$$s_i s_{i+1}\partial_i = \partial_{i+1}s_i s_{i+1}, \qquad \partial_i s_{i+1}s_i = s_i s_{i+1}\partial_{i+1}, \tag{5.3}$$

$$\partial_i\partial_{i+1}s_i + s_i\partial_{i+1}\partial_i = \partial_{i+1}s_i\partial_{i+1}, \qquad \partial_{i+1}\partial_i s_{i+1} + s_{i+1}\partial_i\partial_{i+1} = \partial_i s_{i+1}\partial_i, \tag{5.4}$$

$$\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}. \tag{5.5}$$

Proof. We prove $s_i s_{i+1}\partial_i = \partial_{i+1}s_i s_{i+1}$ by induction. It is simple to verify equality for $f \in \{x_i, x_{i+1}, x_k, \mathbf{a}\}$. Now for induction, we have

$$\begin{aligned} s_i s_{i+1}\partial_i(fg) &= s_i s_{i+1}\partial_i(f)s_i s_{i+1}(g) + s_i s_{i+1}(f)s_i s_{i+1}\partial_i(g)) \\ &= \partial_{i+1}s_i s_{i+1}(f)s_i s_{i+1}(g) + s_i s_{i+1}(f)\partial_{i+1}s_i s_{i+1}(g) \\ &= \partial_{i+1}(s_i s_{i+1}(f)s_i s_{i+1}(g)) = \partial_{i+1}s_i s_{i+1}(fg). \end{aligned}$$

We multiply both sides by $s_{i+1}s_i$ to obtain our next result:

$$s_i s_{i+1}\partial_i = \partial_{i+1}s_i s_{i+1} \implies \partial_i s_{i+1}s_i = s_i s_{i+1}\partial_{i+1}.$$

Moving on, we show $\partial_i\partial_{i+1}s_i + s_i\partial_{i+1}\partial_i = \partial_{i+1}s_i\partial_{i+1}$ by induction. It is zero on all generators x_k with $1 \le k \le n$ and $\mathbf{a} \in A^{\otimes n}$. For the induction, we have

$$\begin{aligned} \partial_i\partial_{i+1}s_i(fg) + s_i\partial_{i+1}\partial_i(fg) &= \partial_i(\partial_{i+1}s_i(f)s_i(g) + s_{i+1}s_i(f)\partial_{i+1}s_i(g)) + s_i\partial_{i+1}(\partial_i(f)g + s_i(f)\partial_i(g)) \\ &= \partial_i\partial_{i+1}s_i(f)s_i(g) + s_i\partial_{i+1}s_i(f)\partial_i s_i(g) + \partial_i s_{i+1}s_i(f)\partial_{i+1}s_i(g) \\ &\quad + s_i s_{i+1}s_i(f)\partial_i\partial_{i+1}s_i(g) + s_i\partial_{i+1}\partial_i(f)s_i(g) + s_i s_{i+1}\partial_i(f)s_i\partial_{i+1}(g) \\ &\quad + s_i\partial_{i+1}s_i(f)s_i\partial_i(g) + s_i s_{i+1}s_i(f)s_i\partial_{i+1}\partial_i(g) \\ &= \partial_{i+1}s_i\partial_{i+1}(f)s_i(g) + s_{i+1}s_i\partial_{i+1}(f)\partial_{i+1}s_i(g) \\ &\quad + \partial_{i+1}s_i s_{i+1}(f)s_i\partial_{i+1}(g) + s_{i+1}s_i s_{i+1}(f)\partial_{i+1}s_i\partial_{i+1}(g) \\ &= \partial_{i+1}(s_i\partial_{i+1}(f)s_i(g) + s_i s_{i+1}(f)s_i\partial_{i+1}(g)) = \partial_{i+1}s_i\partial_{i+1}(fg). \end{aligned}$$

We use the above result to deduce the next one:

$$\begin{aligned} \partial_i\partial_{i+1}s_i + s_i\partial_{i+1}\partial_i = \partial_{i+1}s_i\partial_{i+1} &\implies \partial_i\partial_{i+1}s_i s_{i+1} + s_i\partial_{i+1}\partial_i s_{i+1} = \partial_{i+1}s_i\partial_{i+1}s_{i+1}, \\ &\implies \partial_i s_i s_{i+1}\partial_i + s_i\partial_{i+1}\partial_i s_{i+1} = -\partial_{i+1}s_i s_{i+1}\partial_{i+1}, \\ &\implies -s_i\partial_i s_{i+1}\partial_i + s_i\partial_{i+1}\partial_i s_{i+1} = -s_i s_{i+1}\partial_i\partial_{i+1}, \\ &\implies \partial_{i+1}\partial_i s_{i+1} + s_{i+1}\partial_i\partial_{i+1} = \partial_i s_{i+1}\partial_i. \end{aligned}$$

Finally we check the braid relation $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$. It is zero on all the generators, and we have

$$\begin{aligned} \partial_i\partial_{i+1}\partial_i(gh) &= \partial_i\partial_{i+1}(s_i(g)\partial_i(h) + \partial_i(g)h) \\ &= \partial_i\left(s_{i+1}s_i(g)\partial_{i+1}\partial_i(h) + \partial_{i+1}s_i(g)\partial_i(h) + s_{i+1}\partial_i(g)\partial_{i+1}(h) + \partial_{i+1}\partial_i(g)h\right) \\ &= s_i s_{i+1}s_i(g)\partial_i\partial_{i+1}\partial_i(h) + \partial_i s_{i+1}s_i(g)\partial_{i+1}\partial_i(h) + s_i\partial_{i+1}s_i(g)\partial_i\partial_i(h) \\ &\quad + \partial_i\partial_{i+1}s_i(g)\partial_i(h) + s_i s_{i+1}\partial_i(g)\partial_i\partial_{i+1}(h) + \partial_i s_{i+1}\partial_i(g)\partial_{i+1}(h) \\ &\quad + s_i\partial_{i+1}\partial_i(g)\partial_i(h) + \partial_i\partial_{i+1}\partial_i(g)h \\ &= s_{i+1}s_i s_{i+1}(g)\partial_{i+1}\partial_i\partial_{i+1}(h) + \partial_{i+1}s_i s_{i+1}(g)\partial_i\partial_{i+1}(h) + s_{i+1}\partial_i s_{i+1}(g)\partial_{i+1}\partial_{i+1}(h) \\ &\quad + \partial_{i+1}\partial_i s_{i+1}(g)\partial_{i+1}(h) + s_{i+1}s_i\partial_{i+1}(g)\partial_{i+1}\partial_i(h) + \partial_{i+1}s_i\partial_{i+1}(g)\partial_i(h) \\ &\quad + s_{i+1}\partial_i\partial_{i+1}(g)\partial_{i+1}(h) + \partial_{i+1}\partial_i\partial_{i+1}(g)h \\ &= \partial_{i+1}\partial_i\partial_{i+1}(gh). \end{aligned}$$

This completes the demonstration. □

Proposition 5.5. *In* $\mathrm{End}_{\Bbbk}(\mathrm{Pol}_n(A))$, *we have the following relations:*

$$s_i(\mathbf{a})\partial_i = \partial_i\mathbf{a}, \qquad x_j\partial_i = \partial_i x_j \quad (j \neq i, i+1), \tag{5.6}$$

$$x_{i+1}\partial_i - \partial_i x_i = \tau_{i,i+1}, \qquad \partial_i x_{i+1} - x_i\partial_i = \tau_{i+1,i}. \tag{5.7}$$

Proof. We verify these statements by direct computation. □

Proposition 5.6. *In* $\mathrm{End}(\mathrm{Pol}_n(A))$*, the following relations hold:*

$$\partial_i x_i x_{i+1} = x_i x_{i+1}\partial_i, \qquad \partial_i x_{i+1} x_i = x_{i+1} x_i \partial_i.$$

Proof. For any $f \in \mathrm{Pol}_n(A)$, we get

$$\partial_i(x_i x_{i+1} f) = \partial_i(x_i x_{i+1}) f + s_i(x_i x_{i+1})\partial_i(f) = -x_{i+1}x_i\partial_i(f) = x_i x_{i+1}\partial_i(f),$$

and $\partial_i x_{i+1} x_i = x_{i+1} x_i \partial_i$ follows since $x_i x_{i+1} = -x_{i+1} x_i$. □

Definition 5.7 (Symmetric)**.** A Frobenius superalgebra (A, tr) is *symmetric* when $\mathrm{tr}(ab) = (-1)^{\bar{a}\bar{b}}\,\mathrm{tr}(ba)\ \forall a, b \in A$.

Proposition 5.8. *If A is symmetric, then*

$$\partial_i(x_i^a) = -\tau_{i,i+1}\sum_{k=0}^{a-1}(-1)^{\varepsilon k} x_{i+1}^k x_i^{a-1-k},$$

$$\partial_i(x_{i+1}^b) = \tau_{i+1,i}\sum_{k=0}^{b-1}(-1)^{\varepsilon k} x_i^k x_{i+1}^{b-1-k}.$$

Proof. The results follow by an induction on a and b. □

Proposition 5.9. *If A is symmetric and the trace map is even, then we have*

$$\partial_i(\mathbf{a}p) = s_i(\mathbf{a})\tau_{i,i+1}\frac{p - s_i(p)}{x_{i+1} - x_i},$$

where $\mathbf{a} \in A^{\otimes n}$ and $p \in \Bbbk[x_1, \dots, x_n]$.

Proof. The result follows by induction on the degree of the polynomial. □

6. Basis Theorem for an Arbitrary Frobenius Superalgebra

The Frobenius Demazure operators that we defined in the last chapter have the appropriate behavior to give us a faithful action on $\mathrm{Pol}_n(A) \otimes \mathrm{nilCox}_n(A)$. We use this to prove injectivity for the basis theorem for $\mathrm{nilHecke}_n(A)$, along with a filtration argument to prove surjectivity. We also obtain a basis for the subalgebra $\mathrm{nilCox}_n(A)$.

Definition 6.1. For $\alpha \in \mathbb{N}^n$, denote $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. Next, for $w \in S_n$, denote $\partial_w := \partial_{a_1}\cdots\partial_{a_k}$ where $w = s_{a_1}\cdots s_{a_k}$ in reduced form. Note that ∂_w is well-defined since $\partial_i\partial_j = \partial_j\partial_i$ and $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$ when $|i-j| > 1$. Likewise, define $u_w := u_{a_1}\cdots u_{a_k}$. We choose $\partial_1 = \mathrm{id}_{\mathrm{Pol}_n(A)}$.

Theorem 6.2. $\mathrm{nilHecke}_n(A)$ *acts on* $\mathrm{Pol}_n(A) \otimes \mathrm{nilCox}_n$ *by*

$$f \cdot (g \otimes u_w) := fg \otimes \partial_w \quad (f \in \mathrm{Pol}_n(A)),$$
$$u_i \cdot (g \otimes u_w) := s_i(g) \otimes u_i u_w + \partial_i(g) \otimes u_w,$$

extended by $\Bbbk$-linearity.

Proof. To ensure that this action is well defined, we must check that it satisfies the relations of $\mathrm{nilHecke}_n(A)$. If we denote $g \otimes u_w$ by gu_w, then the proofs of relations (3.3)–(3.5) are nearly identical to the inductive steps in Propositions 5.3 and 5.4. Finally, using Proposition 5.5, we verify the remaining relations, namely (3.6)–(3.9):

$$\begin{aligned}
au_i(g\otimes u_w) &= a(s_i(g)\otimes u_i(u_w) + \partial_i(g)\otimes u_w) = as_i(g)\otimes u_i(u_w) + a\partial_i(g)\otimes u_w\\
&= s_i(s_i(a)g)\otimes u_i(u_w) + \partial_i(s_i(a)g)\otimes u_w = u_i(s_i(a)g\otimes u_w) = u_i(s_i(a)(g\otimes u_w)),\\
x_j u_i(g\otimes u_w) &= x_j(s_i(g)\otimes u_i u_w + \partial_i(g)\otimes u_w) = x_j s_i(g)\otimes u_i u_w + x_j\partial_i(g)\otimes u_w\\
&= s_i(x_j g)\otimes u_i u_w + \partial_i(x_j g)\otimes u_w = u_i(x_j g\otimes u_w) = u_i x_j(g\otimes u_w),\\
(x_{i+1}u_i - u_i x_i)(g\otimes u_w) &= x_{i+1}\partial_i(g)\otimes u_w + x_{i+1}s_i(g)\otimes u_i u_w - \partial_i(x_i g)\otimes u_w - s_i(x_i g)\otimes u_i u_w
\end{aligned}$$

$$
\begin{aligned}
&= (x_{i+1}\partial_i(g) - \partial_i(x_i g)) \otimes u_w + (x_{i+1}s_i(g) - s_i(x_i g)) \otimes u_i u_w \\
&= \tau_{i,i+1} g \otimes u_w + 0 = \tau_{i,i+1}(g \otimes u_w), \\
(u_i x_{i+1} - x_i u_i)(g \otimes u_w) &= \partial_i x_{i+1}(g) \otimes u_w + s_i(x_{i+1}g) \otimes u_i u_w - x_i \partial_i(g) \otimes u_w - x_i s_i(g) \otimes u_i u_w \\
&= (\partial_i x_{i+1}(g) - x_i \partial_i(g)) \otimes u_w + (s_i(x_{i+1}g) - x_i s_i(g)) \otimes u_i u_w \\
&= \tau_{i+1,i} g \otimes u_w + 0 = \tau_{i+1,i}(g \otimes u_w).
\end{aligned}
$$

This completes the proof. □

Definition 6.3 (Filtration)**.** Denoting the free product of $\mathrm{Pol}_n(A)$ and nilCox_n as F, we get a quotient map $\varphi : F \to \mathrm{nilHecke}_n(A)$ given by the quotient of the relations (3.6)–(3.9). In F, we define $\deg(x_i) = 1$, $\deg(u_i) = 0$, $\deg(\mathbf{a}) = 0$, and the degree of a product to be the sum of the degrees. This gives us a decomposition $F = \bigoplus_{r=0}^{\infty} F_r$, where F_r is the set of elements of F of degree r. For each k, define $\mathrm{nilHecke}_n(A)_{\leq k} := \varphi(\bigoplus_{r=0}^{k} F_r)$.

Remark 6.4. The relations $x_i x_j = (-1)^{\varepsilon} x_j x_i$ and $\mathbf{a} x_i = (-1)^{\varepsilon \bar{\mathbf{a}}} x_i \psi_i(\mathbf{a})$ in $\mathrm{Pol}_n(A)$ imply that every element is equal to a sum of terms of the form $\mathbf{a} x^{\alpha}$ where $\alpha \in \mathbb{N}^n$. This is used in the following basis theorem.

Theorem 6.5 (Basis Theorem)**.** *The map*

$$\Phi : \mathrm{Pol}_n(A) \otimes \mathrm{nilCox}_n \to \mathrm{nilHecke}_n(A), \ \Phi(g \otimes u_w) = g u_w,$$

is an isomorphism of vector spaces.

Proof. Let

$$B_1 = \{\mathbf{a} x^{\alpha} \otimes u_w \mid \mathbf{a} \in B^{\otimes n}, \ \alpha \in \mathbb{N}^n, \ w \in S_n\} \subseteq \mathrm{Pol}_n(A) \otimes \mathrm{nilCox}_n,$$

where B is a basis for A. Therefore B_1 is a basis for $\mathrm{Pol}_n(A) \otimes \mathrm{nilCox}_n$ since $\{u_w \mid w \in S_n\}$ is a basis for nilCox_n, described in Section 2 of [FS94] by Fomin and Stanley. We will prove that Φ is surjective by induction on the filtration of $\mathrm{nilHecke}_n(A)$ from Definition 6.3. First, we have $\mathrm{nilHecke}_n(A)_{\leq 0} \subseteq \mathrm{Im}(\Phi)$ since any polynomial with only generators u_i and elements from $A^{\otimes n}$ is equal to a sum of elements of the form $\mathbf{a} u_w$ since we have $u_i \mathbf{a} = s_i(\mathbf{a}) u_i$. Next, assume $\mathrm{nilHecke}_n(A)_{\leq r} \subseteq \mathrm{Im}(\Phi)$ for some $r \in \mathbb{N}$. We wish to show that $\mathrm{nilHecke}_n(A)_{\leq (r+1)} \subseteq \mathrm{Im}(\Phi)$. Suppose we have an element $f_1 z_1 f_2 z_2 \cdots f_k z_k \in \mathrm{nilHecke}_n(A)_{\leq (r+1)}$, for some $k \in \mathbb{N}$ with f_i in $\mathrm{Pol}_n(A)$ and $z_i \in \mathrm{nilCox}_n$. Then each variable can be moved over to the left using relations (3.7)–(3.9). When $u_i x_{i+1} = x_i u_i + \tau_{i,i+1}$ or $u_i x_i = x_{i+1} u_i - \tau_{i+1,i}$ are used, then the second term containing $\tau_{i,j}$ will be in $\mathrm{nilHecke}_n(A)_{\leq r}$, so by the induction hypothesis, it is in $\mathrm{Im}(\Phi)$. As for the leading term, once all the variables are on the left, then it is clearly in $\mathrm{Im}(\Phi)$. Since $\mathrm{Im}(\Phi)$ is closed under addition, the sum of all the terms, each of which we've shown to be in $\mathrm{Im}(\Phi)$, is in $\mathrm{Im}(\Phi)$. This completes the induction, so Φ is surjective.

Next, let 1_C denote 1_{nilCox_n}. Since we have $\partial_i(1_A^{\otimes n}) = 0 \ \forall i$, we get

$$\mathbf{a} x^{\alpha} u_w((1_A^{\otimes n}) \otimes 1_C) = \mathbf{a} x^{\alpha}(w(1_A^{\otimes n}) \otimes u_w 1_C) = \mathbf{a} x^{\alpha}(1_A^{\otimes n} \otimes u_w) = \mathbf{a} x^{\alpha} \otimes u_w.$$

Therefore the map $u : \mathrm{nilHecke}_n(A) \to \mathrm{Pol}_n(A) \otimes \mathrm{nilCox}_n$ defined by $u(w) = w \cdot ((1_A^{\otimes n}) \otimes 1_C)$ is left inverse of Φ since $u \circ \Phi = \mathrm{id}_{\mathrm{Pol}_n(A) \otimes \mathrm{nilCox}_n}$. This means that Φ is injective, so we have an isomorphism as desired, and conclude that $\Phi(B_1)$ is a basis for $\mathrm{nilHecke}_n(A)$. □

Corollary 6.6. *We have a basis* $\{\mathbf{a} u_w \mid w \in S_n, \, \mathbf{a} \in A^{\otimes n}\}$ *for* $\mathrm{nilCox}_n(A)$ *as a vector space.*

Proof. It is clearly spanning since we have the relation $s_i(\mathbf{a}) u_i = u_i \mathbf{a}$. Linear independence follows from the inclusion map $\mathrm{nilCox}_n(A) \to \mathrm{nilHecke}_n(A)$ and Theorem 6.5. □

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