AN INTRODUCTION TO OPERAD THEORY

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ABSTRACT. We give an introduction to category theory and operad theory aimed at the undergraduate level. We first explore operads in the category of sets, and then generalize to other familiar categories. Finally, we develop tools to construct operads via generators and relations, and provide several examples of operads in various categories. Throughout, we highlight the ways in which operads can be seen to encode the properties of algebraic structures across different categories.

CONTENTS

1. INTRODUCTION

Sets equipped with operations are ubiquitous in mathematics, and many familiar operations share key properties. For instance, the addition of real numbers, composition of functions, and concatenation of strings are all associative operations with an identity element. In other words, all three are examples of monoids. Rather than working with particular examples of sets and operations directly, it is often more convenient to abstract out their common properties and work with algebraic structures instead. For instance, one can prove that in any monoid, arbitrarily long products $x_1 x_2 \cdots x_n$ have an unambiguous value, and thus brackets

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may be omitted. By working at the level of algebraic structures, one can prove theorems in greater generality, and avoid repeating the same arguments for many similar cases.

Much like the sets and operations that they abstract, many common algebraic structures have similar properties. For instance, semigroups, monoids, groups, rings, and vector spaces all feature an associative operation, and all but the first of those structures have an identity element. As such, one can prove similar theorems for many algebraic structures; for example, the aforementioned generalized associativity holds in any structure that has an associative operation. We are thus presented with the opportunity to abstract up another level, and pass to a structure that models other algebraic structures. Operads are precisely this type of meta-algebraic structure.

Operads were first rigorously defined by J. Peter May in his 1972 book [May72], which investigated the applications of operads to loop spaces and homotopy analysis. In this document, we will explore the basic notions of operad theory, with a focus on using operads to model the properties of other algebraic structures via category theoretic constructions. This document is aimed at the undergraduate level. We assume that the reader is familiar with basic linear algebra, group theory, ring theory, and with tensor products of vector spaces. Knowledge of basic graph theory is needed to understand some of the constructions in Section 5.1. No background in category theory or operad theory is required. For a more comprehensive treatment of operads, see, for example, [Lei04].

We begin in Section 2 by recalling the definitions of common algebraic structures, and then give a brief introduction to category theory. In Section 3, we define and give examples of operads in the category of sets, and then prove several theorems that show a correspondence between operads and various algebraic structures. In Section 4, we treat operads defined over more general categories, and prove similar correspondence theorems for them. Finally, in Section 5, we develop the necessary machinery to instantiate operads using generators and relations, and give several examples of the different types of operads that can be constructed using those methods.

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2. Preliminary Definitions

2.1. Algebraic Structures.

In this section, we give definitions for the algebraic structures that will be used throughout the document. Note that graded rings and modules are a useful source of examples, but familiarity with graded structures is not strictly needed for operad theory. For a detailed introduction to abstract algebra, see, for example, [JB18].

Definition 2.1.1 (Semigroup)**.** A *semigroup* is a pair (X, m), where X is a set and m is an associative binary function on X called the semigroup multiplication. That is, $m\colon X^2 \to X$ satisfies $m(m(x, y), z) = m(x, m(y, z))$ for all $x, y, z \in X$.

Let (X, m) and (Y, p) be semigroups. A *semigroup homomorphism* from (X, m) to (Y, p) is a function $f\colon X \to Y$ such that $f(m(x, x')) = p(f(x), f(x'))$ for all $x, x' \in X$. In other words, f commutes with the semigroup multiplication.

Definition 2.1.2 (Monoid)**.** A *monoid* is a triple (X, m, I) such that (X, m) is a semigroup, and $I\colon \{\star\} \to X$ is a function from a singleton set to X that outputs an identity element for (X, m). That is, $m(I(\star), x) = x = m(x, I(\star))$ for all $x \in X$. This definition is essentially equivalent to the traditional definition of a monoid as a triple (X, m, i) where $i \in X$ is itself an identity element, but the formulation in terms of maps is easier to work with in a category-theoretic framework.

Let (X, m, I) and (Y, p, E) be monoids. A *monoid homomorphism* from (X, m, I) to (Y, p, E) is a semigroup homomorphism $f\colon X \to Y$ that additionally satisfies $f \circ I = E$.

Definition 2.1.3 (Associative Algebra)**.** A *associative algebra* over a field $\mathbb{K}$ is a pair (V, m), where V is a vector space over $\mathbb{K}$ and m is an associative binary $\mathbb{K}$-bilinear operation on V.

Let (V, m) and (W, p) be associative algebras over $\mathbb{K}$. A *homomorphism of associative algebras* from (V, m) to (W, p) is a $\mathbb{K}$-linear map $f\colon V \to W$ such that $f(m(v, v')) = p(f(v), f(v'))$ for all $v, v' \in V$.

Definition 2.1.4 (Unital Associative Algebra)**.** A *unital associative algebra* over $\mathbb{K}$ is a triple (V, m, I) such that (V, m) is an associative algebra and $I\colon \mathbb{K} \to V$ is a linear map such that $I(1_{\mathbb{K}})$ is an identity element for (V, m).

Let (V, m, I) and (W, p, E) be unital associative algebras over $\mathbb{K}$. A *homomorphism of unital associative algebras* from (V, m, I) to (W, p, E) is a homomorphism of associative algebras $f\colon V \to W$ that additionally satisfies $f \circ I = E$.

Definition 2.1.5 (Graded Ring, Graded Module)**.** Let Γ be a monoid, and denote its multiplication by juxtaposition. A Γ-*graded ring* is a ring A together with a direct sum decomposition $A = \bigoplus_{\gamma \in \Gamma} R_\gamma$, where each R_γ is an abelian group with respect to the addition in R, and $R_\gamma R_\alpha \subseteq R_{\gamma\alpha}$ for any $\gamma, \alpha \in \Gamma$. That is, for any $g \in R_\gamma, a \in R_\alpha$, we have $ga \in R_{\gamma\alpha}$.

A Γ-*graded A-module* is a module M over A (when considered as a non-graded ring) together with a direct sum decomposition $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$, where each M_γ is an abelian group with respect to the addition in M, and $A_\gamma M_\alpha \subseteq M_{\gamma\alpha}$ for any $\gamma, \alpha \in \Gamma$.

Definition 2.1.6. Let A be a Γ-graded ring, and let $\gamma \in \Gamma$. Any nonzero element $g \in A_\gamma$ is called a *homogeneous element of grade* γ, and we write $\overline{g} = \gamma$ to indicate g's grade. We use the same notation for elements of graded modules.

Example 2.1.7. Any ring A can be given the structure of a Γ-graded ring by setting $A_e = A$ for the identity element e of Γ, and $A_\gamma = 0$ (denoting the zero subring) for all other $\gamma \in \Gamma$. That is, all nonzero elements of A have grade e. This grading is called the trivial grading.

A polynomial ring $A[x]$ is naturally graded by $\mathbb{N}$; take $(A[x])_i$ to be $Ax^i = \{ax^i \mid a \in A\}$ for each $i \in \mathbb{N}$. For instance, $\overline{x^2} = 2$. Note that not all elements of $A[x]$ are homogeneous; one example of an inhomogeneous element is $x + x^2$, as it doesn't belong to any $(A[x])_i$.

Definition 2.1.8 (Graded Submodule, Graded Quotient Module)**.** Let A be a Γ-graded ring, M a Γ-graded A-module, and N a (non-graded) submodule of M. For each $\gamma \in \Gamma$, define $N_\gamma = N \cap M_\gamma$. If N is a Γ-graded A-module when equipped with the choice of grading $N = \bigoplus_{\gamma \in \Gamma} N_\gamma$, we say that N (together with this grading) is a *graded submodule of* M.

If N is a graded submodule of M, the quotient module M/N is itself a graded A-module, where the grading is given by $(M/N)_\gamma = M_\gamma / N_\gamma$.

Definition 2.1.9 (Grade-Preserving Module Homomorphism)**.** Let A be a Γ-graded ring, and M and N two Γ-graded A-modules. A *grade-preserving module homomorphism*, also called a *graded module homomorphism*, is a module homomorphism $f\colon M \to N$ such that $f(M_\gamma) \subseteq N_\gamma$ for all $\gamma \in \Gamma$.

Definition 2.1.10 (Super Vector Space)**.** Let $\mathbb{K}$ be a field. Then a *super* $\mathbb{K}$*-vector space* is a $\mathbb{K}$-vector space V equipped with a subspace decomposition $V = V_0 \oplus V_1$. In other words, a super $\mathbb{K}$-vector space is a $\mathbb{Z}_2$-graded $\mathbb{K}$-module, where $\mathbb{K}$ is given the trivial grading.

2.2. **Category Theory.**

Category theory is perhaps the most abstract branch of mathematics. Rather than dealing directly with objects like sets or vector spaces, category theoretic constructions focus on the maps between those objects. By working at this level of abstraction, the similarities between different mathematical structures can be more readily and rigorously identified. In this section, we provide an introduction to basic category theory, with a focus on symmetric monoidal categories, an essential component of operad theory. For a broader introduction to category theory at the undergraduate level, see, for example, [Lei14].

Definition 2.2.1 (Category)**.** A *category* $\mathcal{C}$ consists of the following:

- A collection (e.g. a set or proper class) of *objects*, denoted $\mathrm{Ob}(\mathcal{C})$,
- For each pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, a collection of *morphisms from* x *to* y, denoted $\mathrm{Hom}_{\mathcal{C}}(x, y)$ or just $\mathrm{Hom}(x, y)$ when the category is clear,
- For any objects $x, y, z \in \mathrm{Ob}(\mathcal{C})$, a function $\circ\colon \mathrm{Hom}(y, z) \times \mathrm{Hom}(x, y) \to \mathrm{Hom}(x, z)$ called *composition*, which we usually write as $\circ(g, f) = g \circ f$,
- For each object $x \in \mathrm{Ob}(\mathcal{C})$, an element $\mathrm{id}_x \in \mathrm{Hom}(x, x)$ called the *identity morphism* for x.

Morphisms from x to y are also called morphisms with domain x and codomain y, morphisms with source x and target y, or arrows from x to y. For any morphism $f \in \mathrm{Hom}(x, y)$, we write $f\colon x \to y$ to indicate its source and target.

We additionally require that the following axioms are satisfied:

- Composition is associative. That is, for each quadruple of objects $w, x, y, z \in \mathrm{Ob}(\mathcal{C})$ and triple of morphisms $f \in \mathrm{Hom}(y, z), g \in \mathrm{Hom}(x, y), h \in \mathrm{Hom}(w, x)$, we have $(f \circ g) \circ h = f \circ (g \circ h)$.
- Identity morphisms act as identity elements for composition. That is, for any pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$ and a morphism $f \in \mathrm{Hom}(x, y)$, we have $\mathrm{id}_y \circ f = f = f \circ \mathrm{id}_x$.

Definition 2.2.2 (Inverse Morphism, Isomorphism)**.** Let $\mathcal{C}$ be a category, and $f \in \mathrm{Hom}(x, y)$ any morphism in $\mathcal{C}$. A morphism $g \in \mathrm{Hom}(y, x)$ is said to be the *inverse morphism* of f if $g \circ f = \mathrm{id}_x$ and $f \circ g = \mathrm{id}_y$. Inverse morphisms, when they exist, are unique. We write $g = f^{-1}$ for the inverse of f. A morphism that has an inverse is called an *isomorphism*.

Example 2.2.3 (Category of Sets)**.** The category of sets, denoted Set, is defined by taking Ob(Set) to be the collection of all sets, and each $\mathrm{Hom}(x, y)$ to be the set of functions from

x to y. The composition in this category is given by the usual composition of functions, and the identity morphisms are set-theoretic identity functions. Isomorphisms in Set are bijections.

Example 2.2.4 (Category of $\mathbb{K}$-Vector Spaces)**.** For any field $\mathbb{K}$, the category of $\mathbb{K}$-vector spaces, $\text{Vect}_{\mathbb{K}}$, is defined by taking $\text{Ob}(\text{Vect}_{\mathbb{K}})$ to be the collection of all $\mathbb{K}$-vector spaces, and each $\text{Hom}(x, y)$ to be the set of linear maps from x to y. The composition is given by the composition of linear maps, and the identity morphisms are identity linear maps. Isomorphisms in $\text{Vect}_{\mathbb{K}}$ are bijective linear maps.

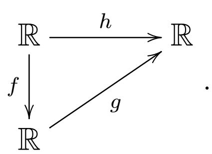
Example 2.2.5 (Category of Γ-graded A-modules)**.** Let Γ be a monoid, and A a commutative Γ-graded ring. The category of Γ-graded A-modules, Mod_A^Γ, is defined by taking $\text{Ob}(\text{Mod}_A^\Gamma)$ to be the collection of all Γ-graded A-modules, and each $\text{Hom}(x, y)$ to be the set of grade-preserving module homomorphisms from x to y. One can also consider the category of graded modules where the hom-sets consist of *all* module homomorphisms from x to y, but it is more convenient to use only grade-preserving maps in the context of operad theory. The composition is given by the composition of module homomorphisms, and the identity morphisms are identity module homomorphisms. Isomorphisms in Mod_A^Γ are bijective grade-preserving module homomorphisms.

Remark 2.2.6. Let A be a commutative ring, and let $\mathbb{Z}_1$ denote the trivial monoid. Then $\text{Mod}_A^{\mathbb{Z}_1}$ is essentially the same as the category of A-modules with no grading; simply forget the grading on any given module in $\text{Mod}_A^{\mathbb{Z}_1}$. As such, any of the examples in this document that deal with the category of Γ-graded A-modules can also be applied to ordinary A-modules. In the same way, as $\mathbb{K}$-vector spaces are simply $\mathbb{K}$-modules, we can view $\text{Mod}_{\mathbb{K}}^{\mathbb{Z}_1}$ as being the same as $\text{Vect}_{\mathbb{K}}$.

Definition 2.2.7 (Locally Small Category)**.** A category $\mathcal{C}$ is called *locally small* if $\text{Hom}_{\mathcal{C}}(x, y)$ is a set (i.e. not a proper class or some other larger collection) for all $x, y \in \text{Ob}(\mathcal{C})$. These sets of morphisms are called *external hom-sets*, or just *hom-sets*. Set, $\text{Vect}_{\mathbb{K}}$, and Mod_A^Γ are all examples of locally small categories.

Definition 2.2.8 (Commutative Diagram)**.** When dealing with equations that involve the composition of morphisms in a category, we often write the compositions diagrammatically to serve as a visual aid. In these diagrams, vertices represent objects in a category, and the arrows between vertices represent morphisms between those objects. A directed path in a diagram represents the composition of morphisms in the order that the arrows are traversed. To say that a diagram is *commutative* means that for any two directed paths with the same start and endpoints, the composite morphisms corresponding to those paths are equal. For instance, consider the set of real numbers $\mathbb{R} \in \text{Ob}(\text{Set})$, and the real functions (i.e. morphisms in Set) given by $f(x) = 2x, g(x) = 3x$, and $h(x) = 6x$. Then the following commutative diagram indicates the fact that $g \circ f = h$:

(1)



Note that the order of composition, $g \circ f$, is opposite to the order that the arrows in the diagram are traversed, corresponding to the fact that $g \circ f$ is the function that first applies f, then g. For larger diagrams, no bracketing is required due to the fact that the composition of morphisms in any category is associative.

For another example, take any associative $\mathbb{K}$-algebra (V, m). Let $m \times \mathrm{id}_V \colon V^3 \to V^2$ denote the function with action $(x, y, z) \mapsto (m(x, y), z)$, and similarly let $\mathrm{id}_V \times m \colon V^3 \to V^2$ denote the function with action $(x, y, z) \mapsto (x, m(y, z))$. Then the following commutative diagram reflects the associativity of m—namely that $m(m(x, y), z) = m(x, m(y, z))$:

$$\begin{array}{ccc} V^3 & \xrightarrow{m \times \mathrm{id}_V} & V^2 \\ {\scriptstyle \mathrm{id}_V \times m}\downarrow & & \downarrow{\scriptstyle m} \\ V^2 & \xrightarrow[m]{} & V \end{array} . \tag{2}$$

Definition 2.2.9 (Linear Diagram)**.** A diagram is called *linear* or *formal* if each variable appears at most once in each vertex. For instance, diagram (1) is linear, but diagram (2) is not, since, for example, the variable V appears three times in the top left vertex: $V^3 = V \times V \times V$.

Definition 2.2.10 (Opposite Category)**.** Let $\mathcal{C}$ be a category. The *opposite category of* $\mathcal{C}$, denoted $\mathcal{C}^{\mathrm{op}}$, is defined as follows:

- Set $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$. That is, the opposite category has the same objects as $\mathcal{C}$. For any object $x \in \mathrm{Ob}(\mathcal{C})$, we sometimes denote the corresponding object in $\mathcal{C}^{\mathrm{op}}$ as x^{op} to emphasize the fact we are working in a different category.
- For each pair of objects $x^{\mathrm{op}}, y^{\mathrm{op}} \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, set $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x^{\mathrm{op}}, y^{\mathrm{op}}) = \mathrm{Hom}_{\mathcal{C}}(y, x)$. That is, the morphisms in $\mathcal{C}^{\mathrm{op}}$ are the same as those in $\mathcal{C}$, but with the source and target of each morphism being formally swapped. While objects in opposite categories are only sometimes written in the form x^{op} in practice, we almost always use the notation $f^{\mathrm{op}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y)$ to denote the opposite morphism corresponding to a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(y, x)$.
- For any triple of objects $x^{\mathrm{op}}, y^{\mathrm{op}}, z^{\mathrm{op}} \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, define the composition function $\circ_{\mathrm{op}} \colon \mathrm{Hom}(y^{\mathrm{op}}, z^{\mathrm{op}}) \times \mathrm{Hom}(x^{\mathrm{op}}, y^{\mathrm{op}}) \to \mathrm{Hom}(x^{\mathrm{op}}, z^{\mathrm{op}})$ by $\circ_{\mathrm{op}}(g^{\mathrm{op}}, f^{\mathrm{op}}) = (\circ(f, g))^{\mathrm{op}}$ for any $f^{\mathrm{op}} \in \mathrm{Hom}(x^{\mathrm{op}}, y^{\mathrm{op}})$ and $g^{\mathrm{op}} \in \mathrm{Hom}(y^{\mathrm{op}}, z^{\mathrm{op}})$, where $\circ$ denotes the composition in $\mathcal{C}$. That is, for any such pair of morphisms we set $g^{\mathrm{op}} \circ_{\mathrm{op}} f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$; note that we swap the order of f and g to obtain a pair of composable morphisms in $\mathcal{C}$.
- The identity morphisms in the opposite category are the same as those in $\mathcal{C}$. In other words, $\mathrm{id}_{x^{\mathrm{op}}} = (\mathrm{id}_x)^{\mathrm{op}}$ for all $x^{\mathrm{op}} \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$.

Definition 2.2.11 (Terminal Object)**.** Let $\mathcal{C}$ be a category. An object $t \in \mathrm{Ob}(\mathcal{C})$ is called a *terminal object* if for every object $x \in \mathrm{Ob}(\mathcal{C})$, there is a unique morphism from x to t. In Set, any singleton set is a terminal object. In $\mathrm{Vect}_{\mathbb{K}}$, the zero vector space is a terminal object. In Mod_A^{Γ}, the trivial graded module is a terminal object.

Definition 2.2.12 (Initial Object)**.** Let $\mathcal{C}$ be a category. An object $i \in \mathrm{Ob}(\mathcal{C})$ is called an *initial object* if for every object $x \in \mathrm{Ob}(\mathcal{C})$, there is a unique morphism from i to x. Equivalently, i is an initial object if and only it is a terminal object in $\mathcal{C}^{\mathrm{op}}$. In Set, the empty set is the unique initial object. In $\mathrm{Vect}_{\mathbb{K}}$, the zero vector space is an initial object. In Mod_A^{Γ}, the trivial graded module is an initial object.

Definition 2.2.13 (Product Category)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *product category* $\mathcal{C} \times \mathcal{D}$ is defined as follows:

- The collection of objects in $\mathcal{C} \times \mathcal{D}$ consists of the ordered pairs (c, d), where $c \in \mathrm{Ob}(\mathcal{C})$ and $d \in \mathrm{Ob}(\mathcal{D})$.
- For any $(x, x'), (y, y') \in \mathrm{Ob}(\mathcal{C} \times \mathcal{D})$, define $\mathrm{Hom}((x, x'), (y, y'))$ to consist of the ordered pairs (f, a), where $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ and $a \in \mathrm{Hom}_{\mathcal{D}}(x', y')$.
- For any morphisms $(f, a) \in \mathrm{Hom}((x, x'), (y, y'))$ and $(g, b) \in \mathrm{Hom}((y, y'), (z, z'))$, define the composition in $\mathcal{C} \times \mathcal{D}$ by: $(g, b) \circ (f, a) = (g \circ f, b \circ a)$. That is, compose the morphisms component-wise.
- The identity morphisms in the product category are the pairs of identity morphisms from $\mathcal{C}$ and $\mathcal{D}$. That is, $\mathrm{id}_{(c,d)} = (\mathrm{id}_c, \mathrm{id}_d)$ for any $(c, d) \in \mathrm{Ob}(\mathcal{C} \times \mathcal{D})$.

Definition 2.2.14 (Functor, Identity Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* F from $\mathcal{C}$ to $\mathcal{D}$, denoted $F : \mathcal{C} \to \mathcal{D}$, consists of:

- A map $F_{\mathrm{Ob}} \colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,
- For each pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, a map $F_{x,y} \colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F_{\mathrm{Ob}}(x), F_{\mathrm{Ob}}(y))$.

We usually suppress the subscripts and write $F(x) = F_{\mathrm{Ob}}(x)$ and $F(f) = F_{x,y}(f)$ for the action of a functor on either objects or morphisms. We require that these maps respect composition and identities in the following sense:

- For any composition of morphisms $g \circ f$ in $\mathcal{C}$, we have $F(g \circ f) = F(g) \circ F(f)$,
- For any object $x \in \mathrm{Ob}(\mathcal{C})$, we have $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

One can easily verify that the composition of two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ (that is, where each of the component maps are composed) results in another functor, $G \circ F : \mathcal{C} \to \mathcal{E}$. The *identity functor on* $\mathcal{C}$, denoted $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$, is defined by $\mathrm{id}_{\mathcal{C}}(x) = x$ and $\mathrm{id}_{\mathcal{C}}(f) = f$ for all objects x and morphisms f.

Definition 2.2.15 (External Hom-Functor)**.** Let $\mathcal{C}$ be a locally small category. Then the *external hom-functor* for $\mathcal{C}$ is the functor $\mathrm{Hom} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ that sends a pair of objects (x, y) to the hom-set $\mathrm{Hom}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y)$, and sends any pair of morphisms (f^{op}, g) with $f^{\mathrm{op}} \colon x \to x', g \colon y \to y'$ to the function $\mathrm{Hom}(f^{\mathrm{op}}, g) \colon \mathrm{Hom}(x, y) \to \mathrm{Hom}(x', y')$ defined as follows: for any morphism $h \colon x \to y$, set $\mathrm{Hom}(f^{\mathrm{op}}, g)(h) = g \circ h \circ f$. Note that f is a morphism from x' to x since f^{op} is a morphism from x to x', and thus this composition has the desired source and target.

Definition 2.2.16 (Isomorphism of Categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called an *isomorphism of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$. In such a case, we call G the inverse functor to F, write $G = F^{-1}$, and say that $\mathcal{C}$ and $\mathcal{D}$ are isomorphic categories. Isomorphic categories are essentially identical in terms of their structure as categories.

Definition 2.2.17 (Natural Transformation, Natural Isomorphism)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F, G : \mathcal{C} \to \mathcal{D}$ two functors between them. A *natural transformation* α from F to G, denoted $\alpha \colon F \to G$, is a map $\alpha \colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Hom}_{\mathcal{D}}(F(-), G(-))$, with action denoted $\alpha(x) = \alpha_x \colon F(x) \to G(x)$, such that for any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, the following diagram commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle\alpha_x}\downarrow & & \downarrow{\scriptstyle\alpha_y} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array} .$$

That is, we require $\alpha_y \circ F(f) = G(f) \circ \alpha_x$. One can easily verify that the composition of two natural transformations results in another natural transformation. For any functor $F: \mathcal{C} \to \mathcal{D}$, the *identity natural transformation* is denoted $\mathrm{id}_F : F \to F$ and defined by $\mathrm{id}_F(x) = \mathrm{id}_{F(x)}$ for all objects $x \in \mathrm{Ob}(\mathcal{C})$.

For any natural transformation $\alpha: F \to G$, a natural transformation $\beta: G \to F$ is said to be the inverse natural transformation of α if $\beta \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \beta = \mathrm{id}_G$. Such inverses are unique, and we write $\beta = \alpha^{-1}$. Natural transformations that have an inverse are called *natural isomorphisms*.

Definition 2.2.18 (Monoidal Category)**.** A *monoidal category* is a category $\mathcal{C}$ equipped with the following:

- A functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the tensor product and written $\otimes(-,-) = - \otimes -$,
- An object $u \in \mathrm{Ob}(\mathcal{C})$, called the unit object or monoidal unit,
- A natural isomorphism $\alpha: (-\otimes-)\otimes- \to -\otimes(-\otimes-)$ called the associator, natural in all three of its arguments, with components of the form $\alpha_{x,y,z}: (x\otimes y)\otimes z \to x\otimes(y\otimes z)$,
- A natural isomorphism $\lambda: u \otimes - \to -$ called the left unitor, with components of the form $\lambda_x: u \otimes x \to x$,
- A natural isomorphism $\rho: - \otimes u \to -$ called the right unitor, with components of the form $\rho_x: x \otimes u \to x$,

such that the following two diagrams, respectively called the triangle and pentagon diagrams, commute for any objects $w, x, y, z \in \mathrm{Ob}(\mathcal{C})$:

(3)
$$\begin{array}{ccc} (x \otimes u) \otimes y & \xrightarrow{\rho_x \otimes \mathrm{id}_y} & x \otimes y \\ {\scriptstyle\alpha_{x,u,y}}\downarrow & \nearrow{\scriptstyle \mathrm{id}_x \otimes \lambda_y} & \\ x \otimes (u \otimes y) & & \end{array} ,$$

(4)
$$\begin{array}{ccccc} & & (w \otimes x) \otimes (y \otimes z) & & \\ & \nearrow{\scriptstyle\alpha_{w\otimes x,y,z}} & & \searrow{\scriptstyle\alpha_{w,x,y\otimes z}} & \\ ((w \otimes x) \otimes y) \otimes z & & & & (w \otimes (x \otimes (y \otimes z))) \\ {\scriptstyle\alpha_{w,x,y}\otimes \mathrm{id}_z}\downarrow & & & & \uparrow{\scriptstyle\mathrm{id}_w \otimes \alpha_{x,y,z}} \\ (w \otimes (x \otimes y)) \otimes z & & \xrightarrow[\alpha_{w,x\otimes y,z}]{} & & w \otimes ((x \otimes y) \otimes z) \end{array} .$$

Example 2.2.19. There is a canonical monoidal category structure on each of Set, $\mathrm{Vect}_{\mathbb{K}}$, and Mod_A^Γ. In the first case, the tensor product is defined by sending a pair of sets (X, Y) to their cartesian product $X \times Y$, and a pair of functions $f: X \to X', g: Y \to Y'$ to the function $f \times g: X \times Y \to X' \times Y'$, defined coordinate-wise: $(f \times g)(x, y) = (f(x), g(y))$ for any $x \in X, y \in Y$. Any singleton set $\{\star\}$ may be selected as the unit object. The associator sends any element $((x, y), z) \in (X \times Y) \times Z$ to $(x, (y, z)) \in X \times (Y \times Z)$. The left unitor has action $(\star, x) \mapsto x$, and the right unitor has action $(x, \star) \mapsto x$.

For the category of $\mathbb{K}$-vector spaces, the tensor product is given by the usual tensor product of vector spaces acting on both elements of the vector spaces and on linear maps between those spaces. The unit object is the field $\mathbb{K}$ considered as a one-dimensional $\mathbb{K}$-vector space. The associator acts on simple tensors by $(v \otimes w) \otimes u \mapsto v \otimes (w \otimes u)$, and is extended linearly to the rest of $(V \otimes W) \otimes U$. The left unitor has action $k \otimes v \mapsto k \cdot v$, and the right unitor has action $v \otimes k \mapsto k \cdot v$, where $\cdot$ denotes scaling by $k \in \mathbb{K}$.

Finally, in Mod_A^Γ, the tensor product is given by the usual tensor product of graded modules—recall from Definition 2.2.5 that A is required to be commutative, so this tensor product is well-defined. The unit object is the ring A considered as a trivially-graded module over itself, and the associator and unitors are analogous to those in the case of $\mathbb{K}$-vector spaces.

Definition 2.2.20 (Symmetric Monoidal Category)**.** A *symmetric monoidal category* is a monoidal category $\mathcal{C}$ equipped with a natural isomorphism $B_{-,-} \colon - \otimes - \to - \otimes -$ called the (symmetric) braiding, with components of the form $B_{x,y} \colon x \otimes y \to y \otimes x$. This braiding must satisfy $B_{y,x} \circ B_{x,y} = \mathrm{id}_{x \otimes y}$ for all $x, y \in \mathrm{Ob}(\mathcal{C})$, and the following diagram, called the first hexagon diagram (or just the hexagon diagram) must commute:

(5)
$$\begin{array}{ccccccc} & & x \otimes (y \otimes z) & \xrightarrow{B_{x,y\otimes z}} & (y \otimes z) \otimes x & & \\ & \nearrow^{\alpha_{x,y,z}} & & & & \searrow^{\alpha_{y,z,x}} & \\ (x \otimes y) \otimes z & & & & & & y \otimes (z \otimes x) \\ & \searrow_{B_{x,y} \otimes \mathrm{id}_z} & & & & \nearrow_{\mathrm{id}_y \otimes B_{x,z}} & \\ & & (y \otimes x) \otimes z & \xrightarrow{\alpha_{y,x,z}} & y \otimes (x \otimes z) & & \end{array} .$$

Example 2.2.21. There is a canonical symmetric braiding in Set given by simply swapping the order of elements: $B_{X,Y}(x, y) = (y, x)$ for any $x \in X, y \in Y$. Similarly, the braiding in $\mathrm{Vect}_\mathbb{K}$ is given by $B_{V,W}(v \otimes w) = (w \otimes v)$, and extended linearly to the rest of $V \otimes W$. The braiding in Mod_A^Γ is also given by the linear extension of $B_{N,M}(n \otimes m) = m \otimes n$.

It is important to note that a given category may admit multiple distinct symmetric monoidal structures, even if the underlying monoidal structure is fixed. For instance, consider the case of $\mathrm{Mod}_\mathbb{K}^{\mathbb{Z}_2}$, the category of $\mathbb{Z}_2$-graded $\mathbb{K}$-modules for some field $\mathbb{K}$. The general braiding defined for any Mod_A^Γ could be used here, but there is another natural choice. $\mathrm{Mod}_\mathbb{K}^{\mathbb{Z}_2}$ can also be seen as the category of $\mathbb{K}$-super vector spaces, which has more structure than a general category of graded modules. When considering $\mathrm{Mod}_\mathbb{K}^{\mathbb{Z}_2}$ as the category of super vector spaces, the braiding is taken to be the one given by $B_{V,W}(v \otimes w) = (-1)^{\overline{w}\,\overline{v}} w \otimes v$ for homogeneous elements $v \in V$ and $w \in W$, and extended linearly to the rest of $V \otimes W$. The definition of this braiding encodes the sign terms that naturally arise when working with super vector spaces.

Theorem 2.2.22 (Coherence Theorem for (Symmetric) Monoidal Categories)**.** *Let $\mathcal{C}$ be a monoidal category. Let D be a diagram in $\mathcal{C}$ where each arrow is a composition of identity* *morphisms, associativity morphisms α, unitors λ and ρ, their inverses, and/or arbitrary tensor products thereof. Then D commutes.*

If $\mathcal{C}$ is a symmetric monoidal category, a similar result holds. Let D be a linear diagram in $\mathcal{C}$ where each arrow is a composition of identity morphisms, α, λ, ρ, the braiding B, their inverses, and/or arbitrary tensor products thereof. Then D commutes.

Proof. See [ML63, Proof of Theorem 5.1]. □

Remark 2.2.23. The restriction to only linear diagrams in the symmetric case is necessary. To illustrate, consider the following non-linear diagram:

(6)
$$X \otimes X \underset{\mathrm{id}_{X\otimes X}}{\overset{B_{X,X}}{\rightrightarrows}} X \; .$$

In general, the braiding in a symmetric monoidal category is not the identity, so this diagram does not commute. However, the coherence theorem for symmetric monoidal categories can be used to prove that certain non-linear diagrams commute. For instance, since the following diagram is linear, the coherence theorem tells us it is commutative for any objects X and Y:

(7)
$$\begin{array}{ccc} (X\otimes Y)\otimes u & \xrightarrow{\alpha_{X,Y,u}} & X\otimes(Y\otimes u) \\ \Big\downarrow{\scriptstyle B_{X\otimes Y,u}} & & \Big\downarrow{\scriptstyle \mathrm{id}_X\otimes\rho_Y} \\ u\otimes(X\otimes Y) & \xrightarrow[\lambda_{X\otimes Y}]{} & X\otimes Y \end{array} \; .$$

In particular, the diagram commutes if you pick $X = Y$, giving a non-linear diagram; the requirement of linearity serves to eliminate diagrams where different paths permute the factors via the braiding in a fundamentally different way, masked by the fact that two or more of the factors are equal.

Definition 2.2.24. Let $\mathcal{C}$ be a monoidal category, $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n \in \mathrm{Ob}(\mathcal{C})$ be objects in $\mathcal{C}$, and $f_1\colon X_1 \to Y_1, f_2\colon X_2 \to Y_2, \ldots, f_n\colon X_n \to Y_n$ be morphisms. We define the unbracketed tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ to be interpreted with brackets nested from left-to-right; for instance, if $n = 4$, we set $X_1 \otimes X_2 \otimes X_3 \otimes X_4 = ((X_1 \otimes X_2) \otimes X_3) \otimes X_4$. We define the unbracketed tensor product of morphisms in the same way. Note then that $f_1 \otimes \cdots \otimes f_n$ is a morphism from $X_1 \otimes \cdots \otimes X_n$ to $Y_1 \otimes \cdots \otimes Y_n$. We often use the "big tensor" notation $\bigotimes_{i=1}^{n} X_i := X_1 \otimes X_2 \otimes \cdots \otimes X_n$ for both types of unbracketed tensors.

Now let X and Y be two objects in $\mathcal{C}$, and $f\colon X \to Y$ a morphism. We define tensor powers by $X^{\otimes n} = \underbrace{X \otimes \cdots \otimes X}_{n \text{ copies}}$. In the special case $n = 0$, we define $X^{\otimes 0} = u$, the monoidal unit. Similarly, we set $f^{\otimes n} = \underbrace{f \otimes \cdots \otimes f}_{n \text{ copies}}$, with the special case $f^{\otimes 0} = \mathrm{id}_u$. Note then that for any $n \in \mathbb{N}$, the morphism $f^{\otimes n}$ has domain $X^{\otimes n}$ and codomain $Y^{\otimes n}$, even in the $n = 0$ case.

Remark 2.2.25. Suppose $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$ are objects in a monoidal category $\mathcal{C}$. Let T_X be a tensor product of the X_i in the order $X_1 \otimes X_2 \otimes \cdots \otimes X_n$, but with brackets inserted arbitrarily between the factors. Similarly, let T_Y be a tensor product of the Y_i in the order $Y_1 \otimes Y_2 \otimes \cdots \otimes Y_n$ with an arbitrary bracketing. It is clear that there is at least one morphism a built up as a composition of associators with domain $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ (with the left-to-right bracketing specified in Definition 2.2.24) and codomain T_X, and similarly at least one morphism b from T_Y to $Y_1 \otimes Y_2 \otimes \cdots \otimes Y_n$ (again, bracketed left-to-right). Theorem 2.2.22 guarantees that a and b are unique. Thus for any morphism $f\colon T_X \to T_Y$, the composite $b \circ f \circ a$ has domain $X_1 \otimes \cdots \otimes X_n$ and codomain $Y_1 \otimes \cdots \otimes Y_n$. Throughout the rest of the document, we identify the morphisms f and $b \circ f \circ a$. This identification together with the notation of Definition 2.2.24 allows us to completely identify any two iterated tensor products of the same objects in the same order, and avoid writing any associators explicitly. In any case where the precise objects and morphisms involved need to be recovered, the aforementioned uniqueness of a and b ensures that one can simply insert associators in any way that makes domains and codomains of the morphisms formally match up.

Definition 2.2.26 (Semigroup Object)**.** Let $\mathcal{C}$ be a monoidal category. A *semigroup object* in $\mathcal{C}$ is a pair (O, m), where O is any object in $\mathcal{C}$ and m is a morphism from $O \otimes O$ to O such that the following associativity diagram commutes:

$$\begin{array}{ccc} O \otimes O \otimes O & \xrightarrow{\mathrm{id}_O \otimes m} & O \otimes O \\ {\scriptstyle m \otimes \mathrm{id}_O}\downarrow & & \downarrow{\scriptstyle m} \\ O \otimes O & \xrightarrow[m]{} & O \end{array} \quad . \tag{8}$$

Definition 2.2.27 (Monoid Object)**.** Let $\mathcal{C}$ be a monoidal category. A *monoid object* in $\mathcal{C}$ is a triple (O, m, I), such that (O, m) is a semigroup object and I is a morphism from the monoidal unit u to O such that the following identity diagrams commute:

$$\begin{array}{ccc} u \otimes O & \xrightarrow{I \otimes \mathrm{id}_O} & O \otimes O \\ & {\scriptstyle \lambda_O}\searrow & \downarrow{\scriptstyle m} \\ & & O \end{array} \qquad \begin{array}{ccc} O \otimes u & \xrightarrow{\mathrm{id}_O \otimes I} & O \otimes O \\ & {\scriptstyle \rho_O}\searrow & \downarrow{\scriptstyle m} \\ & & O \end{array} \quad . \tag{9}$$

Definition 2.2.28 (Commutative Semigroup Object, Commutative Monoid Object)**.** Let $\mathcal{C}$ be a symmetric monoidal category. A *commutative semigroup object* in $\mathcal{C}$ is a semigroup object (O, m) such that the following commutativity diagram commutes:

$$\begin{array}{ccc} O \otimes O & \xrightarrow{B_{O,O}} & O \otimes O \\ & {\scriptstyle m}\searrow & \downarrow{\scriptstyle m} \\ & & O \end{array} \quad . \tag{10}$$

A *commutative monoid object* in $\mathcal{C}$ is a monoid object (O, m, I) such that diagram 10 commutes.

Example 2.2.29. In the category of sets with the usual monoidal structure, semigroup objects are just semigroups, and monoid objects are monoids. In $\mathrm{Vect}_{\mathbb{K}}$, semigroup objects are associative algebras, and monoid objects are unital associative algebras. In the category of super vector spaces, semigroup objects are superalgebras, and monoid objects are unital superalgebras. The commutative versions of these objects give commutative semigroups, monoids, algebras, etc. If one chooses the general Mod_A^Γ braiding for $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{Z}_2}$, commutative semigroup and monoid objects are commutative superalgebras and commutative unital superalgebras, respectively. Taking the super vector space braiding instead results in supercommutative superalgebras and supercommutative unital superalgebras.

Definition 2.2.30 (Morphism of Semigroup Objects, Morphism of Monoid Objects)**.** Let $\mathcal{C}$ be a monoidal category, and $(O, m), (P, n)$ two semigroup objects in $\mathcal{C}$. A *morphism of semigroup objects* from (O, m) to (P, n) is a morphism $f\colon O \to P$ such that the following diagram commutes:

$$\begin{array}{ccc} O \otimes O & \xrightarrow{f \otimes f} & P \otimes P \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ O & \xrightarrow[f]{} & P \end{array} \quad . \tag{11}$$

For any monoid objects (O, m, I) and (P, n, J) in $\mathcal{C}$, a *morphism of monoid objects* from (O, m, I) to (P, n, J) is a morphism $f\colon O \to P$ such that diagram (11) commutes and the following diagram commutes:

$$\begin{array}{ccc} u & \xrightarrow{I} & O \\ & {\scriptstyle J}\searrow & \downarrow{\scriptstyle f} \\ & & P \end{array} \quad . \tag{12}$$

It can easily be verified that morphisms of semigroup and monoid objects are each closed under composition, and that identity morphisms are morphisms of both semigroup and monoid objects.

Definition 2.2.31 (Category of (Commutative) Semigroup Objects, Category of (Commutative) Monoid Objects)**.** Let $\mathcal{C}$ be a monoidal category. The category of semigroup objects in $\mathcal{C}$, denoted $\mathrm{Semi}_{\mathcal{C}}$, is the category with $\mathcal{C}$-semigroup objects as objects, morphisms of semigroup objects as morphisms, and composition given by the composition of those morphisms in $\mathcal{C}$.

Similarly, the category of monoid objects in $\mathcal{C}$, denoted $\mathrm{Mon}_{\mathcal{C}}$, is the category with $\mathcal{C}$-monoid objects as objects, morphisms of monoid objects as morphisms, and composition given by the composition in $\mathcal{C}$.

The categories of commutative semigroup and commutative monoid objects in $\mathcal{C}$ are denoted $\mathrm{CSemi}_{\mathcal{C}}$ and $\mathrm{CMon}_{\mathcal{C}}$. They have commutative semigroup or monoid objects as objects, morphisms of semigroup or monoid objects as morphisms, and composition given by the composition in $\mathcal{C}$.

When working in the category of sets, we omit the symbol $\mathcal{C}$ and write just Semi, Mon, CSemi, or CMon.

3. Operads in the Category of Sets

In this section, we explore operads in the familiar context of sets and functions. We examine the most fundamental examples of operads, and use the language of category theory to prove that they encode the properties of different algebraic structures. Along the way, we use tree diagrams to visualize and better understand the axioms of an operad.

3.1. Basic Definitions.

Definition 3.1.1 (Nonsymmetric Operad)**.** A *nonsymmetric operad* X, sometimes called an *operad without permutations*, is a family $\{X_n\}_{n\in\mathbb{N}}$ of sets whose elements are called n-ary operations, together with a distinguished element $I \in X_1$ and a collection of composition functions $\circ_{i_1,\dots,i_k}\colon X_k \times X_{i_1} \times \cdots \times X_{i_k} \to X_{i_1+\cdots+i_k}$ (one for each $k \geq 1$ and sequence of natural numbers $i_1,\dots,i_k$) satisfying the associativity and identity axioms below. We denote the action of the composition maps by:

$$(f, g_1, \dots, g_k) \mapsto f \circ_{i_1,\dots,i_k} (g_1, \dots, g_k),$$

and sometimes write just $\circ$ for a composition map when the subscript is clear from context.

(Associativity) Let $n \in \mathbb{N}$, and $f \in X_n$. For each $i \in \{1, 2, \dots, n\}$, let $a_i \in \mathbb{N}$ and $g_i \in X_{a_i}$. Then for each a_i, for each $j \in \{1, 2, \dots, a_i\}$, let $h_{i,j} \in X_{k_{i,j}}$ for some arbitrary $k_{i,j} \in \mathbb{N}$. Then:

$$f \circ_{(k_{1,1}+\cdots+k_{1,a_1}),\cdots,(k_{n,1}+\cdots+k_{n,a_n})} (g_1 \circ_{k_{1,1},\dots,k_{1,a_1}} (h_{1,1},\dots,h_{1,a_1}),\dots,g_n \circ_{k_{n,1},\dots,k_{n,a_n}} (h_{n,1},\dots,h_{n,a_n}))$$
$$= (f \circ_{a_1,\dots,a_n} (g_1,\dots,g_n)) \circ_{k_{1,1},\dots,k_{1,n},k_{2,1},\dots,k_{n,a_n}} (h_{1,1},\dots,h_{1,a_i},h_{2,1},\dots,h_{n,a_n}).$$

(Identity) For any $n \in \mathbb{N}$ and $f \in X_n$, we have $f \circ_{\underbrace{1,\dots,1}_{n \text{ copies}}} (\underbrace{I,\dots,I}_{n \text{ copies}}) = f = I \circ_n (f)$.

A graphical interpretation of the operad axioms is outlined in Section 3.2.

Definition 3.1.2 (Symmetric Operad)**.** A *symmetric operad*, sometimes just called an *operad*, is a nonsymmetric operad X equipped with a right action $*$ of the symmetric group S_n on each X_n that satisfies the following equivariance axioms:

(Equivariance 1) Let $n \in \mathbb{N}, f \in X_n$, and $g_1 \in X_{a_1},\dots,g_n \in X_{a_n}$ for some arbitrary $a_i \in \mathbb{N}$. Let $\tau \in S_n$, and let σ be its inverse. Then:

$$(f * \tau) \circ_{a_1,\dots,a_n} (g_1,\dots,g_n) = (f \circ_{a_{\sigma(1)},\dots,a_{\sigma(n)}} (g_{\sigma(1)},\dots,g_{\sigma(n)})) * \tau',$$

where $\sigma \in S_n$ acts on the subscripts $\{1, 2, \dots, n\}$ as usual, and $\tau' \in S_{a_1+\cdots+a_n}$ is the block permutation that applies the action of τ, but treats the first $a_{\sigma(1)}$ elements as the first element, treats the next $a_{\sigma(2)}$ elements as the second element, and so on. For instance, if $\tau = (12)(34) \in S_4$ and the $a_{\sigma(i)}$ values in order are $(3, 4, 2, 1)$, then $\tau' = \left(\begin{smallmatrix} 1&2&3&4&5&6&7&8&9&10 \\ 4&5&6&7&1&2&3&10&8&9 \end{smallmatrix}\right)$.

(Equivariance 2) Let n, f, and the g_i and a_i be as above. Let $\sigma_1 \in S_{a_1},\dots\sigma_n \in S_{a_n}$. Then:

$$f \circ_{a_1,\dots,a_n} (g_1 * \sigma_1,\dots,g_n * \sigma_n) = (f \circ_{a_1,\dots,a_n} (g_1,\dots,g_n)) * (\sigma_1,\dots,\sigma_n),$$

where $(\sigma_1,\dots,\sigma_n) \in S_{a_1,\dots,a_n}$ is the disjoint union of the σ_i. For instance, if we set $\sigma_1 = (12) \in S_2, \sigma_2 = (123) \in S_3$, and $\sigma_3 = (12)(34) \in S_4$, then the disjoint union is $(\sigma_1, \sigma_2, \sigma_3) = (12)(345)(67)(89) \in S_9$.

Example 3.1.3 (Endomorphism Operad over a Set)**.** Let X be a set. For each $n \in \mathbb{N}$, define $\mathrm{End}_X(n) = \mathrm{Hom}(X^n, X)$, the set of all functions from X^n to X. By the convention outlined in Definition 2.2.24, this means that $\mathrm{End}_X(0) = \mathrm{Hom}(X^0, X) = \mathrm{Hom}(\{\star\}, X)$, where $\{\star\}$ is an arbitrary singleton set. We consider the elements of $\mathrm{End}_X(0)$ as zero-argument functions, identifying a function with the unique element in its image: for any $f \in \mathrm{End}_X(0)$, we denote

$f(\star) = f()$, with no argument. The notion of a zero-argument function is made more precise in Remark 4.1.8. Equip the family $\mathrm{End}_X = (\mathrm{End}_X(0), \mathrm{End}_X(1), \dots)$ with compositions maps given by the usual composition of multivariable functions. That is, for n, f, g_i, and a_i as in the definition of an operad, we define:

$$f \circ_{a_1,\dots,a_n} (g_1, \dots, g_n) = f(g_1, \dots, g_n),$$

where $f(g_1, \dots, g_n)$ represents the function with action given by:

$$(x_1, \dots, x_{i_1}, \dots, x_{i_1+\dots i_n}) \mapsto f(g_1(x_1, \dots, x_{i_1}), \dots g_k(x_{i_{n-1}+1}, \dots, x_{i_n})).$$

That is, the function is f composed with the g_i in each argument. The identity element in End_X is $I = \mathrm{id}_X$, the set-theoretic identity function. Finally, define the group action $*$ of S_n on $\mathrm{End}_X(n)$ as permutation of the arguments of a given function. Concretely, if $f\colon X^n \to X$ is a function and $\tau \in S_n$ has inverse σ, then $f * \tau$ is the function that maps $(x_1, \dots, x_n) \mapsto f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Equipped with this group action and collection of composition maps, End_X is a symmetric operad, called the *endomorphism operad over* X. The definition of an operad is designed to mimic the key properties of this kind of multivariable function composition. As such, some authors refer to endomorphism operads as “tautological operads” or “canonical operads”.

Example 3.1.4 (Associative Operad of Sets)**.** For each $n \geq 1$, define $\mathrm{Assoc}(n) = \{\alpha_n\}$, where each α_n is a formal symbol that we think of as an n-input operation on some set. For the special case $n = 0$, set $\mathrm{Assoc}(0) = \emptyset$. Define the composition maps by:

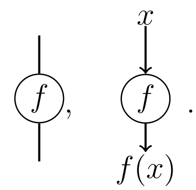
$$\alpha_n \circ_{i_1,\dots,i_n} (\alpha_{i_1}, \dots, \alpha_{i_n}) = \alpha_{i_1+\dots+i_n} \in \mathrm{Assoc}_{i_1+\dots+i_n}.$$

That is, there is only one element in each Assoc_n, so composition outputs the unique element in the codomain of the composition map. Equivalently, composition in Assoc is given by adding the subscripts of the α_i in the parentheses. With the identity being $I = \alpha_1$, the family $\mathrm{Assoc} = (\mathrm{Assoc}(0), \mathrm{Assoc}(1), \dots)$ forms a nonsymmetric operad called the *associative operad*. One can easily see that Assoc is indeed an operad; see Theorem 4.1.13 for a proof in a more general context.

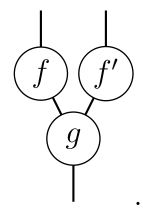
The associative operad encodes the notion of a binary operation being associative: by the definition of composition in Assoc, we know that $\alpha_2 \circ_{1,2} (\alpha_1, \alpha_2) = \alpha_2 \circ_{2,1} (\alpha_2, \alpha_1)$. If we interpret each α_n as an actual n-ary operation in some endomorphism operad End_X and assume that α_1 is the identity on X, (a correspondence made formal in Theorem 3.3.6), this tells us that $\alpha_2(x, \alpha_2(y, z)) = \alpha_2(\alpha_2(x, y), z)$ for any $x, y, z \in X$. In other words, α_2 is associative.

3.2. **Tree Diagram Visualizations.**

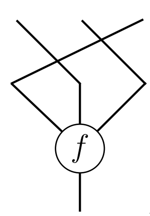
Operads can be visualized using tree diagrams. Here, we introduce such diagrams informally in the context of endomorphism operads, but the same types of diagrams can be used for any operad. The connection between trees and general operads is made more precise in Definition 5.1.6. Working in the endomorphism operad over a set X, let $f\colon X \to X$ be some unary function on X. In the usual way, we draw f and picture its action on some element $x \in X$ as follows:



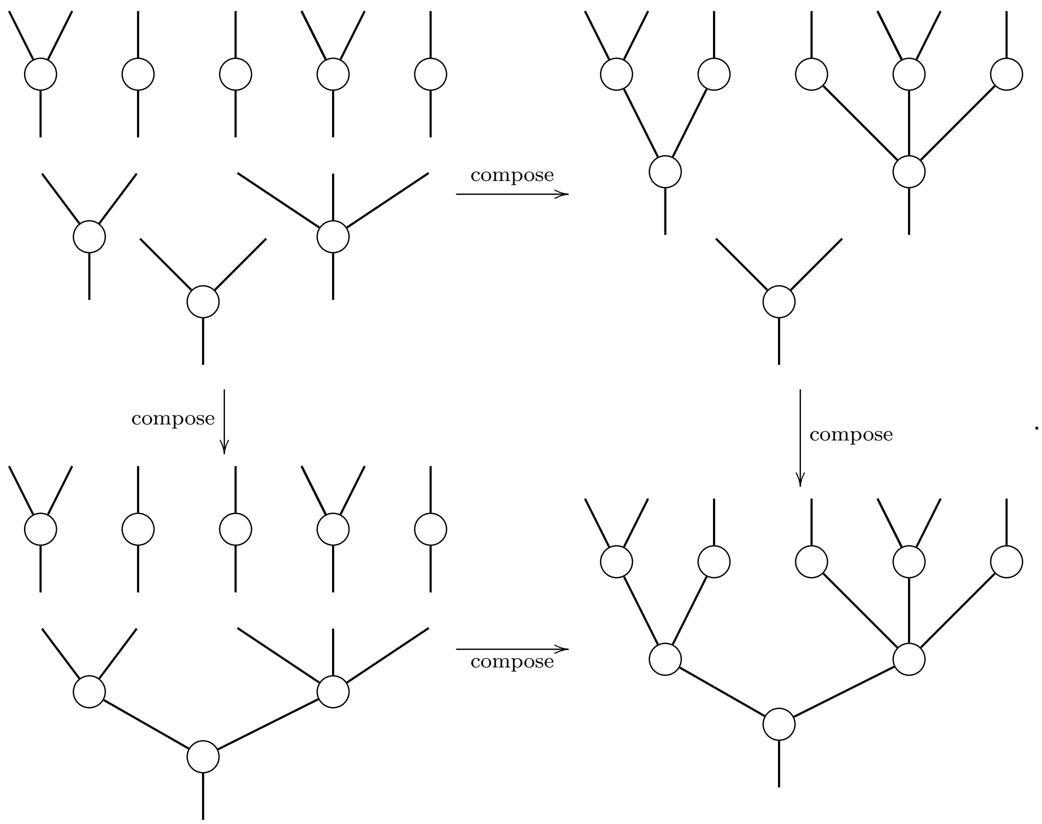
We can extend this notation to multivariable functions and arbitrary compositions in End_X. For example, if $f, f' \colon X \to X$ are unary functions and $g \colon X^2 \to X$ is a binary function, we would draw the composition $g \circ (f, f')$ like this:



In general, one views these tree diagrams as being n-ary functions whose action is given by inserting n inputs at the top of the diagram, moving them downwards, and applying the functions indicated at each vertex. The symmetric action is shown by the permutation of the topmost edges. For instance, if f is a ternary function and we take the permutation $(123) \in S_3$, we would draw $f * (123)$ as:

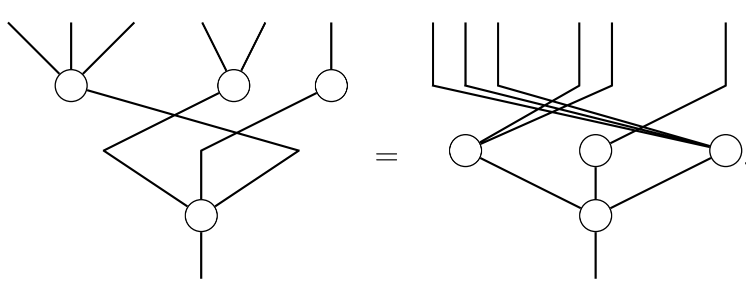


For simplicity, we omit the vertex labels in the following diagrams. The axioms of symmetric and nonsymmetric operads can be visualized using trees. For example, a general instance of the associativity axiom asserts that the following diagram commutes:

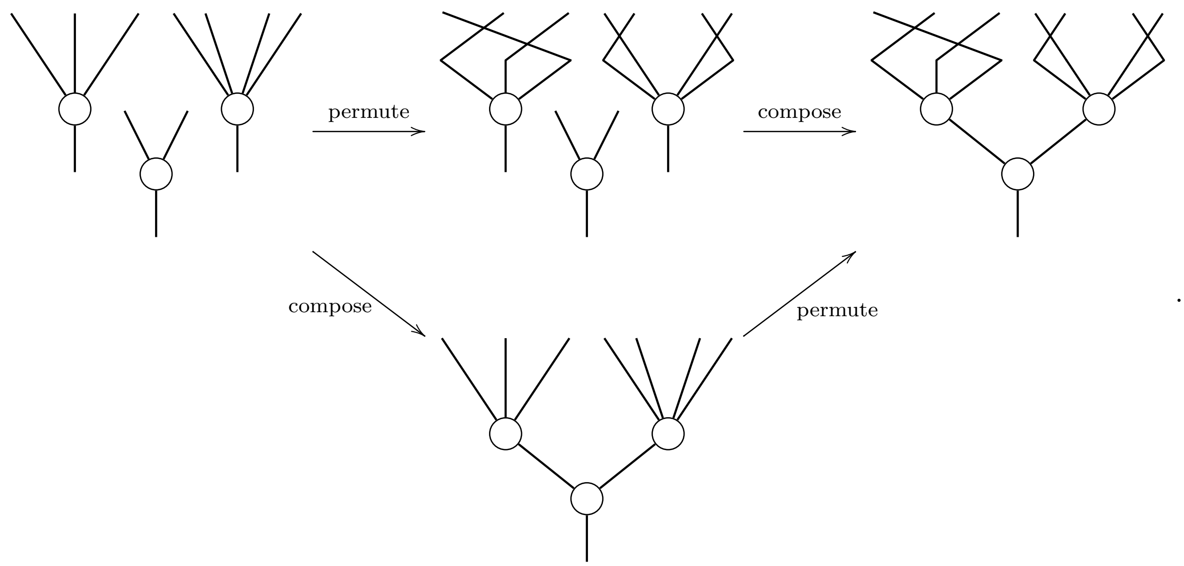


That is, it is equivalent to compose the three rows of functions from either top-to-bottom or from bottom-to-top. In much the same way as the associativity in a monoid allows one to unambiguously write an iterated product $x_1x_2 \cdots x_n$ without brackets, the associativity axiom for an operad allows us to write these multi-step compositions without specifying the order of composition. Graphically, this means that one can draw arbitrarily large tree diagrams without the need for brackets.

A generic instance of the first equivariance axiom looks like this:



One can verify that these two trees indeed represent the same composite function by tracing the inputs as they move through the diagram. The second equivariance axiom tells us that it is equivalent to first permute the edges on trees and then compose them, or compose and then permute. One instance of this axiom says that the following diagram commutes:



3.3. **Morphisms and Algebras over Operads of Sets.**

Much like linear maps in linear algebra, homomorphisms in ring theory, and continuous functions in the study of topology, structure-preserving maps between operads are an integral part of operad theory. In this section, we define morphisms of nonsymmetric and symmetric operads, and then see how these morphisms can encode the data of various algebraic structures.

Definition 3.3.1 (Morphism of Nonsymmetric Operads)**.** Let X and Y be nonsymmetric operads, with identities I and J and families of composite maps $\circ$ and $\circ'$ respectively. Then a morphism of nonsymmetric operads $F : X \to Y$ is a family of functions $\{F_n \colon X_n \to Y_n\}_{n\in\mathbb{N}}$ that satisfy the following:

(MO1) $F_1(I) = J$. That is, F preserves the identity.

(MO2) For n, f, g_i, and a_i as in the definition of an operad,

$$F_{a_1+\cdots+a_n}(f \circ_{a_1,\ldots,a_n} (g_1, \ldots, g_n)) = F_n(f) \circ'_{a_1,\ldots,a_n} (F_{a_1}(g_1), \ldots, F_{a_n}(g_n)).$$

That is, F commutes with the composition maps in X and Y.

We often write $F(f)$ without a subscript when the context is clear.

Definition 3.3.2 (Morphism of Symmetric Operads)**.** Let X and Y be symmetric operads with group actions $*$ and $\star$ respectively. Then a morphism of operads $F\colon X \to Y$ is a morphism of nonsymmetric operads that additionally satisfies:

(MO3) For any $n \in \mathbb{N}, f \in X_n$, and $\tau \in S_n$, we have $F_n(f * \tau) = F_n(f) \star \tau$.

Definition 3.3.3 (Algebra over an Operad)**.** Let X be an operad. An *algebra over* X, also called an X*-algebra*, is a morphism of operads $F\colon X \to \mathrm{End}_O$ for some set O. This may either be a morphism of nonsymmetric or symmetric operads, depending on the type of X. Such a map associates to each formal $n-$ary operation in X a concrete function in the endomorphism operad.

Definition 3.3.4 (Morphism of Algebras over an Operad)**.** Let X be an operad, and let $F\colon X \to \mathrm{End}_O$ and $G\colon X \to \mathrm{End}_P$ be two X-algebras. A morphism of algebras $M\colon F \to G$ is a function $M\colon O \to P$ such that for all $f \in X_n$ and $o_1, o_2, \ldots, o_n \in O$, the following equivariance property holds:

$$M([F(f)](o_1, o_2, \ldots, o_n)) = [G(f)](M(o_1), M(o_2), \ldots, M(o_n)).$$

Definition 3.3.5 (Category of Algebras over an Operad)**.** Let X be an operad. Then the *category of algebras over* X, denoted Alg_X, is the category with algebras over X as objects, morphisms between X-algebras as arrows, and composition of arrows given by the composition of those morphisms.

Theorem 3.3.6. $\mathrm{Alg}_{\mathrm{Assoc}}$ *is isomorphic to the category of semigroups. The isomorphism functor is denoted* $\varphi\colon \mathrm{Alg}_{\mathrm{Assoc}} \to \mathrm{Semi}$*, with inverse* $\sigma\colon \mathrm{Semi} \to \mathrm{Alg}_{\mathrm{Assoc}}$*. The actions on objects and arrows are as follows: let* $F\colon \mathrm{Assoc} \to \mathrm{End}_X$ *and* $G\colon \mathrm{Assoc} \to \mathrm{End}_Y$ *be two algebras over* Assoc *(for some sets* X, Y*), and let* $M\colon F \to G$ *be a morphism of* Assoc*-algebras. Then:*

$$\varphi(F) = \big(X, F(\alpha_2)\big),$$
$$\varphi(M) = M,$$

where the image of M *is interpreted as the underlying map* $M\colon X \to Y$*. Next, let* (X, m) *and* (Y, p) *be semigroups, and* $M\colon X \to Y$ *a semigroup homomorphism. Then:*

$$\sigma(X, m) = F,$$
$$\sigma(M) = M,$$

where the image of M *is interpreted as a morphism of the* Assoc*-algebras* $\sigma(X, m)$ *and* $\sigma(Y, p)$*, and* F *is an algebra over* Assoc *defined inductively as follows:* $F_1(\alpha_1) = \mathrm{id}_X$*, and* $F_n(\alpha_n) = m \circ_{n-1,1} (F_{n-1}(\alpha_{n-1}), \mathrm{id}_X)$ *for all* $n \geq 2$*. For instance,* $F_2(\alpha_2) = m$*, and* $F_3(\alpha_3)$ *is the ternary function on* X *that maps* $(x, y, z) \mapsto m(m(x, y), z)$*.*

Proof. We will use $\circ$ to denote composition in End_X and $\circ'$ to denote composition in Assoc. We first need to verify that φ and σ actually map objects into the claimed categories. Let $F\colon \mathrm{Assoc} \to \mathrm{End}_X$ be an algebra over Assoc for some set X. We need to show that $(X, F(\alpha_2))$ is a semigroup. For each $n \in \mathbb{N}$, we know that $F_n(\alpha_n) \in \mathrm{Hom}(X^n, X)$. Thus $m := F(\alpha_2)$ is automatically a binary operation on X, and it remains to show that m is associative. Let $x, y, z \in X$. Then we have:

$$\begin{aligned}
m\big(m(x,y), z\big) &= m\big(m(x,y), \mathrm{id}_X(z)\big) \\
&= [F_2(\alpha_2)]\Big([F_2(\alpha_2)](x,y), [F_1(\alpha_1)](z)\Big) && \text{By (MO1)} \\
&= \big[F_2(\alpha_2) \circ_{2,1} \big(F_2(\alpha_2), F_1(\alpha_1)\big)\big](x,y,z) \\
&= \big[F_3\big(\alpha_2 \circ'_{2,1} (\alpha_2, \alpha_1)\big)\big](x,y,z) && \text{By (MO2)} \\
&= [F_3(\alpha_3)](x,y,z) && \text{By the definition of } \circ' \\
&= \big[F_3\big(\alpha_2 \circ'_{1,2} (\alpha_1, \alpha_2)\big)\big](x,y,z) && \text{By the definition of } \circ' \\
&= \big[F_2(\alpha_2) \circ_{1,2} \big(F_1(\alpha_1), F_2(\alpha_2)\big)\big](x,y,z) && \text{By (MO2)} \\
&= [F_2(\alpha_2)]\big([F_1(\alpha_1)](x), [F_2(\alpha_2)](y,z)\big)
\end{aligned}$$

$$= m\big(\operatorname{id}_X(x), m(y,z)\big) \qquad \text{By (MO1)}$$
$$= m\big(x, m(y,z)\big).$$

Thus m is associative, and φ indeed maps algebras over Assoc to semigroups over X.

Now suppose that (X, m) is any semigroup. Set $F = \sigma(X, m)$. Then F preserves the identity by the definition of σ. It also respects composition since all possible compositions of m with the same number of arguments are equal by the assumption that m is associative. Thus F is a morphism of operads, and is in turn an algebra over Assoc.

Let $M\colon (X, m) \to (Y, p)$ be a homomorphism of semigroups. We need to show that $\sigma(M)$ is a morphism of Assoc-algebras. In particular, we must verify that any semigroup homomorphism satisfies the equivariance property of Definition 3.3.3. So let $n \in \mathbb{N}, \alpha_n \in \mathrm{Assoc}_n$, and $x_1, \dots, x_n \in X$. Write $F = \sigma(X, m)$ and $G = \sigma(Y, p)$. We first check the $n = 1$ case of the equivariance property:

$$\begin{aligned}
M\big([F(\alpha_1)](x_1)\big) &= M(\operatorname{id}_X(x_1)) && \text{By the definition of } F\\
&= M(x_1)\\
&= \operatorname{id}_Y(M(x_1)\\
&= [G(\alpha_1)]\big(M(x_1)\big), && \text{By the definition of } G
\end{aligned}$$

and so the equivariance property holds. We prove the other cases inductively. Let $n \geq 2$, and suppose the equivariance property holds for all smaller natural numbers. Then we have the following:

$$\begin{aligned}
&M\Big(\big[F(\alpha_n)\big](x_1, \dots, x_n)\Big)\\
&= M\Big(\big[F\big(\alpha_2 \circ'_{n-1,1} (\alpha_{n-1}, \alpha_1)\big)\big](x_1, \dots, x_n)\Big) && \text{By the definition of } \circ'\\
&= M\Big(\big[F(\alpha_2) \circ_{n-1,1} \big(F(\alpha_{n-1}), F(\alpha_1)\big)\big](x_1, \dots, x_n)\Big) && \text{By (MO2)}\\
&= M\Big(\big[m \circ_{n-1,1} (F(\alpha_{n-1}), \operatorname{id}_X)\big](x_1, \dots, x_n)\Big) && \text{By the definition of } F\\
&= M\Big(m\big([F(\alpha_{n-1})](x_1, \dots, x_{n-1}), x_n\big)\Big) && \text{By the definition of } \circ\\
&= p\Big(M\big([F(\alpha_{n-1})](x_1, \dots, x_{n-1})\big), M(x_n)\Big) && \text{Since } M \text{ is a semigroup homomorphism}\\
&= p\Big([G(\alpha_{n-1})]\big(M(x_1), \dots, M(x_{n-1})\big), [G(\alpha_1)]\big(M(x_n)\big)\Big) && \text{By the inductive hypothesis}\\
&= [G(\alpha_2)]\Big([G(\alpha_{n-1})]\big(M(x_1), \dots, M(x_{n-1})\big), [G(\alpha_1)]\big(M(x_n)\big)\Big) && \text{By the definition of } G\\
&= [G(\alpha_2) \circ_{n-1,1} \big(G(\alpha_{n-1}), G(\alpha_1)\big)]\big(M(x_1), \dots, M(x_n)\big)\\
&= [G\big(\alpha_2 \circ'_{n-1,1} (\alpha_{n-1}, \alpha_1)\big)]\big(M(x_1), \dots, M(x_n)\big) && \text{By (MO2)}\\
&= [G(\alpha_n)]\big(M(x_1), \dots, M(x_n)\big), && \text{By the definition of } \circ'
\end{aligned}$$

and so the equivariance property holds for n, completing the inductive proof that M is a morphism of Assoc-algebras.

Finally, we need to show that φ maps morphisms as claimed, sending morphisms of Assoc-algebras to semigroup homomorphisms. Let $M\colon F \to G$ be a morphism of Assoc-algebras, write $(X,m)=\varphi(F)$ and $(Y,p)=\varphi(G)$, and let $x,y\in X$. Then:

$$\begin{aligned}
M\big(m(x,y)\big) &= M\big([F(\alpha_2)](x,y)\big) && \text{By the definition of } m\\
&= [G(\alpha_2)]\big(M(x),M(y)\big) && \text{By the equivariance property}\\
&= p\big(M(x),M(y)\big). && \text{By the definition of } p
\end{aligned}$$

Thus M is a semigroup morphism. We conclude that φ and σ are indeed functors between $\mathrm{Alg}_{\mathrm{Assoc}}$ and Semi, as claimed.

The functors φ and σ are clearly inverses by their definition. Thus φ is an isomorphism of categories, as desired. □

Definition 3.3.7 (Unital Associative Operad)**.** For each $n\geq 0$, define $\mathrm{Assocu}(n)=\{\alpha_n\}$, where each α_n is a formal symbol as in the case of the non-unital associative operad; the only difference is that we include a formal nullary operation α_0. Define the composition maps and identity as in the non-unital case. Then the family $\mathrm{Assocu}=(\mathrm{Assocu}(0),\mathrm{Assocu}(1),\dots)$ forms a nonsymmetric operad called the *unital associative operad*.

Theorem 3.3.8. $\mathrm{Alg}_{\mathrm{Assocu}}$ *is isomorphic to the category of monoids. The isomorphism functors φ and σ are defined as in Theorem 3.3.6, with the following modifications:*

$$\begin{aligned}
\varphi(F) &= \big(X, F_2(\alpha_2), F_0(\alpha_0)\big),\\
\sigma(X,m,I) &= F,
\end{aligned}$$

where (X,m,I) is a monoid, and F is defined as in the aforementioned theorem, but with the addition $F(\alpha_0)=I$.

Proof. By the same arguments as in the proof of Theorem 3.3.6, φ maps algebras over Assocu to semigroups. It remains to show that all of these semigroups are monoids; that is, we have to prove that the product induced by any algebra over Assocu has an identity. So let $F\colon \mathrm{Assocu}\to\mathrm{End}_X$ be an algebra over Assoc. Denote the induced product on X by $m\colon X^2\to X$. Note that, by definition, $F(\alpha_0)$ is a nullary function mapping into X. Write $I=F(\alpha_0)$. We claim that (X,m,I) is a monoid. Let $x\in X$. Then we have:

$$\begin{aligned}
m\big(x,I(\star)\big) &= F(\alpha_2)\big(\mathrm{id}_X(x),I()\big)\\
&= [F(\alpha_2)]\big([F(\alpha_1)](x),[F(\alpha_0)]()\big)\\
&= \big[F(\alpha_2)\circ_{1,0}\big(F(\alpha_1),F(\alpha_0)\big)\big](x)\\
&= [F\big(\alpha_2\circ'_{1,0}(\alpha_1,\alpha_0)\big)](x) && \text{By (MO2)}\\
&= [F(\alpha_1)](x) && \text{By the definition of } \circ' \text{ in Assocu}\\
&= \mathrm{id}_X(x)\\
&= x.
\end{aligned}$$

An analogous argument shows that $m(I(\star),x)=x$, and thus that I is indeed an identity. So φ does map into the category of monoids.

We also need to show that σ has the correct target. Let (X,m,I) be a monoid, and write $\sigma(X,m,I)=F$. The same arguments as in the proof of Theorem 3.3.6 show that F is an algebra over Assocu.

Next, we have to verify that φ sends morphisms of Assocu-algebras to monoid homomorphisms. Let $M\colon F\to G$ be a morphism of Assocu-algebras, write $(X,m,I)=\varphi(F)$ and

$(Y, p, E) = \varphi(G)$, and let $x \in X$. The proof of Theorem 3.3.6 shows that M respects the monoid multiplication, so it remains to check that M preserves the identity. We have:

$$\begin{aligned} M(I) &= M(F(\alpha_0)) \\ &= G(\alpha_0) && \text{By the equivariance property} \\ &= E. \end{aligned}$$

So φ does indeed output monoid homomorphisms.

Finally, we need to check that σ sends monoid homomorphisms to morphisms of Assocu-algebras. Let $M\colon (X, m, I) \to (Y, p, E)$ be a monoid homomorphism. The proof of Theorem 3.3.6 shows that $\sigma(M)$ satisfies the equivariance property for $n \geq 1$. It remains to show that the property is also satisfied for $n = 0$:

$$\begin{aligned} M(F(\alpha_0)) &= M(I) \\ &= E && \text{Since } M \text{ is a monoid homomorphism} \\ &= G(\alpha_0). \end{aligned}$$

So M is indeed a morphism of Assocu-algebras. We conclude that φ and σ are functors as claimed. They are clearly inverses by their definition, so φ is an isomorphism of categories, as desired. □

Example 3.3.9 ((Unital) Commutative Operad)**.** Define $\mathrm{Comm}(n) = \mathrm{Assoc}(n)$ for each $n \in \mathbb{N}$, and equip each with the trivial group action $*$ of S_n; that is, $f * \tau = f$ for all $n \in \mathbb{N}, f \in \mathrm{Comm}(n)$, and $\tau \in S_n$. Then $\mathrm{Comm} = (\mathrm{Comm}(0), \mathrm{Comm}(1), \dots)$ together with the same composition maps and identity as in Assoc forms a symmetric operad called the *commutative operad*.

Define Commu identically to Comm, but with $\mathrm{Commu}(n) = \mathrm{Assocu}(n)$. This forms a symmetric operad called the *unital commutative operad*.

Theorem 3.3.10.

(A) $\mathrm{Alg}_{\mathrm{Comm}}$ *is isomorphic to the category of commutative semigroups.*
(B) $\mathrm{Alg}_{\mathrm{Commu}}$ *is isomorphic to the category of commutative monoids.*

In both cases, the isomorphisms φ and σ are the same as in the respective non-commutative theorems.

Proof. **Part (A):** Proceeding from what has been proved in Theorem 3.3.6, it remains to show that φ produces *commutative* semigroups, and that the algebras σ produces are morphisms of symmetric operads. Let $F\colon \mathrm{Comm} \to \mathrm{End}_X$ be an algebra over Comm, and write $\varphi(F) = (X, m)$. Let $x, y \in X$, and let $\sigma = (12)$ be the nontrivial element in S_2. Then we have:

$$\begin{aligned} m(x, y) &= \big[F_2(\alpha_2)\big](x, y) \\ &= \big[F_2(\alpha_2 * \sigma)\big](x, y) && \text{Since the action of } S_n \text{ on Comm is trivial} \\ &= \big[F_2(\alpha_2) * \sigma\big](x, y) && \text{By (MO3)} \\ &= [m * \sigma](x, y) \\ &= m(y, x). && \text{By the definition of the group action on } \mathrm{End}_X \end{aligned}$$

Thus m is commutative. Now suppose (X, m) is a commutative semigroup. We need to show that $\sigma(X, m) = F$ is a morphism of symmetric operads. The proof of Theorem 3.3.6 shows that it is a morphism of nonsymmetric operads, so it remains to check property (MO3). By assumption, m is commutative, so the trivial group action of Comm is indeed respected.

The actions of φ and σ on morphisms are exactly the same as in Theorem 3.3.6, so we again conclude that they are inverse functors, and that φ is an isomorphism.

Part (B) Combine the arguments of Part (A) and Theorem 3.3.8. □

Theorems 3.3.6, 3.3.8, and 3.3.10 tell us that algebras over the associative, unital associative, commutative, and unital commutative operads are essentially the same thing as semigroups, monoids, commutative semigroups, and commutative monoids, respectively. The existence of these theorems raises a natural question: can one construct operads that model other common algebraic structures like algebras and superalgebras? In fact, as we will see in the following section, *the same* operads can be used for this purpose.

4. General Operads

All of the definitions, examples, and theorems from Section 3 can be directly transferred into more general categories. We first work in the context of a symmetric monoidal category, and then direct our focus to a handful of familiar categories with extra structure for the remainder of the document. For a treatment of operads in categories without that extra structure, see, for example, [Sun07].

4.1. Basic Definitions.

Definition 4.1.1 (Action of S_n on Symmetric Monoidal Categories)**.** Let $\mathcal{C}$ be a symmetric monoidal category with braiding $B_{x,y} \colon x \otimes y \to y \otimes x$. For any iterated product $\bigotimes_{i=1}^{n} x_i$ and any $\sigma \in S_n$, we want to define a morphism $\mu(\sigma) \colon \bigotimes_{i=1}^{n} x_i \to \bigotimes_{i=1}^{n} x_{\sigma^{-1}(i)}$, which we think of as permuting the factors of the iterated product as σ permutes the elements of $\{1, 2, \dots, n\}$. We want these morphisms to form a group with respect to composition, and for μ to then be a group homomorphism.

First, recall that the symmetric group S_n is generated by the set of simple transpositions $\{\sigma_k \mid 1 \leq k \leq n-1\}$, where σ_k swaps the k and $k+1$'th elements. Further, S_n is totally specified by that set of generators and the following relations:

(SYM1) $\sigma_k^2 = 1$ for all $1 \leq k \leq n-1$,
(SYM2) $\sigma_k\sigma_j = \sigma_j\sigma_k$ for all $1 \leq k \leq n-1$ and $1 \leq j \leq n-1$ such that $j \neq k \pm 1$,
(SYM3) $\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}$ for all $1 \leq k \leq n-2$.

Thus to show that μ is a group homomorphism, we will simply need to map the generators and show that those morphisms $\mu(\sigma_k)$ satisfy the three relations above.

Define $\mu(\sigma_k) = \left(\bigotimes_{i=1}^{k-1} \mathrm{id}_{x_i}\right) \otimes B_{x_k,x_{k+1}} \otimes \left(\bigotimes_{i=k+2}^{n} \mathrm{id}_{x_i}\right)$; that is, use the braiding to swap the k and $k+1$'th factors, and apply the identity morphism to the other factors. For any S_n, it's clear that the set of all finite sequences of compositions of the morphisms $\mu(\sigma_k)$ does indeed form a group, where the multiplication is given by composition, and the identity element is simply $\bigotimes_{i=1}^{n} \mathrm{id}_{x_i}$. Each $\mu(\sigma_k)$ is its own inverse since we're assuming that the braiding is symmetric, so (SYM1) holds. Further, one can see that $\mu(\sigma_k)\mu(\sigma_j) = \mu(\sigma_j)\mu(\sigma_k)$ holds for all $j \neq k \pm 1$ by noting that $\mu(\sigma_k)$ leaves all but the k and $k+1$'th factors unchanged, and thus relation (SYM2) is satisfied.

Working towards the third relation, note that if we ignore the associators in the hexagon diagram (5) as outlined in Remark 2.2.25, we obtain:

$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{B_{x,y\otimes z}} & y \otimes z \otimes x \\ {\scriptstyle B_{x,y}\otimes \mathrm{id}_z}\downarrow & \nearrow_{\mathrm{id}_y \otimes B_{x,z}} & \\ y \otimes x \otimes z & & \end{array} , \tag{13}$$

For (SYM3) to hold, we want the following diagram to commute:

$$\begin{array}{ccccccc} & & y \otimes x \otimes z & \xrightarrow{\mathrm{id}_y \otimes B_{x,z}} & y \otimes z \otimes x & & \\ & \nearrow^{B_{x,y}\otimes \mathrm{id}_z} & & & & \searrow^{B_{y,z}\otimes \mathrm{id}_x} & \\ x \otimes y \otimes z & & & & & & z \otimes y \otimes x \\ & \searrow_{\mathrm{id}_x \otimes B_{y,z}} & & & & \nearrow_{\mathrm{id}_z \otimes B_{x,y}} & \\ & & x \otimes z \otimes y & \xrightarrow[B_{x,z}\otimes \mathrm{id}_y]{} & z \otimes x \otimes y & & \end{array} , \tag{14}$$

since this corresponds to the equality $\mu(\sigma_k) \circ \mu(\sigma_{k+1}) \circ \mu(\sigma_k) = \mu(\sigma_{k+1}) \circ \mu(\sigma_k) \circ \mu(\sigma_{k+1})$. Using the equality (13), we can replace the first two arrows on the upper path and the second two arrows on the lower path to obtain the following equivalent diagram:

$$\begin{array}{ccccc} & & y \otimes z \otimes x & & \\ & \nearrow^{B_{x,y\otimes z}} & & \searrow^{B_{y,z}\otimes \mathrm{id}_x} & \\ x \otimes y \otimes z & & & & z \otimes y \otimes x \\ & \searrow_{\mathrm{id}_x \otimes B_{y,z}} & & \nearrow_{B_{x,z\otimes y}} & \\ & & x \otimes z \otimes y & & \end{array} . \tag{15}$$

Consider functors $F, G\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ given by $F(x,y) = x \otimes y$ and $G(x,y) = y \otimes x$, and the morphism $(\mathrm{id}_x, B_{y,z})\colon (x, y \otimes z) \to (x, z \otimes y)$ in $\mathcal{C} \times \mathcal{C}$. Then the above diagram can equivalently be written (flipped along the diagonal) as:

$$\begin{array}{ccc} F(x, y \otimes z) & \xrightarrow{F(\mathrm{id}_x, B_{y,z})} & F(x, z \otimes y) \\ {\scriptstyle B_{x,y\otimes z}}\downarrow & & \downarrow{\scriptstyle B_{x,z\otimes y}} \\ G(x, y \otimes z) & \xrightarrow[G(\mathrm{id}_x, B_{y,z})]{} & G(x, z \otimes y) \end{array} , \tag{16}$$

and the fact that this diagram commutes is precisely given by the fact that B is natural. We conclude that the group generated by the $\mu(\sigma_k)$ satisfies all three of the generating relations of S_n, and thus μ is a group homomorphism. We take these morphisms $\mu(\sigma)$ to be the canonical action of the symmetric groups S_n on any symmetric monoidal category. For brevity, we usually omit the symbol μ and just write $\sigma\colon \bigotimes_{i=1}^{n} x_i \to \bigotimes_{i=1}^{n} x_{\sigma^{-1}(i)}$.

Definition 4.1.2 (Nonsymmetric Operad)**.** Let $\mathcal{C}$ be a symmetric monoidal category. A *nonsymmetric operad X over $\mathcal{C}$* is a family $\{X_n\}_{n\in\mathbb{N}}$ of objects from $\mathcal{C}$ together with a unit morphism $I\colon u \to X_1$ and a collection of morphisms:

$$\gamma_{i_1,\ldots,i_k}\colon X_k \otimes X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_k} \to X_{i_1+\cdots+i_k},$$

where the indices range as in Definition 3.1.1 (we use γ rather than $\circ$ to avoid confusion with categorical composition) satisfying the associativity and identity axioms below.

(Associativity) Let $n \in \mathbb{N}$, let $a_1, \ldots, a_n \in \mathbb{N}$, and for each a_i, for each $j \in \{1, 2, \ldots, a_i\}$ let $k_{i,j} \in \mathbb{N}$. For brevity, we denote the list $a_1, \ldots, a_n$ by A and the lists $k_{i,1}, \ldots, k_{i,a_i}$ by K_i. Write ΣA and ΣK_i for the sum of the entries in those lists. Let σ be the permutation that rearranges factors as follows:

$$\sigma\colon X_n \otimes (X_{a_1} \otimes X_{k_{1,1}} \otimes \cdots \otimes X_{k_{1,a_1}}) \otimes (X_{a_2} \otimes \cdots \otimes X_{k_{2,a_2}}) \otimes (\cdots \otimes X_{k_{n,a_n}}) \\ \to (X_n \otimes X_{a_1} \otimes X_{a_2} \otimes \cdots \otimes X_{a_n}) \otimes X_{k_{1,1}} \otimes \cdots \otimes X_{k_{n,a_n}}.$$

Then the following diagram commutes:

$$\begin{array}{ccc}
X_n \otimes \bigotimes_{i=1}^{n} \left(X_{a_i} \otimes \bigotimes_{j=1}^{a_i} X_{k_{i,j}}\right) & \xrightarrow{\sigma} & X_n \otimes X_{a_1} \otimes \cdots \otimes X_{a_n} \otimes \left(\bigotimes_{i=1}^{n}\bigotimes_{j=1}^{a_i} X_{k_{i,j}}\right) \\
 & & \downarrow{\scriptstyle \gamma_A \otimes \mathrm{id}_{X_{k_{1,1}}} \otimes\cdots\otimes \mathrm{id}_{X_{k_{n,a_n}}}} \\
\downarrow{\scriptstyle \mathrm{id}_{X_n} \otimes \gamma_{K_1} \otimes\cdots\otimes \gamma_{K_n}} & & X_{\sum A} \otimes \left(\bigotimes_{i=1}^{n}\bigotimes_{j=1}^{a_i} X_{k_{i,j}}\right) \\
 & & \downarrow{\scriptstyle \gamma_{K_1,\ldots,K_n}} \\
X_n \otimes X_{\sum K_1} \otimes \cdots \otimes X_{\sum K_n} & \xrightarrow[\gamma_{\sum K_1,\ldots,\sum K_n}]{} & X_{\sum K_1+\cdots+\sum K_n}
\end{array}.$$

(Identity) For every $n \in \mathbb{N}$, the following diagrams commute:

$$\begin{array}{ccc}
u \otimes X_n & \xrightarrow{I \otimes \mathrm{id}_{X_n}} & X_1 \otimes X_n \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \gamma_n} \\
X_n & \xrightarrow[\mathrm{id}_{X_n}]{} & X_n
\end{array},$$

$$\begin{array}{ccccccc}
X_n \otimes u^{\otimes n} & \xrightarrow{\rho} & X_n \otimes u^{\otimes n-1} & \xrightarrow{\rho} & \ldots & \xrightarrow{\rho} & X_n \\
\downarrow{\scriptstyle \mathrm{id}_{X_n} \otimes I^{\otimes n}} & & & & & & \downarrow{\scriptstyle \mathrm{id}_{X_n}} \\
X_n \otimes X_1^{\otimes n} & & & \xrightarrow[\gamma_{1,\ldots,1}]{} & & & X_n
\end{array}.$$

Definition 4.1.3 (Group Action)**.** Let $\mathcal{C}$ be any category, and G a group. Let x be an object in $\mathcal{C}$. A *right group action* of G on x is a group antihomomorphism $*\colon G \to \mathrm{Aut}(x)$, where $\mathrm{Aut}(x)$ is the automorphism group of x in $\mathcal{C}$. That is, if e is the identity in G and $a, b \in G$ are any elements, then $*(e) = \mathrm{id}_x$ and $*(ab) = *(b) \circ *(a)$, where $\circ$ denotes the composition of morphisms in $\mathcal{C}$. Throughout this document, we refer to right group actions simply as group actions.

Definition 4.1.4 (Symmetric Operad)**.** A *symmetric operad over* $\mathcal{C}$ is a nonsymmetric operad X over $\mathcal{C}$ together with a group action $*\colon S_n \to \mathrm{Aut}(X_n)$ for each $n \in \mathbb{N}$ that satisfies the equivariance axioms below. As in the case for iterated tensor products, we often write just σ instead of $*(\sigma)$ for the action of the symmetric group on each X_n.

(Equivariance 1) Let $n \in \mathbb{N}$, and $a_1, \ldots, a_n \in \mathbb{N}$. Let $\tau \in S_n$ have inverse σ, and let $\tau' \in S_{a_1+\cdots+a_n}$ denote the associated block permutation as defined in Definition 3.1.2. Then the following diagram commutes:

$$\begin{CD}
X_n \otimes X_{a_1} \otimes \cdots \otimes X_{a_n} @>{\gamma_{a_1,\ldots,a_n}}>> X_{a_1+\cdots+a_n} \\
@V{\tau \otimes \mathrm{id}_{X_{a_1}} \otimes \cdots \otimes \mathrm{id}_{X_{a_n}}}VV @. \\
X_n \otimes (X_{a_1} \otimes \cdots \otimes X_{a_n}) @. @V{\tau'}VV \\
@V{\mathrm{id}_{X_n} \otimes \mu(\tau)}VV @. \\
X_n \otimes X_{a_{\sigma(1)}} \otimes \cdots \otimes X_{a_{\sigma(n)}} @>>{\gamma_{a_{\sigma(1)},\ldots,a_{\sigma(n)}}}> X_{a_1+\cdots+a_n}
\end{CD} .$$

(Equivariance 2) Let n and the a_i be as above. Let $\sigma_1 \in S_{a_1}, \ldots \sigma_n \in S_{a_n}$. Then the following diagram commutes:

$$\begin{CD}
X_n \otimes X_{a_1} \otimes \cdots \otimes X_{a_n} @>{\gamma_{a_1,\ldots,a_n}}>> X_{a_1+\cdots+a_n} \\
@V{\mathrm{id}_{X_n} \otimes \sigma_1 \otimes \cdots \otimes \sigma_n}VV @V{(\sigma_1,\ldots,\sigma_n)}VV \\
X_n \otimes X_{a_1} \otimes \cdots \otimes X_{a_n} @>>{\gamma_{a_1,\ldots,a_n}}> X_{a_1+\cdots+a_n}
\end{CD} ,$$

where $(\sigma_1, \ldots, \sigma_n)$ is the disjoint union defined in Definition 3.1.2.

Remark 4.1.5. The categories $\mathrm{Set}, \mathrm{Vect}_{\mathbb{K}}$, and Mod_A^Γ have extra structure beyond that of a general symmetric monoidal category. We use the following:

(Underlying Sets) The objects in this category can naturally be regarded as structured sets. This means that for any object O, there is some *underlying set* $O' \in \mathrm{Ob}(\mathrm{Set})$ for O. For instance, the underlying set of a vector space $(V, \mathbb{K}, +, \cdot)$ is the set of its vectors, V. Similarly, the underlying set of a Γ-graded A-module is the set of elements in that module. Objects in Set are their own underlying sets. We identify objects in these categories with their underlying sets; for instance, we refer to a vector space $(V, \mathbb{K}, +, \cdot)$ simply as V, and write $v \in V$ to mean that v is a vector in the underlying set.

(Underlying Functions) The morphisms from x to y in any of these categories are simply functions from the underlying set of x to the underlying set of y; for instance, linear maps are functions between underlying sets of vectors. We identify morphisms with their underlying functions.

(Internal Hom-Sets) The hom-sets in these categories can canonically be interpreted as objects in the same category. For instance, the set of linear maps between two $\mathbb{K}$-vector spaces is itself a $\mathbb{K}$-vector space when equipped with pointwise operations.

The same is true for the sets of grade-preserving maps in Mod_A^Γ, and the set of all functions between two sets is tautologically an object in Set.

(Tensor Map) There is a tensor map $\otimes\colon X \times Y \to X \otimes Y$, from the cartesian product of (the underlying sets of) any two objects X, Y to (the underlying set of) their tensor product. Thus it makes sense to speak of simple tensors in these categories; for any $x \in X, y \in Y$, we write $\otimes(x, y) = x \otimes y$. We require each of these maps to satisfy the universal property of a tensor product. That is, for any function $f\colon X \times Y \to Z$ such that fixing either coordinate gives a morphism $X \to Z$ or $Y \to Z$ in the underlying category, there exists a unique morphism $\tilde{f}\colon X \otimes Y \to Z$ such that $f = \tilde{f} \circ \otimes$. Note that this tensor map commutes with the tensor product of morphisms in the following sense: for any morphisms $f\colon x \to x'$ and $g\colon y \to y'$, we have $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$. Further, any morphisms $a, b\colon X \otimes Y \to X' \otimes Y'$ that agree on all simple tensors must be equal.

(Unit) The monoidal units in these categories contain a distinguished element $1 \in u$ such that for every object $X \in \mathcal{C}$ and every element $x \in X$, there is exactly one morphism $f\colon u \to X$ such that $f(1) = x$. Further, $\lambda(1 \otimes x) = x = \rho(x \otimes 1)$. In Mod_A^Γ, this distinguished element is the multiplicative unit in A. In Set, the distinguished element is the unique element in the monoidal unit.

In the remainder of this document, we will focus on categories that possess this structure.

Definition 4.1.6 (Closed Symmetric Monoidal Category of Structured Sets)**.** A *closed symmetric monoidal category of structured sets* is a symmetric monoidal category $\mathcal{C}$ that has all of the structure described in Remark 4.1.5.

Example 4.1.7 (Endomorphism Operad over Closed Symmetric Monoidal Categories of Structured Sets)**.** Let $\mathcal{C}$ be a closed symmetric monoidal category of structured sets. For any object O in $\mathcal{C}$, we define the *endomorphism operad over* O as follows: for each $n \in \mathbb{N}$, set $\text{End}_O(n) = \text{Hom}(O^{\otimes n}, O)$. The composition morphisms $\beta_{i_1,\dots,i_k}$ are defined by setting $\beta_{i_1,\dots,i_k}(f \otimes g_1 \otimes \cdots \otimes g_n) = f \circ (g_1 \otimes \cdots \otimes g_n)$ for any given morphisms $f \in \text{End}_O(n)$ and $g_1 \in \text{End}_O(i_1), \dots, g_n \in \text{End}_O(i_n)$. The assumption (Tensor Map) ensures that this uniquely defines β.

In this operad, the identity map $I\colon u \to \text{End}_O(1)$ is defined to be the unique morphism from u to $\text{Hom}(O, O)$ such that $I(1) = \text{id}_O$; the existence and uniqueness of such a morphism is guaranteed by the assumption (Unit).

The action of S_n on $\text{End}_O(n)$ is given by $*(\sigma) = \sigma = \text{Hom}(\mu(\sigma)^{\text{op}}, \text{id}_O)$, the image of the pair of morphisms $(\mu(\sigma)^{\text{op}}, \text{id}_O)$ in the external hom-functor. In other words, the action is given by precomposition with $\sigma = \mu(\sigma)$.

Remark 4.1.8. The definition of β in the previous example, interpreted strictly, does not output the correct type of morphisms when one or more of the subscripts is equal to 0. For instance, $\beta_{0,1}$ should be a morphism from $\text{End}_O(2) \otimes \text{End}_O(0) \otimes \text{End}_O(1)$ to $\text{End}_O(1)$ (that is, to $\text{Hom}(O, O)$), but the definition sends a tensor of morphisms $f \otimes g \otimes h$ to $f \circ (g \otimes h)$, which is a morphism from $u \otimes O$ to O. Note, however, that a natural precomposition with λ_O^{-1} transforms this morphism into an element of $\text{Hom}(O, O)$, the desired hom-set. In general, we set $\beta_{i_1,\dots,i_k}(f \otimes g_1 \otimes \cdots \otimes g_n) = f \circ (g_1 \otimes \cdots \otimes g_n) \circ p$, where p is the morphism from $O^{\otimes i_1 + \cdots + i_k}$ to the domain of $f \circ (g_1 \otimes \cdots \otimes g_n)$ built up from identity morphisms, λ^{-1}, and ρ^{-1}; the existence of such a morphism is clear, and Theorem 2.2.22 guarantees its uniqueness.

Example 4.1.9 (Associative Operad over $\mathcal{C}$)**.** Let $\mathcal{C}$ be a monoidal category with an initial object i. For each $n \geq 1$, define $\text{Assoc}^{\mathcal{C}}(n) = u$ and set $\text{Assoc}^{\mathcal{C}}(0) = i$. We define the composition maps inductively: for any $n \geq 0$, define $\gamma_n \colon \text{Assoc}^{\mathcal{C}}(1) \otimes \text{Assoc}^{\mathcal{C}}(n) \to \text{Assoc}^{\mathcal{C}}(n)$ by $\gamma_n = \lambda$. Then for any iterated product $A = \bigotimes_{i=1}^{k} \text{Assoc}^{\mathcal{C}}(a_i)$ with $k \geq 2$, we define $\gamma_{a_1,\ldots,a_k} \colon \text{Assoc}^{\mathcal{C}}(k) \otimes A \to \text{Assoc}^{\mathcal{C}}(a_1 + \cdots + a_k)$ by $\gamma_{a_1,\ldots,a_k} = \gamma_{a_2,\ldots,a_k} \circ \lambda \otimes (\bigotimes_{i=2}^{k} \text{id}_{\text{Assoc}^{\mathcal{C}}(a_i)})$. The identity is id_u. Then $\text{Assoc}^{\mathcal{C}}$ forms a nonsymmetric operad called the *associative operad over* $\mathcal{C}$.

Example 4.1.10 (Unital Associative Operad over $\mathcal{C}$)**.** Let $\mathcal{C}$ be a monoidal category. For each $n \in \mathbb{N}$, define $\text{Assocu}^{\mathcal{C}}(n) = u$. The composition maps and identity are the same as those in the non-unital associative operad. Then $\text{Assocu}^{\mathcal{C}}$ forms a nonsymmetric operad called them *unital associative operad over* $\mathcal{C}$.

Example 4.1.11 ((Unital) Commutative Operad over $\mathcal{C}$)**.** Let $\mathcal{C}$ be a monoidal category. If $\mathcal{C}$ has an initial object, define $\text{Comm}^{\mathcal{C}} = \text{Assoc}^{\mathcal{C}}$ for each $n \in \mathbb{N}$, and equip each with the trivial group action of S_n. Then $\text{Comm}^{\mathcal{C}}$ together with the same composition maps and identity as in $\text{Assoc}^{\mathcal{C}}$ forms a symmetric operad called the *commutative operad over* $\mathcal{C}$.

Even if $\mathcal{C}$ doesn't have an initial object, define $\text{Commu}^{\mathcal{C}}$ in the same way, but with $\text{Commu}^{\mathcal{C}}(n) = \text{Assocu}^{\mathcal{C}}(n)$. This forms a symmetric operad called the *unital commutative operad over* $\mathcal{C}$.

Lemma 4.1.12. *In both* $\text{Assoc}^{\mathcal{C}}$ *and* $\text{Assocu}^{\mathcal{C}}$*, we have* $\gamma_{a_1,\ldots,a_n} = \gamma_{b_1,\ldots,b_n}$ *for any* $a_i, b_i \in \mathbb{N}$ *such that* $a_1 + \cdots + a_n = b_1 + \cdots + b_n$*. For* $\text{Assoc}^{\mathcal{C}}$*, we also require that* $a_i = 0$ *if any only if* $b_i = 0$ *for all* i*; that is, any zeroes in the subscripts must appear at the same positions.*

Proof. The result follows from Theorem 2.2.22, as both γ morphisms are built up from λ and the identity, and they both have the same domain and codomain. □

Theorem 4.1.13. $\text{Assoc}^{\mathcal{C}}, \text{Assocu}^{\mathcal{C}}, \text{Comm}^{\mathcal{C}}$*, and* $\text{Commu}^{\mathcal{C}}$ *are operads. That is, they all satisfy the axioms of Definition 4.1.2, and the latter two satisfy the axioms of Definition 4.1.4.*

Proof. Note that by the definition of the composition maps in $\text{Assoc}^{\mathcal{C}}$ (which are the same as those in the other three operads in question), the associativity axiom is given by a linear diagram consisting of only braidings, associators, unitors, and tensors and compositions thereof. Thus by Theorem 2.2.22, the diagram commutes. The same is true for the identity and equivariance axioms. □

4.2. Morphisms and Algebras over General Operads.

In this section, we give the definitions of morphisms and algebras for general operads, and then prove general versions of the category isomorphisms from Section 3.

Definition 4.2.1 (Morphism of Nonsymmetric Operads)**.** Let X and Y be two nonsymmetric operads over the symmetric monoidal category $\mathcal{C}$, with identities I and J and families of composition maps γ and β respectively. Then a morphism of nonsymmetric operads $F : X \to Y$ is a family of $\mathcal{C}$-morphisms $\{F_n \colon X_n \to Y_n\}_{n \in \mathbb{N}}$ that satisfy the following:

(MO1) F preserves the identity. That is, the following diagram commutes:

$$\begin{array}{ccc} u & \xrightarrow{J} & Y_1 \\ {\scriptstyle I}\downarrow & \nearrow_{F_1} & \\ X_1 & & \end{array} .$$

(MO2) F commutes with the composition maps in X and Y. Concretely, for any $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{N}$, the following diagram commutes:

$$\begin{array}{ccc} X_n \otimes \left(\bigotimes_{i=1}^{n} X_{a_i} \right) & \xrightarrow{\gamma_{a_1,\ldots,a_n}} & X_{a_1+\cdots+a_n} \\ {\scriptstyle F_n \otimes \left(\bigotimes_{i=1}^{n} F_{a_i} \right)}\downarrow & & \downarrow{\scriptstyle F_{a_1+\cdots+a_n}} \\ Y_n \otimes \left(\bigotimes_{i=1}^{n} Y_{a_i} \right) & \xrightarrow{\beta_{a_1,\ldots,a_n}} & Y_{a_1+\cdots+a_n} \end{array} .$$

Definition 4.2.2 (Morphism of Symmetric Operads)**.** Let X and Y be symmetric operads over the symmetric monoidal category $\mathcal{C}$. Then a morphism of symmetric operads $F\colon X \to Y$ is a morphism of nonsymmetric operads that additionally satisfies:

(MO3) F commutes with the action of S_n. That is, for any $n \in \mathbb{N}$ and $\tau \in S_n$, the following diagram commutes:

$$\begin{array}{ccc} X_n & \xrightarrow{\tau} & X_n \\ {\scriptstyle F_n}\downarrow & & \downarrow{\scriptstyle F_n} \\ Y_n & \xrightarrow{\tau} & Y_n \end{array} .$$

Definition 4.2.3 (Algebra over an Operad)**.** Let X be an operad over a symmetric monoidal closed category of structured sets $\mathcal{C}$. An *algebra over* X, also called an X*-algebra*, is a morphism of operads $F\colon X \to \mathrm{End}_O$ for some object $O \in \mathcal{C}$; this may either be a morphism of nonsymmetric or symmetric operads, depending on which type of operad X is.

Definition 4.2.4 (Morphism of Algebras over an Operad)**.** Let X be an operad over a symmetric monoidal closed category of structured sets $\mathcal{C}$. A morphism $M\colon F \to G$ between two X-algebras $F\colon X \to \mathrm{End}_O$ and $G\colon X \to \mathrm{End}_P$ is a $\mathcal{C}$-morphism $M\colon O \to P$ such that for all $n \in \mathbb{N}$ and any $f \in X_n$, the following diagram commutes:

$$\begin{array}{ccc} O^{\otimes n} & \xrightarrow{F(f)} & O \\ {\scriptstyle M^{\otimes n}}\downarrow & & \downarrow{\scriptstyle M} \\ P^{\otimes n} & \xrightarrow[G(f)]{} & P \end{array} . \tag{17}$$

Definition 4.2.5 (Category of Algebras over an Operad)**.** The definition of Alg_X for an operad over a general symmetric monoidal category is exactly the same as the one given in the context of sets in Definition 3.3.5

Theorem 4.2.6. *For any symmetric monoidal closed category of structured sets* $\mathcal{C}$ *with an initial object,* $\mathrm{Alg}_{\mathrm{Assoc}^{\mathcal{C}}}$ *is isomorphic to* $\mathrm{Semi}_{\mathcal{C}}$*. We denote the isomorphism functor by*

$\varphi\colon \mathrm{Alg}_{\mathrm{Assoc}^{\mathcal{C}}} \to \mathrm{Semi}_{\mathcal{C}}$, with inverse $\sigma\colon \mathrm{Semi}_{\mathcal{C}} \to \mathrm{Alg}_{\mathrm{Assoc}^{\mathcal{C}}}$. The actions on objects and morphisms are as follows: let $F\colon \mathrm{Assoc}^{\mathcal{C}} \to \mathrm{End}_O$ and $G\colon \mathrm{Assoc}^{\mathcal{C}} \to \mathrm{End}_P$ be algebras over $\mathrm{Assoc}^{\mathcal{C}}$, and let $M\colon F \to G$ be a morphism of $\mathrm{Assoc}^{\mathcal{C}}$-algebras. Then:

$$\varphi(F) = (O, F_2(1)),$$
$$\varphi(M) = M.$$

Next, let (O, m) and (P, p) be semigroup objects, and $M\colon O \to P$ a morphism of semigroup objects. Then:

$$\sigma(O, m) = F,$$
$$\sigma(M) = M,$$

where F is an algebra over $\mathrm{Assoc}^{\mathcal{C}}$ defined inductively as follows: set $F_1 = J$, where J is the identity map for End_O. Then, for any $n \geq 2$, define F_n to be the unique map from u to $\mathrm{Hom}(O^{\otimes n}, O)$ such that $F_n(1) = m \circ (F_{n-1}(1) \otimes \mathrm{id}_O)$. In the special case $n = 0$, we have no choice to make; by definition, $\mathrm{Assoc}^{\mathcal{C}}(0)$ is an initial object, and thus there is a unique map $F_0\colon \mathrm{Assoc}^{\mathcal{C}}(0) \to \mathrm{End}_O(0)$.

Proof. As in the proof of Theorem 3.3.6, we first need to verify that φ and σ actually map objects into the claimed categories. Let $F\colon \mathrm{Assoc}^{\mathcal{C}} \to \mathrm{End}_O$ be an algebra over $\mathrm{Assoc}^{\mathcal{C}}$. We label the claimed semigroup object product $m = F_2(1)$. By construction, we have $m \in \mathrm{End}_O(2) = \mathrm{Hom}(O \otimes O, O)$. So m is a morphism from $O \otimes O$ to O. It remains to show that the associativity diagram (8) commutes—that is, $m \circ (\mathrm{id}_O \otimes m) = m \circ (m \otimes \mathrm{id}_O)$. Here, we use γ and I to denote the composition and identity maps in $\mathrm{Assoc}^{\mathcal{C}}$, and β and J for the maps in End_O. Note that since F is a morphism of operads, we must have $F_1 \circ I = J$ due to (MO1). But $I = \mathrm{id}_u$, so this condition says that $F_1 = J$. We then have the following:

$$
\begin{aligned}
m \circ (\mathrm{id}_O \otimes m) &= F_2(1) \circ \big(J(1) \otimes F_2(1)\big) && \text{By the definition of } m \text{ and } J\\
&= F_2(1) \circ \big(F_1(1) \otimes F_2(1)\big)\\
&= \beta_{1,2}\big(F_2(1) \otimes F_1(1) \otimes F_2(1)\big) && \text{By the definition of } \beta\\
&= \beta_{1,2}\big([F_2 \otimes F_1 \otimes F_2](1 \otimes 1 \otimes 1)\big) && \text{By (Tensor Map)}\\
&= [\beta_{1,2} \circ (F_2 \otimes F_1 \otimes F_2)](1 \otimes 1 \otimes 1) && \text{By the definition of the composition of functions}\\
&= [F_3 \circ \gamma_{1,2}](1 \otimes 1 \otimes 1) && \text{By (MO2)}\\
&= [F_3 \circ \gamma_{2,1}](1 \otimes 1 \otimes 1) && \text{By Lemma 4.1.12}\\
&= \big[\beta_{2,1} \circ (F_2 \otimes F_2 \otimes F_1)\big](1 \otimes 1 \otimes 1) && \text{By (MO2)}\\
&= \beta_{2,1}\big([F_2 \otimes F_2 \otimes F_1](1 \otimes 1 \otimes 1)\big)\\
&= \beta_{2,1}\big(F_2(1) \otimes F_2(1) \otimes F_1(1)\big)\\
&= F_2(1) \circ \big(F_2(1) \otimes F_1(1)\big)\\
&= m \circ (m \otimes \mathrm{id}_O),
\end{aligned}
$$

as desired.

Now suppose that (O, m) is a semigroup object. Set $F = \sigma(O, m)$. Then F satisfies (MO1) since, by definition, $F_1 = J$ and $I = \mathrm{id}_u$, and so $F_1 \circ I = J$ holds trivially. Property (MO2) amounts to $F_{a_1+\cdots+a_n} = \beta_{a_1+\cdots+a_n} \circ F_n \otimes \left(\bigotimes_{i=1}^{n} F_{a_i}\right)$ based on the definition of composition in $\mathrm{Assoc}^{\mathcal{C}}$. By assumption, m is associative, so any iterated compositions of m are equal. Thus by the definition of F, the desired equality holds, and F is indeed an algebra over $\mathrm{Assoc}^{\mathcal{C}}$.

Now let $M\colon (O,m) \to (P,p)$ be a morphism of semigroup objects. We need to verify that $\sigma(M) = M$ is a morphism of $\mathrm{Assoc}^{\mathcal{C}}$-algebras. In particular, we need to check that diagram (17) commutes. Set $F = \sigma(O,m)$ and $G = \sigma(P,p)$. An identical argument to that in the proof of Theorem 3.3.6 shows that $M \circ F_n(1) = G_n(1) \circ M^{\otimes n}$; simply replace each α_n with 1. Note then that $M \circ F_n(-)$ and $G_n(-) \circ M^{\otimes n}$ are both functions from u to $\mathrm{Hom}(O^{\otimes n}, P)$, and they agree on the input $1 \in u$. By the assumption (Unit), the two functions are equal. Thus $M \circ F_n(f) = G_n(f) \circ M^{\otimes n}$ holds for all $f \in u = \mathrm{Assoc}^{\mathcal{C}}(n)$, as desired.

Next, let $M\colon F \to G$ be a morphism of $\mathrm{Assoc}^{\mathcal{C}}$-algebras, and set $(O,m) = \varphi(F)$ and $(P,p) = \varphi(G)$. Diagram (11) (with $f = M$) is the case $n = 2$ of diagram (17) with $f = 1$, as $F_2(1) = m$ and $G_2(1) = p$ by construction. Thus M is a morphism of semigroup objects.

Finally, we show that φ and σ are inverse functors. They act as the identity on morphisms, so it suffices to check the action on objects. Let (O,m) be a semigroup object. Then $\varphi(\sigma(O,m)) = \varphi(F) = (O, F_2(1)) = (O,m)$, since $F_2(1) = m$ by the definition of σ. Now let $F\colon \mathrm{Assoc}^{\mathcal{C}} \to \mathrm{End}_O$ be an algebra over $\mathrm{Assoc}^{\mathcal{C}}$. Write $\sigma(\varphi(F)) = \sigma(O, F_2(1)) = G$. We've already verified that G is an algebra over $\mathrm{Assoc}^{\mathcal{C}}$, so we know that $G_1(1) = F_1(1) = \mathrm{id}_O$, and $G_2(1) = F_2(1)$ holds by the definition of σ. Further, $G_0(1) = F_0(1)$ must hold, since both are morphisms from the initial object. Let $n > 2$, and suppose that $G_{n-1}(1) = F_{n-1}(1)$ holds. We then have the following:

$$
\begin{aligned}
G_n(1) &= F_2(1) \circ \big(G_{n-1}(1) \otimes \mathrm{id}_O\big) && \text{By the definition of } \sigma \\
&= F_2(1) \circ \big(F_{n-1}(1) \otimes F_1(1)\big) \\
&= \beta_{n-1,1}\big(F_2(1) \otimes F_{n-1}(1) \otimes F_1(1)\big) && \text{By the definition of } \beta \\
&= \beta_{n-1,1}\big([F_2 \otimes F_{n-1} \otimes F_1](1 \otimes 1 \otimes 1)\big) \\
&= [\beta_{n-1,1} \circ (F_2 \otimes F_{n-1} \otimes F_1)](1 \otimes 1 \otimes 1) \\
&= [F_n \circ \gamma_{n-1,1}](1 \otimes 1 \otimes 1) && \text{By (MO2)} \\
&= F_n\big(\gamma_{n-1,1}(1 \otimes 1 \otimes 1)\big) \\
&= F_n\Big(\gamma_1\big([\lambda \otimes \mathrm{id}_u](1 \otimes 1 \otimes 1)\big)\Big) && \text{By the definition of } \gamma \\
&= F_n\Big(\lambda\big([\lambda \otimes \mathrm{id}_u](1 \otimes 1 \otimes 1)\big)\Big) && \text{By the definition of } \gamma \\
&= F_n\big(\lambda(1 \otimes 1)\big) && \text{By (Unit)} \\
&= F_n(1), && \text{By (Unit)}
\end{aligned}
$$

and thus $G_n(1) = F_n(1)$ holds for all n by induction. Then F_n and G_n are morphisms with domain u that agree on the input $1 \in u$, so (Unit) tells us that they are equal. We conclude that $F = G$, and that φ and σ are indeed inverses. □

Theorem 4.2.7. *For any closed symmetric monoidal concrete category $\mathcal{C}$, $\mathrm{Alg}_{\mathrm{Assocu}^{\mathcal{C}}}$ is isomorphic to $\mathrm{Mon}_{\mathcal{C}}$. The isomorphism functors φ and σ are the same as in Theorem 4.2.6,* *with the following modifications:*

$$\varphi(F) = \big(O, F_2(1), F_0(1)\big),$$
$$\sigma(O, m, H) = F,$$

where F is defined as in the aforementioned theorem except in the $n = 0$ case, where we define $F_0\colon \mathrm{Assocu}^{\mathcal{C}}(0) \to \mathrm{End}_O(0)$ to be the unique morphism such that $F_0(1) = H$.

Proof. By the same arguments as in the proof of Theorem 4.2.6, we know that $\varphi(F)$ is a semigroup object. It remains to show that $F_0(1)$ is indeed an identity map, or in other words, that the diagrams (9) commute. We will prove that the first diagram commutes; the proof for the other diagram is completely analogous. We need to show that $m \circ (H \otimes \mathrm{id}_O) = \lambda_O$, or, equivalently, $m \circ (H \otimes \mathrm{id}_O) \circ \lambda_O^{-1} = \mathrm{id}_O$. We have the following:

$$\begin{aligned}
m \circ (H \otimes \mathrm{id}_O) \circ \lambda_O^{-1} &= F_2(1) \circ \big(F_0(1) \otimes F_1(1)\big) \circ \lambda_O^{-1} && \text{By the definition of } F\\
&= \beta_{0,1}\big(F_2(1) \otimes F_0(1) \otimes F_1(1)\big) && \text{By the definition of } \beta\\
&= \beta_{0,1}\big([F_2 \otimes F_0 \otimes F_1](1 \otimes 1 \otimes 1)\big)\\
&= \big[\beta_{0,1} \circ (F_2 \otimes F_0 \otimes F_1)\big](1 \otimes 1 \otimes 1)\\
&= [F_1 \circ \gamma_{0,1}](1 \otimes 1 \otimes 1) && \text{Using (MO2)}\\
&= F_1\big(\gamma_{0,1}(1 \otimes 1 \otimes 1)\big)\\
&= F_1\Big(\big[\gamma_1 \circ (\lambda \otimes \mathrm{id}_u)\big](1 \otimes 1 \otimes 1)\Big) && \text{By the definition of } \gamma\\
&= F_1\Big(\big[\lambda \circ (\lambda \otimes \mathrm{id}_u)\big](1 \otimes 1 \otimes 1)\Big) && \text{By the definition of } \gamma\\
&= F_1\Big(\lambda\big([\lambda \otimes \mathrm{id}_u](1 \otimes 1 \otimes 1)\big)\Big)\\
&= F_1\Big(\lambda\big(\lambda(1 \otimes 1) \otimes \mathrm{id}_u(1)\big)\Big) && \text{Using (Tensor Map)}\\
&= F_1\big(\lambda(1 \otimes 1)\big) && \text{Using (Unit)}\\
&= F_1(1) && \text{Using (Unit)}\\
&= \mathrm{id}_O, && \text{By the definition of } F
\end{aligned}$$

as desired.

Analogous arguments to those in the proof of Theorem 4.2.6 show that the action of σ sends monoid objects to algebras over $\mathrm{Assocu}^{\mathcal{C}}$.

Now let $M\colon (O, m, H) \to (P, p, K)$ be a morphism of monoid objects. The same argument as in the proof of Theorem 4.2.6 (where the proof of $M \circ F_n(1) = G_n(1) \circ M^{\otimes n}$ is now inherited from Theorem 3.3.8 to account for the $n = 0$ case) shows that $\sigma(M)$ is a morphism of $\mathrm{Assocu}^{\mathcal{C}}$-algebras.

Next, let $M\colon F \to G$ be a morphism of $\mathrm{Assocu}^{\mathcal{C}}$-algebras. Write $\varphi(F) = (O, m, H)$ and $\varphi(G) = (P, p, K)$. We need to show that diagram (12) commutes, where M is the morphism of monoid objects in this case. That is, we want to show $M \circ H = K$. We have:

$$\begin{aligned}
M \circ H &= M \circ F_0(1) && \text{By the definition of } H\\
&= G_0(1) \circ M^{\otimes 0} && \text{Using the } n = 0 \text{ case of diagram (17)}
\end{aligned}$$

$= G_0(1)$	Since $M^{\otimes 0} = \mathrm{id}_u$ by definition
$= K,$	By the definition of K

as desired.

Finally, we must show that φ and σ are inverse functors. Let (O, m, H) be a semigroup object. Then $\varphi(\sigma(O, m, H)) = \varphi(F) = (O, m, H)$, since $F_2(1) = m$ and $F_0(1) = H$ by the definition of σ. Now let $F\colon \mathrm{Assocu}^{\mathcal{C}} \to \mathrm{End}_O$ be an algebra over $\mathrm{Assocu}^{\mathcal{C}}$. Write $\sigma(\varphi(F)) = \sigma(O, F_2(1), F_0(1)) = G$. We know that $F_1(1) = G_1(1) = \mathrm{id}_O$ since F and G are both $\mathrm{Assocu}^{\mathcal{C}}$-algebras. Further, $F_0(1) = G_0(1)$ and $F_2(1) = G_2(1)$ hold by the definition of σ. By the same inductive argument as in the proof of Theorem 4.2.6, we conclude that $F = G$, as desired. Thus φ and σ are indeed inverses. □

Theorem 4.2.8. *For any closed symmetric monoidal concrete category $\mathcal{C}$, $\mathrm{Alg}_{\mathrm{Comm}^{\mathcal{C}}}$ is isomorphic to $\mathrm{CSemi}_{\mathcal{C}}$. The isomorphism functors φ and σ are the same as those in Theorem 4.2.6.*

Proof. First, note that φ and σ are inverses since they have exactly the same actions as the functors in Theorem 4.2.6. Similarly, since the morphisms in $\mathrm{CSemi}_{\mathcal{C}}$ are just morphisms of semigroup objects like in $\mathrm{Semi}_{\mathcal{C}}$, we don't need to do any further work to verify that these two functors map morphisms appropriately. We thus proceed by checking only that the targets for objects are correct.

Let $F\colon \mathrm{Comm}^{\mathcal{C}} \to \mathrm{End}_O$ be an algebra over $\mathrm{Comm}^{\mathcal{C}}$. The aforementioned theorem shows that $\varphi(F) = (O, m)$ is a semigroup object, so it remains to show that (O, m) is commutative in the sense of diagram (10). That is, we want to show $m \circ B_{O,O} = m$. Denote the permutation $(12) \in S_2$ by τ. We have:

$m \circ B_{O,O} = F_2(1) \circ B_{O,O}$	By the definition of F
$= F_2(1) \circ \tau$	By the definition of the symmetric action in End_O
$= \tau \circ F_2(1)$	Using (MO3)
$= F_2(1)$	Since the symmetric action in $\mathrm{Comm}^{\mathcal{C}}$ is trivial
$= m,$	

as desired.

Now let (O, m) be a commutative semigroup object. The proof of 4.2.6 shows that $\sigma(O, m)$ satisfies conditions (MO1) and (MO2), so it remains to show that (MO3) also holds. The fact that m is commutative means that the trivial group action of $\mathrm{Comm}^{\mathcal{C}}$ is respected, and thus we conclude that φ and σ are indeed isomorphism functors. □

Theorem 4.2.9. *For any closed symmetric monoidal concrete category $\mathcal{C}$, $\mathrm{Alg}_{\mathrm{Commu}^{\mathcal{C}}}$ is isomorphic to $\mathrm{CMon}_{\mathcal{C}}$. The isomorphism functors φ and σ are the same as those in Theorem 4.2.7.*

Proof. Combine the arguments used in the proofs of Theorems 4.2.7 and 4.2.8. □

When applied to $\mathrm{Vect}_{\mathbb{K}}$, the previous four theorems provide a characterization of various types of algebras in terms of operads. Specifically, Theorem 4.2.6 says that algebras over the associative operad correspond exactly to associative algebras, Theorem 4.2.7 says that algebras over the unital associative operad correspond to unital associative algebras, and Theorems 4.2.8 and 4.2.9 give the same correspondence for commutative algebras. The same is true in the category of super vector spaces, where these theorems give correspondences with superalgebras, unital superalgebras, and their commutative versions.

The operads $\mathrm{Assoc}^{\mathcal{C}}, \mathrm{Assocu}^{\mathcal{C}}, \mathrm{Comm}^{\mathcal{C}}$, and $\mathrm{Commu}^{\mathcal{C}}$ abstract out the basic notions of an algebraic structure being associative, having a unit, and being commutative. To properly treat structures with more complex properties, we need more powerful operad theoretic techniques.

5. Operads via Generators and Relations

The concept of "free" mathematical objects appears often in abstract algebra. Examples include free groups, free semigroups, and free modules. Free objects are the most generic examples of a particular structure, in a sense usually made precise by some sort of universal property. Some algebraic structures also admit the notion of quotient objects—for example, quotient vector spaces and quotient groups. In this section, we develop the tools necessary to construct both quotient operads and free operads, and then use those objects to instantiate various operads via generators and relations.

5.1. **Quotient Operads and Free Operads.**

Throughout this section, we write $\mathcal{V} = \mathrm{Mod}_A^\Gamma$ for brevity. In any applications of general results, we will specify A and Γ explicitly.

Definition 5.1.1 (Operadic Ideals)**.** Let X be a symmetric operad over $\mathcal{V}$. Let $\{Y_n\}_{n\in\mathbb{N}}$ be a family of objects in $\mathcal{V}$ such that Y_n is a graded submodule of X_n for all $n \in \mathbb{N}$, and each Y_n is S_n-invariant with respect to the symmetric action from X. Suppose further that $(f \circ_{a_1,\dots,a_n} (g_1, \dots, g_n)) \in Y$ for any $f, g_1, \dots, g_n \in X$ where at least one of those elements is in Y. Then Y is called an *operadic ideal* of X.

Definition 5.1.2 (Quotient Operad)**.** Let Y be an operadic ideal of X. Define the *quotient operad* X/Y as follows: set $(X/Y)_n = X_n/Y_n$ for each $n \in \mathbb{N}$, where this represents a quotient of graded modules. We designate $I + Y_1$ as the identity, where I is the identity of X, and define the composition maps by:

$$(f + Y_n) \circ_{a_1,\dots,a_n} (g_1 + Y_{a_1}, \dots, g_n + Y_{a_n}) = (f \circ_{a_1,\dots,a_n} (g_1, \dots, g_n)) + Y_{a_1+\cdots+a_n}.$$

If X is a symmetric operad, define a symmetric action on X/Y by: $(f + Y_n) * \sigma = (f * \sigma) + Y_n$ for all $\sigma \in S_n$.

We often write elements $f + Y_n \in (X/Y)_n$ simply as $f \in (X/Y)_n$ when it is clear that we are working in the quotient operad.

Theorem 5.1.3 (Quotient Operads are Operads)**.** *The quotient operad X/Y defined above is an operad.*

Proof. We need to show that the composition maps are well-defined. Associativity then follows immediately from the associativity of composition in X, and the same is true for the identity axiom. We will show that *partial compositions*, that is, those of the form $f \circ_i g := f \circ (I, \dots, I, \underbrace{g}_{\text{position } i}, I, \dots, I)$ are well-defined. Any composition can be expressed as a sequence of partial compositions, so it suffices to consider this special case. Let $f, f' \in X_n$, and $g, g' \in X_k$ for some $n, k \in \mathbb{N}$, and suppose that $f \sim f'$ and $g \sim g'$, where $\sim$ is the usual equivalence relation of coset membership used in a quotient module. That is, $f - f' \in Y_n$, and $g - g' \in Y_k$. Then we need to show that:

$$(f \circ_i g) + Y_{n+k-1} = (f' \circ_i g') + Y_{n+k-1},$$

or equivalently,

$$(f \circ_i g) - (f' \circ_i g') \in Y_{n+k-1}.$$

Note that $\circ$ is a grade-preserving multi-A-linear map by the definition of an operad, as we are working in the category of graded A-modules. We thus proceed by expansion:

$$\begin{aligned}(f - f') \circ_i (g - g') &= (f \circ_i g) - (f' \circ_i g) - (f \circ_i g') + (f' \circ_i g') \\ &= (f \circ_i g) - (f' \circ_i g) - (f \circ_i g') + (f' \circ_i g') - (f' \circ_i g') + (f' \circ_i g') \\ &= \big[(f \circ_i g) - (f' \circ_i g')\big] + \big[(f' \circ_i g') - (f' \circ_i g)\big] + \big[(f' \circ_i g') - (f \circ_i g')\big] \\ &= \big[(f \circ_i g) - (f' \circ_i g')\big] + \big[f' \circ_i (g' - g)\big] + \big[(f' - f) \circ_i g'\big].\end{aligned}$$

Re-writing this equality, we obtain:

$$\begin{aligned}\big[(f \circ_i g) - (f' \circ_i g')\big] &= \big[(f - f') \circ_i (g - g')\big] - \big[f' \circ (g' - g)\big] - \big[(f' - f) \circ_i g'\big] \\ &= \big[(f - f') \circ_i (g - g')\big] + \big[f' \circ (g - g')\big] + \big[(f - f') \circ_i g'\big].\end{aligned}$$

Note that all three of the terms on the final line are elements of Y_{n+k-1}: by assumption, $f - f'$ and $g - g'$ are elements of Y, which is an operadic ideal. By the closure of rings under addition, we conclude that $\big[(f \circ_i g) - (f' \circ_i g')\big]$ is an element of Y_{n+k-1}, as desired.

Now we check that the symmetric action is well-defined. The equivariance properties of a symmetric operad then follow from those in X. Let $f, f' \in X_n$ with $f \sim f'$, and $\sigma \in S_n$. We need to verify that $(f * \sigma) + Y_n = (f' * \sigma) + Y_n$, or equivalently, $(f * \sigma) - (f' * \sigma) \in Y_n$. By assumption, $f = f' + y$ for some $y \in Y_n$. Then:

$$\begin{aligned}(f * \sigma) - (f' * \sigma) &= (f' + y) * \sigma - (f' * \sigma) \\ &= (f' * \sigma) + (y * \sigma) - (f' * \sigma) && \text{Since } *\sigma \text{ is an automorphism} \\ &= y * \sigma \\ &= y', && \text{For some } y' \in Y_n, \text{ since } Y_n \text{ is } S_n\text{-invariant}\end{aligned}$$

so this action is indeed well-defined. □

Theorem 5.1.4. *Let X and Y be $\mathcal{V}$-operads, and $F\colon X \to Y$ a morphism of operads. Then the kernel of F, defined as* $\ker_n(F) := \{f \in X_n \mid F(f) = 0 \in Y_n\}$, *is an operadic ideal.*

Proof. By the definition of a morphism of operads, F is required to be a family of grade-preserving linear maps. Thus each $\ker_n(F)$ is a graded submodule of X_n. It remains to check the absorption property. Suppose that $f \in \ker_n(F)$, and use Z_k to denote the zero map in Y_k for each $k \in \mathbb{N}$. Then:

$$\begin{aligned}F\big(f \circ_{a_1,\ldots,a_n} (g_1, \ldots, g_n)\big) &= F(f) \circ_{a_1,\ldots,a_n} \big(F(g_1), \ldots, F(g_n)\big) && \text{Since } F \text{ is a morphism of operads} \\ &= Z_n \circ_{a_1,\ldots,a_n} \big(F(g_1), \ldots, F(g_n)\big) && \text{Since } f \in \ker_n(F) \\ &= (0_A Z_n) \circ_{a_1,\ldots,a_n} \big(F(g_1), \ldots, F(g_n)\big)\end{aligned}$$

$$= 0_A \cdot \Big(Z \circ_{a_1,\dots,a_n} \big(F(g_1), \dots, F(g_n) \big) \Big) \quad \text{Since } \circ \text{ is multilinear}$$
$$= Z_{a_1+\cdots+a_n}.$$

So this composition is in $\ker(F)$. The case where one of the g_i is in the kernel is analogous. Thus $\ker(F)$ is indeed an operadic ideal. □

Definition 5.1.5 (Rooted Trees)**.** Recall that a *tree* is an acyclic connected graph. Equivalently, trees are those graphs such that there is a unique path between any two vertices.

A *rooted tree* is a tree together with a distinguished vertex called the root.

The *depth* of a vertex in a rooted tree is the length of its path to the root.

The *height* of a rooted tree is the greatest depth of any vertex in that tree.

In a rooted tree, an edge joining two vertices u and v is called an *outgoing edge* from u if the depth of u is less than the depth of v. In such a case, v is called an *out-neighbour* of u.

Definition 5.1.6 (Free Operad)**.** For each $n \in \mathbb{N}$, let P_n be a (potentially empty) set, such that the P_n are pairwise disjoint. We require P_1 to contain a distinguished element that we denote I. In practice, since every free operad must contain such an element, we usually do not explicitly define P_1 to contain I. Let $P = (P_0, P_1, \dots)$ denote the family of these sets. Define $\mathcal{T}(P)$ to be the set of all finite rooted trees subject to the following extra structure:

The root must be labeled with some $p \in P_k$, where k is the degree of the root. Further, every vertex of degree $d \geq 2$ must be labeled with some $p \in P_{d-1}$. Leaves (other than the root) may either be labeled with an element of P_0 or be unlabeled, in which case we call it an *input vertex*. All input vertices are required to have depth equal to the height of the tree. Every non-input vertex must be equipped with a total ordering on its outgoing edges. Finally, each tree in $\mathcal{T}(P)$ must be equipped with a total ordering on its input vertices. We consider these trees up to graph isomorphisms that preserve the total orders and labels.

Let $f \in \mathcal{T}(P)$. If all vertices in f with depth $n \geq 1$ are labeled with I, note that each of these vertices must have exactly one out-neighbour. Consider the tree obtained by identifying each vertex of depth n with its unique out-neighbour, replacing each label I with the labels of those out-neighbours. In the special case where the out-neighbours are input vertices (and are thus not labeled), we instead leave the identified vertices unlabeled and maintain the total order on input vertices that previously existed in f. Call the tree obtained by repeating this reduction process until no depth $n \geq 1$ of vertices are all labeled with I the *identity-reduced form* of f.

Let $T(P)$ be the set of trees from $\mathcal{T}(P)$ that have been transformed to identity-reduced form. Partition $T(P)$ into the collection $\{T_n(P) \mid n \in \mathbb{N}\}$, where each $T_n(P)$ consists of the trees in $T(P)$ with exactly n input vertices.

Define the composition $f \circ (g_1, \dots, g_n)$ of trees by concatenation, replacing each input vertex of f with the root of the g_i in order of the total ordering on the input vertices of f. Define the total ordering on the input vertices of $f \circ (g_1, \dots, g_n)$ to be the lexicographic one. That is, all of the input vertices from g_i are smaller than those from g_j precisely when $i < j$, and the vertices in each g_i are locally ordered as they were before composition. After performing this concatenation, reduce the resulting tree to identity-reduced form. By this definition, the composition in $T(P)$ is clearly associative since concatenation of trees is associative, and it also satisfies the identity axioms of Definition 3.1.1 via the identity-reduction process.

Define the right action $*$ of the symmetric group S_n on $T_n(P)$ by permuting the total order on the input vertices. Explicitly, if $f \in T_n(P)$ and $\tau \in S_n$, then $f * \tau$ is the tree where the least input vertex from f is now the $\tau(1)$'th least in the new ordering, the second least input is now the $\tau(2)$'th least, and so on. This definition can be seen to satisfy the equivariance axioms of Definition 3.1.2.

For each $p \in P_n$, we identify p with the tree in $T_n(P)$ that consists of the root labeled by p and n input vertices, with the order on the outgoing edges from p agreeing with the order on the input vertices at the end of each of those edges. With this structure, the family $T(P) = \{T_0(P), T_1(P), \dots\}$ forms a symmetric operad of sets called the *free operad generated by P*.

For each $n \in \mathbb{N}$, define $T_n^A(P)$ to be the free graded A-module generated by taking $T_n(P)$ as a basis and giving that module the trivial grading. Define the composition and symmetric action by extending the composition and symmetric action from $T(P)$ linearly. Then $T^A(P) = \{T_0^A(P), T_1^A(P), \dots\}$ forms a symmetric operad in $\mathcal{V}$ called the *free graded operad over A generated by P*.

Note that the composition and symmetric action for free operads are the same as the operations on tree diagrams outlined in Section 3.2, with only cosmetic differences. Specifically, in those diagrams, the input vertices are omitted, and an additional edge at the bottom of each tree is included for illustrative purposes. The total order on each set of outgoing edges corresponds to the left-to-right ordering of the edges as drawn in the plane, and the same is true for the order on the input vertices.

Lemma 5.1.7. *Let $T(P)$ be a free operad. Let $f \in T(P)$. Then f can be expressed as a finite sequence of compositions of elements in P followed by a symmetric action.*

The same is true for any free graded operad $T^A(P)$, where elements can be expressed as a linear combination of finite compositions of elements in P, each followed by a symmetric action.

Proof. We proceed by induction on the height of f. If f has height 0, then it consists of just a root vertex. This root must then be labeled with some $p \in P_0$ by the definition of $T(P)$, and is thus equal to p itself via the identification specified in Definition 5.1.6; we consider this to be a zero-step sequence of compositions. If f has height 1, then it consists of a root vertex labeled with some $p \in P_n$ that is adjacent to n input vertices, for some $n \in \mathbb{N}$. Thus f is equal to p followed by the unique symmetric action that results in the desired order on the input vertices.

Now suppose that the statement of this lemma holds for all elements of $T(P)$ with height k or less, for some fixed $k \geq 1$. Suppose f has height $k+1$. First, collect the labels of the vertices with depth k in some arbitrary order. Write this list of labels as $h_1, h_2, \dots, h_n$. Consider the tree obtained by taking the subtree of f consisting of vertices with depth k or less, unlabeling all of the vertices with depth k, and giving them the same order that the labels h_i were collected in. That tree, which we will call g, is then an element of $T(P)$ with depth k, so the induction hypothesis tells us that it can be expressed as a finite sequence of compositions of elements in P followed by a symmetric action. We write this as:

$$g = g_{1,1} \circ (g_{2,1}, g_{2,2}, \dots, g_{2,n_2}) \circ \cdots \circ (g_{k,1}, \dots, g_{k,n_k}) * \tau,$$

for some $g_{i,j} \in P$ and a permutation τ. Then by the definition of composition in $T(P)$, we have $f = g \circ (h_1, \dots, h_n) * \sigma$, where σ is the unique permutation that will reorder the input vertices to match the order in f. The first equivariance axiom of a symmetric operad (and the fact that $*$ is a group action) then tells us that we can rewrite this composition as:

$$f = g_{1,1} \circ (g_{2,1}, \dots, g_{2,n_2}) \circ \dots \circ (g_{k,1}, \dots, g_{k,n_k}) \circ (h_{\tau^{-1}(1)}, \dots, h_{\tau^{-1}(n)}) * \sigma\tau',$$

where each of the $g_{i,j}$ and h_i are in P, as desired.

The result for $T^A(P)$ follows from the fact that $T_n(P)$ is taken as a basis for $T_n^A(P)$. □

Theorem 5.1.8. *The free operads of Definition 5.1.6 are free, in the sense that they satisfy the following universal property: let P be as in the definition of a free operad (without the element I included), and let X be a symmetric operad over the category of sets. For any family of functions $F_n : P_n \to X_n$ (with n ranging over $\mathbb{N}$), there exists a unique morphism of operads $G: T(P) \to X$ that extends the F_n.*

Similarly, if X is instead a symmetric $\mathcal{V}$-operad, there exists a unique morphism of operads $G: T^A(P) \to X$ that extends the F_n.

Proof. Set $G_1(I)$ to be the identity in X_1. For every $f \in P_n$, set $G_n(f) = F_n(f)$. For any other $f \in T_n(P)$, Lemma 5.1.7 tells us that we can write:

$$f = a \circ (a_{1,1}, \dots, a_{1,k_1}) \circ \dots \circ (a_{m,1}, \dots, a_{m,k_m}) * \sigma$$

for some $a, a_{i,j} \in P$ and $\sigma \in S_n$. We define:

$$G_n(f) = F(a) \circ (F(a_{1,1}), \dots, F(a_{1,k_1})) \circ \dots \circ (F(a_{m,1}), \dots, F(a_{m,k_m})) * \sigma,$$

where in each case F stands for the relevant function F_i, according to which P_i each element belongs to. By this definition, G clearly satisfies the properties of a morphism of symmetric operads. Since any other morphism $G' : T(P) \to X$ that extends the F_n must agree with G on elements of P and satisfy the equivariance property used to define G, this morphism is unique.

The morphism is constructed analogously for the case with $T^A(P)$; simply define each G_n to be A-multilinear. The uniqueness of these morphisms follows from uniqueness in the $T(P)$ case and the fact that any other G' would also be required to be A-multilinear. □

Theorem 5.1.9. *The intersection of an arbitrary number of operadic ideals is an operadic ideal.*

Proof. Let X be a $\mathcal{V}$-operad, and let $\{Y(i) \mid i \in I\}$ be a collection of operadic ideals of X for some index set I. Set $Y = \bigcap_{i \in I} Y(i)$, where this denotes the operad given by $Y_n = \bigcap_{i \in I} Y_n(i)$. We know that each Y_n is a graded submodule of X_n by a theorem from abstract algebra, so it remains to check the absorption property. So let $f, g_1, \dots, g_n \in X$ with at least one of them being an element of Y. Without loss of generality, we assume that $f \in Y$. Take any $i \in I$. Then $f \in Y(i)$, so $f \circ (g_1, \dots, g_n) \in Y(i)$ since $Y(i)$ is an operadic ideal. This holds for all i, so we conclude $f \circ (g_1, \dots, g_n) \in Y$, as desired. □

Definition 5.1.10. Let X be a $\mathcal{V}$-operad, and for each $n \in \mathbb{N}$, let R_n be a subset (not necessarily a submodule) of X_n. Consider the collection $\{Y(i) \mid i \in I\}$ of operadic ideals of X satisfying $R_n \subseteq Y_n(i)$ for all n and all i. This collection is non-empty, as X itself is such an operadic ideal. Define $(R) = \bigcap_{i \in I} Y(i)$. Then by Theorem 5.1.9, (R) is an operadic ideal of X, which we call *the ideal generated by* R.

Definition 5.1.11 (Operad from Generators and Relations)**.** Let P be a family of sets as in Definition 5.1.6. For each $n \in \mathbb{N}$, let R_n be a (possibly empty) set of vectors in $T_n^A(P)$. Then define $\langle P \mid R \rangle = T^A(P)/(R)$. We call $\langle P \mid R \rangle$ the *operad generated by* P *subject to* R, and call P the family of generators and R the family of relations.

In practice, the R_n are often specified by sets of equalities rather than vectors. In this case, an equality of vectors $v = w$ is interpreted as the vector $v - w$, since taking the quotient $T^A(P)/(R)$ will then set $v - w = 0$, giving the desired equality.

Theorem 5.1.12. *Let X/Q be a quotient operad, and let Y be any symmetric $\mathcal{V}$-operad. Let $F\colon X \to Y$ be a morphism of operads such that $Q \subseteq \ker(F)$. Then there exists a morphism of operads $G\colon X/Q \to Y$ such that $F = G \circ \pi$, where $\pi\colon X \to X/Q$ is the usual quotient map.*

Proof. Define $G_n(f + Q_n) = F_n(f)$. Note G is then well-defined since if $f + Q_n = f' + Q_n$, then $f = f' + q$ for some $q \in Q_n$, by definition. Then $F_n(f) = F_n(f' + q) = F_n(f') + F_n(q) = F_n(f')$ by assumption on F. It's clear that G extends F, and it inherits all of the morphism properties from F, so it's a morphism of operads itself. It's clear that $F = G \circ \pi$. □

Corollary 5.1.13. *Let P be a family of generators, R a family of relations, and X a symmetric $\mathcal{V}$-operad. Suppose that $F_n\colon P_n \to X_n$ is a family of functions that maps every element of R to zero. Then there exists a morphism of operads $G\colon T^A(P)/(R) \to X$ that extends the F_n.*

Proof. Using Theorem 5.1.8, there is a unique morphism of operads $G\colon T^A(P) \to X$ that extends the F_n. Then Theorem 5.1.4 tells us that $\ker(G)$ is an operadic ideal. By assumption, $R_n \subseteq \ker_n(G)$ for each n. Since (R) is defined to be the intersection of all operadic ideals that contain R, we have $(R) \subseteq \ker(G)$. Thus by Theorem 5.1.12, there exists a morphism of operads $H\colon T^A(P)/(R) \to X$ that extends the F_n, as desired. □

5.2. More Examples of Operads.

In the final two sections, we give several examples to illustrate the power (and limitations) of the techniques developed above.

Definition 5.2.1 (Lie Algebra)**.** A *Lie algebra* over $\mathbb{K}$ is a pair $(V, [\cdot,\cdot])$, where V is a vector space over $\mathbb{K}$ and $[\cdot,\cdot]$ is a bilinear operation on V (called the Lie bracket) that satisfies the following properties for all $x, y, z \in V$:

(Alternativity) $[x, x] = 0$,
(Jacobi Identity) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$.

If $\mathbb{K}$ is a field of any characteristic other than 2, alternativity is equivalent to skew symmetry:

(Skew Symmetry) $[x, y] = -[y, x]$.

Let $(V, [\cdot,\cdot])$ and $(W, [\cdot,\cdot]')$ be Lie algebras over $\mathbb{K}$. A *homomorphism of Lie algebras* from $(V, [\cdot,\cdot])$ to $(W, [\cdot,\cdot]')$ is a $\mathbb{K}$-linear map $f\colon V \to W$ such that $f([v, v']) = [f(v), f(v')]'$ for all $v, v' \in V$.

Example 5.2.2 (Lie Operad)**.** Let m be a formal symbol. Define a family of generators by $P_2 = \{m\}$ with all other $P_i = \emptyset$, and a family of relations by:

$$\begin{aligned}
R_2 &= \{m = -m * (12)\},\\
R_3 &= \{(m \circ (I, m)) + (m \circ (I, m)) * (123) + (m \circ (I, m)) * (132) = 0\},
\end{aligned}$$

with all other $R_i = \emptyset$. Then we define $\mathrm{LieOp} = \langle P \mid R\rangle$, and call it the *Lie operad*.

Theorem 5.2.3. *Let $\mathbb{K}$ be a field of any characteristic other than 2, and fix the underlying category for* LieOp *to be* $\mathrm{Vect}_{\mathbb{K}}$*. Then* $\mathrm{Alg}_{\mathrm{LieOp}}$ *is isomorphic to the category of Lie algebras over $\mathbb{K}$, which we denote $\mathcal{L}$. The isomorphism functor is denoted $\varphi\colon \mathrm{Alg}_{\mathrm{LieOp}} \to \mathcal{L}$, with inverse $\sigma\colon \mathcal{L} \to$* LieOp*. These functors are defined as in Theorem 3.3.6 (where the Lie bracket takes the place of the semigroup multiplication), with the following modification to σ: let $(V, [\cdot,\cdot])$ be a Lie algebra. Then:*

$$\sigma(V, [\cdot,\cdot]) = F,$$

where F is the algebra over LieOp *induced by the mapping $m \mapsto [\cdot,\cdot]$ via Corollary 5.1.13.*

Proof. As in the previous proofs of category isomorphisms, we need to verify that φ and σ map objects and arrows into the claimed categories; that they are inverses is clear from the definitions. So let $F\colon \mathrm{LieOp} \to \mathrm{End}_V$ be an algebra over LieOp. Then $\varphi(F) = (V, F(m))$. We will write $F(m) = [\cdot,\cdot]$ for brevity. We have to verify that $[\cdot,\cdot]$ satisfies the axioms of a Lie bracket. As $\mathbb{K}$ is not of characteristic 2, we can use the characterization of Lie brackets that requires $[\cdot,\cdot]$ to be bilinear, anticommutative, and satisfy the Jacobi identity. Since F is an algebra over LieOp, we know that $[\cdot,\cdot] \in \mathrm{End}_V(2)$, so $[\cdot,\cdot]$ is bilinear. Next we check the Jacobi identity. Let $x, y, z \in V$. Then:

$$\begin{aligned}
&[x,[y,z]] + [z,[x,y]] + [y,[z,x]]\\
=&\Big([\cdot,\cdot] \circ (\mathrm{id}_V, [\cdot,\cdot])\Big)(x,y,z) + \Big([\cdot,\cdot] \circ (\mathrm{id}_V, [\cdot,\cdot]) * (123)\Big)(x,y,z)\\
&+ \Big([\cdot,\cdot] \circ (\mathrm{id}_V, [\cdot,\cdot]) * (132)\Big)(x,y,z)\\
=&\Big([\cdot,\cdot] \circ (\mathrm{id}_V, [\cdot,\cdot]) + [\cdot,\cdot] \circ (\mathrm{id}_V, [\cdot,\cdot]) * (123)\\
&+ [\cdot,\cdot] \circ (\mathrm{id}_V, [\cdot,\cdot]) * (132)\Big)(x,y,z)\\
=&\Big(F_2(m) \circ (F_1(I), F_2(m)) + F_2(m) \circ (F_1(I), F_2(m)) * (123)\\
&+ F_2(m) \circ (F_1(I), F_2(m)) * (132)\Big)(x,y,z)\\
=&\Big(F_3(m \circ (I,m)) + F_3(m \circ (I,m)) * (123)\\
&+ F_3(m \circ (I,m)) * (132)\Big)(x,y,z)
\end{aligned}$$

$$=\Big(F_3(Z_3)\Big)(x,y,z)$$ Where Z_3 denotes the zero in LieOp_3, by the definition of the relations in LieOp

$$=\mathbf{0}(x,y,z)$$ Where $\mathbf{0}$ denotes the zero in $\mathrm{End}_V(3)$, since F is linear

$$=0\in V.$$

Thus the Jacobi identity holds. Now we check anticommutativity.

$$\begin{aligned}
[x,y] &= \big(F(m)\big)(x,y) \\
&= \big(F_2(-m*(12))\big)(x,y) && \text{By the definition of the relations in LieOp} \\
&= \big(-F_2(m)*(12)\big)(x,y) && \text{Since } F \text{ is a morphism of operads} \\
&= \big(-[\cdot,\cdot]*(12)\big)(x,y) \\
&= -[y,x].
\end{aligned}$$

So anticommutativity holds. Thus $\varphi(F)$ is indeed a Lie algebra over $\mathbb{K}$.

Next, let $(V,[\cdot,\cdot])$ be a Lie algebra, and set $\sigma(V,[\cdot,\cdot])=F$, as defined in the theorem statement. By definition, F preserves the identity, so it remains to check that it also commutes with composition and the symmetric action. But F is in fact *defined* as commuting with the composition in LieOp, and the symmetric action is respected due to the properties of a Lie bracket. So F is indeed an algebra over LieOp.

That $\sigma(V,[\cdot,\cdot])$ is an algebra over LieOp is guaranteed by Corollary 5.1.13.

Now let $M\colon (V,[\cdot,\cdot])\to(W,[\cdot,\cdot]')$ be a homomorphism between two Lie algebras, and denote $F=\sigma(V,[\cdot,\cdot])$ and $G=\sigma(W,[\cdot,\cdot]')$. Then by definition, M respects the Lie brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]'$. That is, $M\circ[\cdot,\cdot]=[\cdot,\cdot]'\circ M\otimes M$. Additionally, it's clear that we have $M\circ \mathrm{id}_V=\mathrm{id}_W\circ M$. Considering that LieOp is generated by I and m, this precisely says that $\sigma(M)$ is a morphism of LieOp-algebras.

Finally, le $M\colon F\to G$ be a morphism of LieOp-algebras, and write $\varphi(F)=(V,[\cdot,\cdot])$ and $\varphi(G)=(W,[\cdot,\cdot]')$. Then for any $x,y\in V$, we have:

$$\begin{aligned}
M([x,y]) &= M(F_2(m)(x,y)) \\
&= G_2(m)(M(x),M(y)) && \text{Since } M \text{ is a morphism of LieOp-algebras} \\
&= [M(x),M(y)]'.
\end{aligned}$$

So M respects the Lie brackets, and thus $\varphi(M)$ is a homomorphism of Lie algebras. We conclude that φ is an isomorphism of categories. □

Theorem 5.2.4. *Let $\mathbb{K}$ be a field of characteristic different from 2, and fix the underlying category for* LieOp *to be the category of super $\mathbb{K}$-vector spaces. Then $\mathrm{Alg}_{\mathrm{LieOp}}$ is isomorphic to the category of Lie superalgebras over $\mathbb{K}$. The isomorphism functors φ and σ are defined as in Theorem 5.2.3.*

Proof. Repeat the arguments of Theorem 5.2.3, noting that the symmetric action in the endomorphism operad over a super vector space provides the sign terms needed for super skew-symmetry and the super Jacobi identity. □

Remark 5.2.5. Any type of algebra defined using multilinear equations (that is, where each side of the equality is linear in each of its variables) can be directly translated into a corresponding operad as in Example 5.2.2. Some non-multilinear equations can be transformed into multilinear ones for this purpose. For instance, assuming the Jacobi identity in a field of characteristic other than 2, the non-multilinear equation $[x, x] = 0$ for alternativity is equivalent to the multilinear equation $[x, y] = -[y, x]$ for skew symmetry. Sometimes this conversion into multilinear terms results in a much more complicated expression; for example, see Definition 16 on page 9 of [DMR17] for a multilinear version of the Jordan identity, $(xy)(xx) = x(y(xx))$.

Operads can also model the properties of algebraic structures with operations of more than two arguments, as shown by the following example.

Definition 5.2.6 (Lie Triple System)**.** Let $\mathbb{K}$ be a field and V be a vector space over $\mathbb{K}$. Then a *Lie triple system* is a pair $(V, [\cdot,\cdot,\cdot])$, where $[\cdot,\cdot,\cdot]$ is a $\mathbb{K}$-trilinear map satisfying the following properties, for all $u, v, w, x, y \in V$:

(LT1) $[u, v, w] = -[v, u, w]$,
(LT2) $[u, v, w] + [w, u, v] + [v, w, u] = 0$,
(LT3) $[u, v, [w, x, y]] = [[u, v, w]x, y] + [w, [u, v, x], y] + [w, x, [u, v, y]]$.

Example 5.2.7 (Lie Triple Operad)**.** Let m be a formal symbol. Define a family of generators by $P_3 = \{m\}$ with all other $P_i = \emptyset$, and a family of relations by:

$$R_3 = \{m = -m * (12), m + m * (123) + m * (132) = 0\},$$
$$R_5 = \{m \circ (I, I, m) = m \circ (m, I, I) + m \circ (I, m, I) * (123) + m \circ (I, I, m) * (13)(24)\},$$

with all other $R_i = \emptyset$. Then we define $\mathrm{LieTr} = \langle P \mid R\rangle$, and call it the *Lie triple operad*.

Theorem 5.2.8. *Let $\mathbb{K}$ be a field and fix the underlying category for* LieTr *to be* $\mathrm{Vect}_{\mathbb{K}}$*. Then* $\mathrm{Alg}_{\mathrm{LieTr}}$ *is isomorphic to the category of Lie triple systems over* $\mathbb{K}$*. The isomorphism functors are defined as in Theorem 5.2.3.*

Proof. Use analogous arguments to those in the proof of Theorem 5.2.3. □

The previous theorems suggest a natural definition for "Lie triple super systems"; simply take these systems to be the images of algebras over LieTr via the functor φ in the context of the category of super vector spaces. An explicit definition of these systems follows, which agrees with the standard definition in e.g. [ZWCZ14, Definition 2.1]. In general, one can use operads in this way to determine the natural analogues of familiar algebraic structures in different categories.

Definition 5.2.9 (Lie Triple Super System)**.** Let $\mathbb{K}$ be a field and V be a super vector space over $\mathbb{K}$. Then a *Lie triple super system* is a pair $(V, [\cdot,\cdot,\cdot])$, where $[\cdot,\cdot,\cdot]$ is a grade-preserving $\mathbb{K}$-trilinear map satisfying the following properties, for all homogeneous $u, v, w, x, y \in V$:

(ST1) $[u, v, w] = -(-1)^{\overline{u}\,\overline{v}}[v, u, w]$,
(ST2) $[u, v, w] + (-1)^{(\overline{v}+\overline{u})\overline{w}}[w, u, v] + (-1)^{\overline{u}(\overline{v}+\overline{w})}[v, w, u] = 0$,
(ST3) $[u, v, [w, x, y]] = [[u, v, w], x, y]+(-1)^{(\overline{v}+\overline{u})\overline{w}}[w, [u, v, x], y]+(-1)^{(\overline{v}+\overline{u})(\overline{w}+\overline{x})}[w, x, [u, v, y]]$.

Remark 5.2.10. If $\mathcal{C}$ is a monoidal category, $\mathcal{C}^{\mathrm{op}}$ is also monoidal in a canonical way: the tensor functor $\otimes^{\mathrm{op}}$ is defined by $X^{\mathrm{op}} \otimes^{\mathrm{op}} Y^{\mathrm{op}} = Y \otimes X$, the monoidal unit is the same, the associator and unitors are replaced by their inverses, and the left and right unitors switch roles. If $\mathcal{C}$ is symmetric, the opposite category can also be made symmetric by simply taking $(B_{X,Y})^{\mathrm{op}} = B_{Y,X}$.

Definition 5.2.11 (Cosemigroup Object, Comonoid Object)**.** Let $\mathcal{C}$ be a monoidal category. A *cosemigroup object* is a pair (O, Δ), where O is an object in $\mathcal{C}$ and $\Delta\colon O \to O \otimes O$ is a morphism called the comultiplication such that the following diagram commutes:

$$\begin{array}{ccc} O & \xrightarrow{\Delta} & O \otimes O \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_O \otimes \Delta} \\ O \otimes O & \xrightarrow[\Delta \otimes \mathrm{id}_O]{} & O \otimes O \otimes O \end{array} . \tag{18}$$

A *comonoid object* is a triple (O, Δ, ϵ) such that (O, Δ) is a cosemigroup object and $\epsilon\colon O \to u$ is a morphism called the counit that makes the following diagrams commute:

$$\begin{array}{ccc} O & \xrightarrow{\Delta} & O \otimes O \\ & {\scriptstyle\lambda_O^{-1}}\searrow & \downarrow{\scriptstyle \epsilon\otimes \mathrm{id}_O} \\ & & u \otimes O \end{array} \qquad \begin{array}{ccc} O & \xrightarrow{\Delta} & O \otimes O \\ & {\scriptstyle\rho_O^{-1}}\searrow & \downarrow{\scriptstyle \mathrm{id}_O \otimes \epsilon} \\ & & O \end{array} . \tag{19}$$

Alternatively, one can note that the previous three diagrams are exactly the diagrams that define semigroup and monoid objects according to Definitions 2.2.26 and 2.2.27, only with the arrows reversed and the unitors inverted. Thus cosemigroup and comonoid objects can equivalently be defined as semigroup and monoid objects in $\mathcal{C}^{\mathrm{op}}$. This way of thinking provides an immediate definition of co-commutative objects of both types if $\mathcal{C}$ is symmetric; we simply require the following diagram—namely, Diagram (10) with the arrows reversed—to commute:

$$\begin{array}{ccc} O & \xrightarrow{\Delta} & O \otimes O \\ & {\scriptstyle\Delta}\searrow & \downarrow{\scriptstyle B_{O,O}} \\ & & O \otimes O \end{array} . \tag{20}$$

Applying Theorems 4.2.6 through 4.2.9 to various opposite categories gives a characterization of these types of co-objects in terms of algebras over operads. In general, given any type of algebraic structure characterized by an operad in this way, one can obtain a natural definition of the dual (the "co-version") of that structure by looking at the algebras over that operad in the opposite category.

Definition 5.2.12 (Derivation)**.** Let $\mathbb{K}$ be a field and V an associative $\mathbb{K}$-algebra whose multiplication is denoted by juxtaposition. A *derivation* for V is a linear map $D\colon V \to V$ that satisfies the Leibniz rule: $D(vw) = D(v)w + vD(w)$ for all $v, w \in V$.

If W is an associative $\mathbb{K}$-superalgebra, an *even superderivation* for W is a grade-preserving linear map $D\colon W \to W$ that satisfies the same Leibniz rule. An *odd superderivation* for W is a grade-reversing linear map $D\colon W \to W$ that satisfies the following graded Leibniz rule: $D(vw) = D(a)b + (-1)^{|a|}aD(b)$.

Example 5.2.13 (Derivation Operad)**.** Let m and D be formal symbols. Define a family of generators by $P_1 = \{D\}$ and $P_2 = \{m\}$, with all other $P_i = \emptyset$, and a family of relations by:

$$R_2 = \{D \circ (m) = m \circ (D, I) + m \circ (I, D)\}$$
$$R_3 = \{m \circ (m, I) = m \circ (I, m)\},$$

with all other $R_i = \emptyset$. Then we define $\text{Deriv} = \langle P \mid R \rangle$, and call it the *derivation operad.*

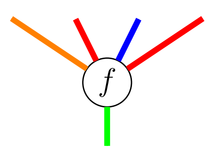
Similarly to the result in Theorem 5.2.3, algebras over Deriv in the category $\text{Vect}_{\mathbb{K}}$ correspond to associative $\mathbb{K}$-algebras equipped with a derivation. The relation in R_3 is the associativity condition, and the relation in R_2 forces the image of D to be a derivation. Likewise, algebras over Deriv in the category of $\mathbb{K}$-super vector spaces correspond to associative $\mathbb{K}$-superalgebras equipped with an even superderivation. Note that a similar result does *not* hold for $\mathbb{K}$-superalgebras equipped with odd superderivations, as by definition, the category of super $\mathbb{K}$-vector spaces that we are working with contains only grade-preserving linear maps.

5.3. **Coloured Operads.**

The techniques developed in this document are insufficient to construct an operad whose algebras correspond to groups, or more generally, to any algebraic structure wherein every element is required to have an inverse. One can view a group (G, m, i) as being a monoid equipped with an inverse map, $-^{-1}\colon G \to G$, such that $m(x, x^{-1}) = i = m(x^{-1}, x)$ holds for all $x \in G$. It would be simple to construct an operad corresponding to the product m, identity i, and a map from G to G. However, the defining property of the inverse map is inherently non-linear. As such, there is no straightforward way to translate the equation into operad theoretic terms. A composition like $m \circ (\text{id}_G, -^{-1})$ gives the two-variable function $(x, y) \mapsto m(x, y^{-1})$, and not the desired one-variable function $x \mapsto m(x, x^{-1})$. The repetition of a variable in a formula cannot be encoded using plain generators and relations.

One generalization of the concept of an operad is that of a *coloured operad.* We will give a brief introduction to coloured operads here; for a more technical definition, see, for example, [nLa18].

Elements of a coloured operad can, as in a usual operad, be thought of as functions with n inputs and one output. Such n-ary functions are drawn similarly to the trees in Section 3.2, but with each edge given a colour. For instance:



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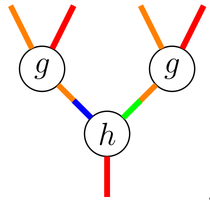
The colours indicate the "type" of variable being input or output along each edge. When working in the category of sets, each colour corresponds to a set. Thus, if we label the sets corresponding to the colours orange, red, blue, and green as O, R, B, and G, respectively, the above diagram represents some function $f\colon O \times R \times B \times R \to G$. The composition in a coloured operad is more restricted than in a non-coloured one—the input and output colours in each composition must match up. To illustrate, consider functions $g\colon O \times R \to O$ and $h\colon B \times G \to R$. The composition $h \circ (g, g)$,



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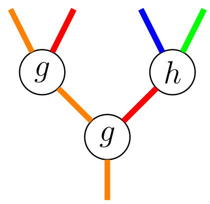
does not make sense. The two instances of g each output a variable from O, but h requires inputs from B and G. Thus the expression $h \circ (g, g)$ is left undefined. On the other hand, the composition $g \circ (g, h)$,



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does make sense, as the domains and codomains of each function (that is, the colours of their associated edges) match up correctly. A coloured operad comes equipped with an identity element for each colour, and may or may not be equipped with a symmetric action. Coloured operads must satisfy associativity, identity, and equivariance axioms analogous to those for non-coloured operads. The definitions for morphisms between coloured operads and other related notions are also analogous to the non-coloured versions. Coloured operads with only one colour are equivalent to ordinary operads.

The archetypal coloured operad is the endomorphism operad over multiple sets. Let $\{X_i\}_{i \in I}$ be a family of sets for some (usually finite) index set I. The endomorphism operad over the X_i consists of all functions whose domain is an iterated cartesian product of the sets X_i, and whose codomain is one of the X_i. There is one colour in this operad for each $i \in I$. The compositions (when defined, as per the note on coloured composition above) and symmetric action in this operad are defined as in Example 3.1.3.

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