HIGHEST WEIGHT CATEGORIES

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ABSTRACT. In this paper, we give a brief introduction to the theory of highest weight categories accessible to undergraduate students.

CONTENTS

INTRODUCTION

The concept of a highest weight category was introduced by Parshall, Scott and Cline [3] as a category theoretic generalization of the BGG category $\mathscr{O}$ of highest weight modules used in the study of the representation theory of Lie algebras. This document will serve as a resource for learning about highest weight categories to students interested in representation theory who have a minimal knowledge of category theory and Lie algebras. The goal of the document is to give the student a compact primer on highest weight categories, focusing on examples other than category $\mathscr{O}$, so that they may be able to read more advanced topics without excessive confusion.

In Section 1 we will cover the prerequisite category theory, including the definition of a category, limits and colimits, projective and injective objects, and subobjects and quotient objects. In Section 2 we will cover rings, associative algebras, and modules over associative algebras. In particular, we will focus on both semisimple associative algebras, and the category of modules over the upper triangular matrices as examples of highest weight categories. Section 3 will cover some fundamental concepts of Lie algebras and modules over Lie algebras, as well as give a brief description of the BGG category $\mathscr{O}$.

Acknowledgements. Thank you to professor Alistair Savage, for his advice and seemingly endless patience. This document was written over the summer of 2016 under the supervision of Professor Alistair Savage, and was funded by the University of Ottawa's work study program and Prof. Savage's NSERC Discovery Grant.

1. Category Theory

1.1. Basics of Categories.

Category theory is used in many fields of mathematics to provide a way of generalizing algebraic concepts by encoding information about different types of algebraic objects in terms of the morphisms, or structure preserving maps, between them. This generalizes the common theme of studying algebraic and analytic structures by looking at the homomorphisms between them.

Definition 1.1.1 (Category)**.** A category $\mathscr{C}$ is a class of objects $\operatorname{Ob}\mathscr{C}$ along with a class $\operatorname{Hom}\mathscr{C}$ of morphisms between objects that satisfy the following axioms:

(i) For every $f\colon A \to B$ and $g\colon B \to C$ in $\mathscr{C}$ there exists another morphism, $g \circ f\colon A \to C$ in $\mathscr{C}$ called the composition of f and g.

(ii) Composition of morphisms is associative, so for any $f\colon A \to B, g\colon B \to C$ and $h\colon C \to D$ in $\mathscr{C}$

$$(h \circ g) \circ f = h \circ (g \circ f).$$

(iii) For any $A \in \mathscr{C}$ there exists a morphism $1_A\colon A \to A$, called the identity morphism of A, such that for every $f\colon A \to B$ and $g\colon B \to A$, we have that:

$$f \circ 1_A = f \text{ and } 1_A \circ g = g.$$

Example 1.1.2. Some common examples of categories are the categories Set of sets with functions, and the categories $\text{Vect}_{\mathbb{K}}$, Grp, Ring of vector spaces over $\mathbb{K}$, groups, and rings respectively, with their usual homomorphisms as morphisms.

Example 1.1.3. If one takes a poset P and interprets the elements of P as objects and say there is a morphism $a \to b$ whenever $a \leq b$, then we get an example of something called a *small category* where the collection of objects forms a set.

Anyone should be familiar with the category Set of sets with functions as morphisms. It is useful to identify functions with additional properties, such as being injective, surjective, bijective etc, which have natural category theoretic generalizations.

Definition 1.1.4 (Types of morphisms)**.** Say $\mathscr{C}$ is a category, and let $f\colon A \to B$, $g\colon A \to B$ be morphisms in $\mathscr{C}$.

- **Monomorphism**: $h\colon B \to C$ is called a monomorphism if it cancels on the left:

$$h \circ g = h \circ f \implies g = f.$$

  We write $h\colon B \hookrightarrow C$ in diagrams.

- **Epimorphism**: $h\colon C \to A$ is called an epimorphism if it cancels on the right:
$$g \circ h = f \circ f \implies g = f.$$
We write $h\colon C \twoheadrightarrow A$ in diagrams.
- **Isomorphism**: $f\colon A \to B$ is called an isomorphism if there is a morphism $g\colon B \to A$ such that:
$$f \circ g = 1_A \text{ and } g \circ f = 1_B.$$
We write $f\colon A \cong B$ in diagrams.
- **Retraction**: $f\colon A \to B$ is called a retraction if there is a morphism $g\colon B \to A$ such that:
$$g \circ f = 1_B.$$
- **Section**: $f\colon A \to B$ is called an isomorphism if there is a morphism $g\colon B \to A$ such that:
$$f \circ g = 1_A.$$

Monomorphisms, or *embeddings*, generalize the idea of embedding an object into a larger object, while epimorphisms and isomorphisms capture the idea of projecting onto a smaller object and of structural identity respectively. Retractions are always monomorphisms and sections are always epimorphisms, they are strictly stronger conditions that are conflated in many concrete categories that are commonly worked with. An important fact that will not be addressed in detail is that being a monomorphism and an epimorphisms does not in general imply being an isomorphism, we call such a morphism a *bimorphism* and call categories where bimorphisms are isomorphisms *balanced*.

Of course, while it is important to study maps between objects in a category, it is equally fundamental to study maps between categories themselves. The concept of maps between categories, called *functors*, is where the power of category theory lies.

Definition 1.1.5 (Functor)**.** We define a *functor* $F\colon \mathscr{C} \to \mathscr{D}$ to be a mapping between categories that:

- Maps each object A in $\mathscr{C}$ to a single object $F(A)$ in $\mathscr{D}$.
- Maps each morphism $f\colon A \to B$ in $\mathscr{C}$ to a single morphism $F(f)\colon F(A) \to F(B)$ in $\mathscr{D}$.
- Preserves composition: $F(f \circ g) = F(f) \circ F(g)$.
- Preserves identity morphisms: $F(1_A) = 1_{F(A)}$.

In addition, we define $F(\mathrm{Hom}(A, B)) = \mathrm{Hom}(F(A), F(B))$.

Example 1.1.6. If $\mathscr{C}$ is a locally small category one can define a functor $F\colon \mathscr{C} \to \mathrm{Set}$, called a *forgetful functor*, which sends objects in $\mathscr{C}$ to their underlying sets and morphisms to their underlying functions, “forgetting” any additional structure.

Example 1.1.7. The functor ${}^*\colon \mathrm{Vect}_{\mathbb{K}} \to \mathrm{Vect}_{\mathbb{K}}$ that sends a vector space V to its dual space $V^* = \mathrm{Hom}(V, \mathbb{K})$, which is given a vector space structure with pointwise addition and scalar multiplication.

Example 1.1.8. For any object A in a locally small category $\mathscr{C}$ we can define a functor $\mathrm{Hom}(A, -)\colon \mathscr{C} \to \mathrm{Set}$ by sending B in $\mathscr{C}$ to $F(B) = \mathrm{Hom}(A, B)$, and sending $f\colon B \to C$ to $F(f)\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ by defining $F(f)(g) = f \circ g$.

Example 1.1.9. Arguably one of the more important examples of a functor is a *diagram* in a category $\mathscr{C}$, which is a functor from a category J, called an index category, into $\mathscr{C}$.

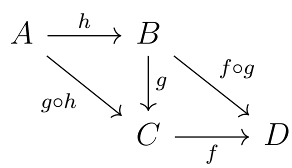
In category theory it is common to write relationships between objects and morphisms with *diagrams* instead of equations. Diagrams concisely represent the relationships between objects and the morphisms between them.

Definition 1.1.10 (Diagrams)**.** Fix a category $\mathscr{C}$; we define a *diagram* of type J in $\mathscr{C}$ to be a functor from J, called the index category, to $\mathscr{C}$: $F\colon J \to \mathscr{C}$. We usually only consider J as a small category, or even more commonly a finite one.

It is usually best to think of the index category J of a diagram as a directed graph in the graph theoretic sense, with its objects as vertices and morphisms as directed edges. A diagram of type J in $\mathscr{C}$ is then just a labelling of the objects and morphisms in J with those in $\mathscr{C}$.

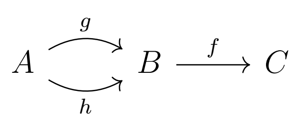
Definition 1.1.11 (Commutative Diagram)**.** Given a diagram F, we say F is a *commutative diagram* if the compositions of morphisms along different paths with the same endpoints are equal.

Example 1.1.12. Earlier we required that composition of morphisms be associative, that $f \circ (g \circ h) = (f \circ g) \circ h$. We can rephrase this in terms of diagrams. In other words, we require that the following diagram commute for all f,g and h:

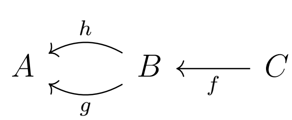


Example 1.1.13. We can rephrase some of our definitions regarding morphisms using diagrams:

We say a morphism $f\colon B \to C$ is a monomorphism if the following diagram commutes whenever $h \circ f = g \circ f$:



We say a morphism $f\colon C \to B$ is an epimorphism if the following diagram commutes whenever $f \circ h = f \circ g$:



Definition 1.1.14 (Terminal, Initial, and Zero Objects)**.** Let $\mathscr{C}$ be a category.

- **Terminal Objects:** An object T in $\mathscr{C}$ is called *terminal* if for any object A, there is exactly one morphism $f\colon A \to T$.
- **Initial Objects:** An object S in $\mathscr{C}$ is called *initial* if for any object A, there is exactly one morphism$f\colon S \to A$.

- **Zero Objects:** An object 0 in $\mathscr{C}$ is called a *zero object* if it is both initial and terminal.

In a category with a zero object we write $0_{AB} \colon A \to B$, or just 0 if it is unambiguous, for the unique morphism $A \to 0 \to B$.

Lemma 1.1.15. *Terminal and Initial objects are unique up to an isomorphism.*

Terminal and Initial objects are unique up to an isomorphism.

(1) Say T_1 and T_2 are terminal objects, then there exists unique morphisms f_1 and f_2 such that the following commutes:

$$1_{T_1} \circlearrowright T_1 \underset{f_1}{\overset{f_2}{\rightleftarrows}} T_2 \circlearrowleft 1_{T_2}$$

Therefore f_1 and f_2 are each other's inverses and $T_1 \cong T_2$.

(2) Say S_1 and S_2 are initial objects, then there exists unique morphisms f_1 and f_2 such that the following commutes:

$$1_{T_1} \circlearrowright S_1 \underset{f_2}{\overset{f_1}{\rightleftarrows}} S_2 \circlearrowleft 1_{T_2}$$

Therefore f_1 and f_2 are each other's inverses and $S_1 \cong S_2$.

■

Corollary 1.1.16. *Zero objects are unique up to isomorphism.*

Example 1.1.17.

(a) The empty set $\emptyset$ is the initial object in the category Set, the singleton set $\{*\}$ is the terminal object.

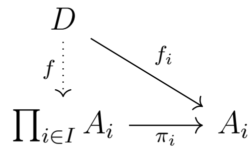
(b) In the category Grp of groups the trivial group 0 is the zero object.

(c) The trivial module 0 is the zero object for the category of R-modules.

In many algebraic structures it is possible to take the *product* of objects, constructions such as the cartesian product of sets, and the various analogous constructions for the algebraic categories. This can be extended to the general construction of products in categories.

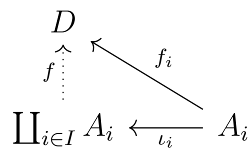
1.2. **Basic Constructions in Categories.**

Definition 1.2.1 (Categorical Product)**.** Let $\{A_i\}_{i\in I}$ be a collection of objects in $\mathscr{C}$ indexed by the set I. Then if it exists we define the *categorial product* to be the unique object $\prod_{i\in I} A_i$ with morphisms $\{\pi_i \colon \prod_{i\in I} A_i \to A_i\}_{i\in I}$ such that for any object D and collection of morphisms $\{f_i \colon D \to A_i\}_{i\in I}$ there exists an $f \colon D \to \prod_{i\in I} A_i$ such that the following diagram commutes:



A concept closely related to the categorical product is the categorial sum, in many categories they are even the same.

Definition 1.2.2 (Categorical Sum (also called the Coproduct))**.** Let $\{A_i\}_{i\in I}$ be a collection of objects in $\mathscr{C}$ indexed by the set I. Then if it exists we define the *categorial sum* to be the unique object $\coprod_{i\in I} A_i$ with morphisms $\{\iota_i\colon A_i \to \coprod_{i\in I} A_i\}_{i\in I}$ such that for any object D and collection of morphisms $\{f_i\colon A_i \to D\}_{i\in I}$ there exists an $f\colon \coprod_{i\in I} A_i \to D$ such that the following diagram commutes:



The importance of the categorial product and sum cannot be understated, they provide a standard way of constructing new objects in a category form collections of old ones.

Example 1.2.3 (Cartesian Product and Direct Sum of Sets)**.** In the category of sets the categorical product corresponds to the cartesian product of sets:

$$A \prod B = A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

And the categorical product corresponds to the direct sum of sets:

$$A \coprod B = A \oplus B = (A \times \{0\}) \cup (B \times \{1\}).$$

Example 1.2.4 (Cartesian Product of Groups)**.** If A, B are groups, the categorical product is the cartesian product of the underlying sets, which along with the operation $(a, b) \cdot_{A\times B} (c, d) = (a \cdot_A c, b \cdot_B d)$ is called the direct product of groups. If A and B are abelian, or more generally for finite collections of abelian groups, the categorical product and sum are isomorphic.

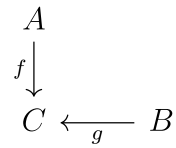
Example 1.2.5. The categorical product of modules is defined in much the same way as the product of groups, with all operations defined componentwise. As in the case with abelian groups, the categorial product and sum are isomorphic for finite collections.

Example 1.2.6. Let P be a poset interpreted as a category, and let $A \subseteq P$.

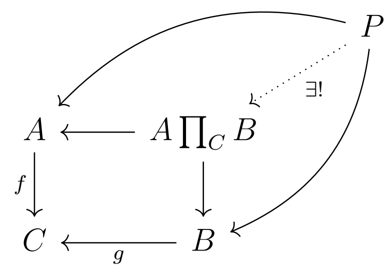
- We call an element $m \in P$ the *meet* of A, written $m = \bigwedge A$, if $m \leq a \quad \forall a \in A$ and if for all $w \in P$ such that $w \leq a \quad \forall a \in A$, we have $m \leq w$.
- We call a $j \in P$ the *join* of A, written $j = \bigvee A$, if $j \geq a \quad \forall a \in A$ and if for all $w \in P$ such that $w \geq a \quad \forall a \in A$, we have $j \geq w$. for $a, b \in P$.

We write the meet and join as $a \wedge b$ and $a \vee b$ respectively. The meet and join as defined above correspond to the categorial product and sum in the poset P interpreted as a category.

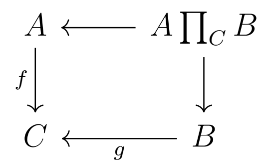
Definition 1.2.7 (Pullbacks)**.** If we have a diagram:



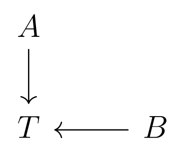
we define the *pullback* of $f\colon A \to C$ and $g\colon B \to C$ to be an object $A \prod_C B$, with morphisms $A \prod_C B \to A$ and $A \prod_C B \to B$ such that for any other object P with morphisms $P \to A$ and $P \to B$ the following diagram commutes:



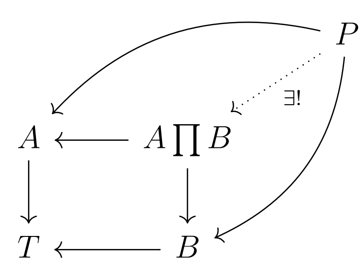
If such an object exists it is unique; we call $A \prod_C B$ the pullback of the diagram and we call the following diagram a pullback square:



Example 1.2.8 (Binary Products)**.** In a category with a terminal object T and pullbacks, the pullback of the following diagram:

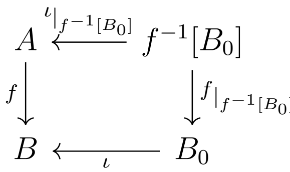


is the product $A \prod B = A \prod_T B$:

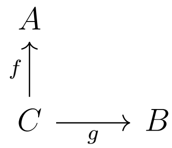


As there is exactly one morphism $X \to T$ for a given X, T can be omitted from the diagram, giving us the *product*.

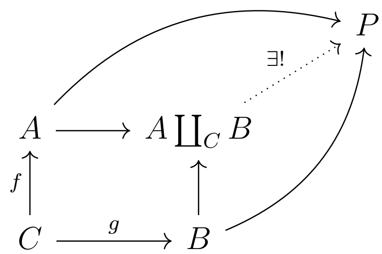
Example 1.2.9 (Preimage)**.** In the category of sets, pick a function $f\colon A \to B$ and $B_0 \subseteq B$ with the inclusion map $\iota\colon B_0 \to B$. Then the pullback of f and ι is the preimage $f^{-1}[B_0]$ of B_0.



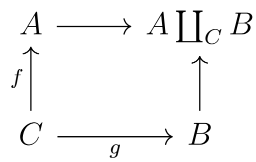
Definition 1.2.10 (Pushouts)**.** If we have a diagram:



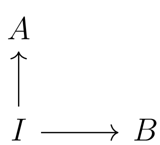
we define the *pushout* of $f\colon C \to A$ and $g\colon C \to B$ to be the object $A \coprod_C B$, with morphisms $A \to A \coprod_C B$ and $B \to A \coprod_C B$ such that for any other object P with morphisms $A \to P$ and $B \to P$ the following diagram commutes:



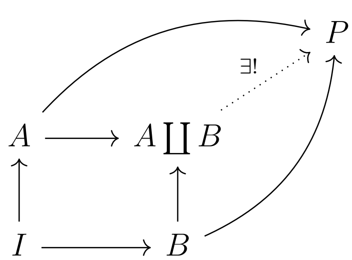
If such an object exists it is unique; we call $A \coprod_C B$ the pushout of the diagram and we call the following diagram a pushout square:



Example 1.2.11 (Coproducts)**.** In a category with an initial object I and pushouts, the pushout of the following diagram:



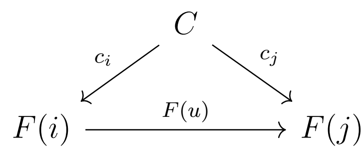
is the coproduct $A \coprod B = A \coprod_I B$:



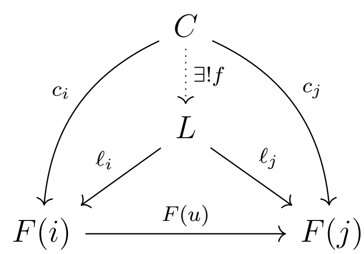
As there is exactly one morphism $I \to X$ for a given X, I can be omitted from the diagram, giving us the *coproduct*.

1.3. More Categorial Constructions.

Definition 1.3.1 (Limits)**.** Let $F\colon J \to \mathscr{C}$ be a diagram in $\mathscr{C}$, we define a *cone* on F as an object C along with a family of morphisms $(c_j\colon C \to F(j))_{j\in J}$ indexed by objects in J, written $(C, c_j)_{j\in J}$, such that the following diagram commutes for any $u\colon i \to j$ in J:



We then define the *limit* of the diagram F to be a cone $(L, \ell_j)_{j\in J}$ such that for any other cone $(C, c_j)_{j\in J}$ there is a unique morphism $f\colon C \to L$ such that the following diagram commutes:



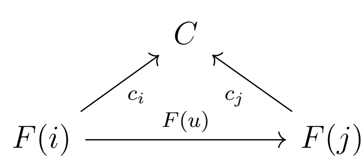
If such an L exists it is unique and we write $L = \lim F$.

Example 1.3.2 (Products)**.** Let J be a discrete category; which is one with only objects and identity morphisms. Then for a functor $F\colon J \to \mathscr{C}$, if the limit exists it is isomorphic to $\prod_{j\in J} F(j)$.

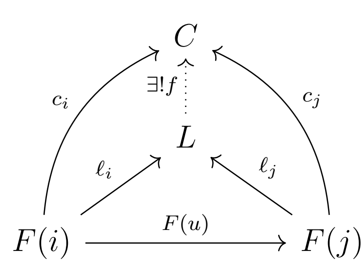
Example 1.3.3.

Limits are one of the unifying concepts in category theory, pullbacks; arbitrary products and terminal objects are all examples of limits of a diagram.

Definition 1.3.4 (Colimits)**.** Let $F\colon J \to \mathscr{C}$ be a diagram in $\mathscr{C}$, we define a *cocone* on F as an object C along with a family of morphisms $(c_j\colon F(j) \to C)_{j\in J}$ indexed by objects in J, written $(C, c_j)_{j\in J}$, such that the following diagram commutes for any $u\colon i \to j$ in J:



We then define the *colimit* of the diagram F to be a cocone $(L, \ell_j)_{j\in J}$ such that for any other cocone $(C, c_j)_{j\in J}$ there is a unique morphism $f\colon L \to C$ such that the following diagram commutes:



If such an L exists it is unique and we write $L = \operatorname{colim} F$.

Example 1.3.5 (Coproducts)**.** Let J be a discrete category; which is one with only objects and identity morphisms. Then for a functor $F\colon J \to \mathscr{C}$, if the colimit exists it is isomorphic to $\coprod_{j\in J} F(j)$.

Definition 1.3.6 (Filtered Category)**.** We call a category J *filtered* if for any objects i,j in J:

- there are morphisms $i \to k$ and $j \to k$ for some third object k.
- for every pair of parallel morphisms $u, v\colon i \to j$ there is a morphism $w\colon j \to k$ for some k such that $w \circ u = w \circ v$.

If we have a diagram $F\colon J \to \mathscr{C}$ whose domain is filtered, we call F a filtered diagram.

Definition 1.3.7 (Cofiltered Category)**.** We call a category J *cofiltered* if for any objects i,j in J:

- there are morphisms $k \to i$ and $k \to j$ for some third object k.
- for every pair of parallel morphisms $u, v\colon j \to i$ there is a morphism $w\colon k \to j$ for some k such that $u \circ w = v \circ w$.

If we have a diagram $F\colon J \to \mathscr{C}$ whose domain is cofiltered, we call F a cofiltered diagram.

Definition 1.3.8 (Direct Limit (Filtered Colimit))**.** Let J be a small, filtered category; a *direct limit* is the colimit of a functor $F\colon J \to \mathscr{C}$. If the limit exists we write $\varinjlim F$.

Example 1.3.9.

Definition 1.3.10 (Inverse Limit (Cofiltered limit))**.** Let J be a small, cofiltered category; an *inverse limit* is the limit of a functor $F\colon J \to \mathscr{C}$. If the limit exists we write $\varprojlim F$.

Example 1.3.11 (p-adic integers)**.** Let $p \in \mathbb{Z}$ be a prime. For $n \geq m$ Define

$$f_{n,m}\colon \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$$

by

$$r + n\mathbb{Z} \quad \mapsto \quad r + m\mathbb{Z}.$$

Then the following diagram is cofiltered:

$$\mathbb{Z}/p\mathbb{Z} \xleftarrow{f_{2,1}} \mathbb{Z}/p^2\mathbb{Z} \xleftarrow{f_{3,2}} \mathbb{Z}/p^3\mathbb{Z} \xleftarrow{f_{4,3}} \cdots$$

and the diagram above has an inverse limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$, called the *p-adic integers*.

Example 1.3.12 (Ring of Formal Power Series over a Ring R)**.** Let R be any commutative ring, and let $R[x]$ be the ring of polynomials with coefficients in R. Define, for $n \geq m$:

$$f_{n,m}\colon R[t]/t^nR[t] \to R[t]/t^mR[t]$$

by

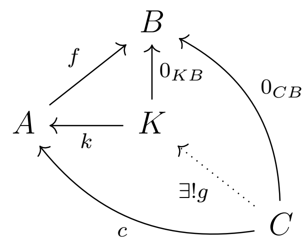
$$p + t^nR[t]/t^nR[t] \quad \mapsto \quad p + t^mR[t]/t^mR[t]$$

Then the following diagram is cofiltered:

$$R[t]/t^mR[t] \xleftarrow{f_{2,1}} R[t]/t^mR[t] \xleftarrow{f_{3,2}} R[t]/t^mR[t] \xleftarrow{f_{4,3}} \cdots$$

And the diagram above has an inverse limit $\varprojlim R[t]/t^nR[t] = R[[t]]$, called the *ring of formal power series* on R.

Definition 1.3.13 (Kernels)**.** Let $f\colon A \to B$ be a morphism in a category $\mathscr{C}$ with a zero object. We define a *kernel* of f to be the map $k\colon K \to A$ such that $f \circ k = 0_{KB}$; and for any other morphism $c\colon C \to A$ where $f \circ c = 0_{CB}$, the following diagram commutes:



If such a k exists it is unique up to isomorphism and we write $k = \ker f$.

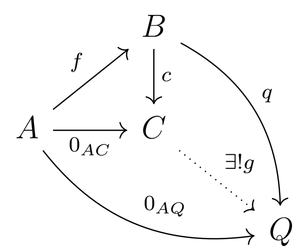
For the kernel $\ker f\colon K \to A$ of f, the kernel as used outside of category theory corresponds to the domain K.

Example 1.3.14. In the category of groups we define the kernel of a homomorphism $f\colon G \to H$ as:

$$\ker f = f^{-1}(0_H) = \{g \in G \mid f(g) = 0_H\}.$$

Them the inclusion map $i\colon \ker f \to G$ is the category theoretic kernel.

Definition 1.3.15 (Cokernels)**.** Let $f\colon A \to B$ be a morphism in a category $\mathscr{C}$ with a zero object. We define a *cokernel* of f to be the map $c\colon B \to C$ such that $c \circ f = 0_{AC}$; and for any other morphism $q\colon A \to Q$ where $f \circ q = 0_{CB}$, the following diagram commutes:



If such a c exists it is unique up to isomorphism and we write $c = \operatorname{coker} f$.

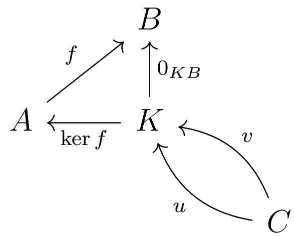
Example 1.3.16. In the category of groups the cokernel of a homomorphism $f\colon G \to H$ is the quotient group $\operatorname{coker} f = H/\operatorname{Im} f$; strictly speaking, the category theoretic cokernel is the quotient map $g \in G \mapsto g \operatorname{Im} f \in H/\operatorname{Im} f$.

Lemma 1.3.17. *Let $f\colon A \to B$ be a morphism, and assume kernels and cokernels exist:*

(a) The cokernel of f is an epimorphism.
(b) If f is a monomorphism then $\ker f = 0 \to A$.
(c) The kernel of f is a monomorphism.
(d) If f is an epimorphism then $\operatorname{coker} f = A \to 0$.

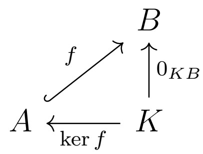
This is a commonly known fact and is proved as an example.

Proof. (a) Let $f\colon A \to B$ be a morphism, and let u, v be morphisms such that $\ker f \circ u = \ker f \circ v$, then the following commutes:



Then we see, if we set $c = k \circ u = k \circ v$, that $f \circ c = f \circ k \circ u = f \circ k \circ v = 0_{CA}$. Therefore there exists a unique morphism $g\colon C \to K$, $u = v = g$, and k is a monomorphism.

(b) Let $f\colon A \to B$ be a monomorphisms, then the following diagram commutes:



Hence $f \circ k = 0_{KB} = f \circ 0_{KA}$ implies $k = 0_{KA}$ by monotonicity, and you can then show that $K \cong 0$ by invoking the universal property of kernels.

■

The proofs of (c) and (d) are dual to those of (a) and (b), i.e. the proofs are obtained by swapping the directions of all morphisms involved.

Definition 1.3.18 (Abelian Category)**.** We say a category is *abelian* if:

- It has a zero object.
- It has all binary products and coproducts.
- It has all kernels and cokernels.
- All monomorphisms are the kernel of a morphism, and all epimorphisms are a cokernel of a morphism (in other words monomorphism and epimorphisms are *normal*).

Alternatively, an abelian category is a category whose hom-sets are abelian groups, has kernels and cokernels, has finite products and coproducts and whose monomorphisms and epimorphisms are all normal.

Example 1.3.19. The canonical example of an abelian category is the category Ab of abelian groups.

Lemma 1.3.20 (See [7, Theorem 2.11])**.** *For any monomorphism f and epimorphism g in an abelian category, $f = \ker \operatorname{coker} f$ and $g = \operatorname{coker} \ker g$.*

When studying an object A in some concrete category, Grp for example, it is often useful to consider the subobjects of A, a subobject B being a subset of A that is a member of the category. Of course, in general one cannot take the sub*set* of an object in a category that is not concrete, so instead one considers the *embeddings* $e\colon B \to A$ into an object A, the monomorphisms with codomain A, and define these to be our subobjects.

Definition 1.3.21 (Subobject)**.** Let $\mathscr{C}$ be an abelian category, and let A be an object in $\mathscr{C}$. Then we define a *subobject* of A to be an equivalence class of monomorphisms $\{u\colon S \to A\}$ under the equivalence relation $(u\colon S \to A) \equiv (v\colon T \to A)$ if there exists an isomorphism $w\colon S \to T$ such that $u = v \circ w$. We denote the collection of subobjects of A as $\operatorname{sub} A$. Often we abuse notation and write $B \subseteq A$ for the subobject $\{B \to A\}$.

To the reader unfamiliar with category theory this definition of subobject may seem strange, it does not always coincide with the usual notion of subobject, instead only considering them unique up to an isomorphism. However, In concrete categories, categories whose objects are sets with additional structure, one can use the usual definition of a subobject S of A as a subset of A that is also a member of the category.

Example 1.3.22. In the category Set of sets, subobjects are subsets, or more strictly, the equivalence classes of subsets of the same cardinality.

Example 1.3.23. In the category Grp of groups, the subobjects are subgroups.

Definition 1.3.24 (Quotient object)**.** Let $\mathscr{C}$ be an abelian category, and let A be an object in $\mathscr{C}$. Then we define a *quotient object* of A to be an equivalence class of epimorphisms $\{u\colon A \to S\}$ under the equivalence relation $(u\colon A \to S) \equiv (v\colon A \to T)$ if there exists an isomorphism $w\colon S \to T$ such that $v = w \circ u$. We denote the collection of quotient objects of A as $\operatorname{quot} A$.

Definition 1.3.25 (Subobject Lattice)**.** Given an object A in an abelian category, one can define a partial order on the collection of subobjects of A $\operatorname{sub} A$. For $u, v \in \operatorname{sub} A$ define $u \leq v$ if there exists a morphism w such that $u = v \circ w$. Then $\leq$ defines a poset structure on $\operatorname{sub} A$, which we will call the *subobject lattice* of A.

Lemma 1.3.26 (Subobject Lattices have all their Meets and Joins [8, p.7])**.**

Since the subobject lattice has all its meets and joins, it is a lattice; this is important because we can take arbitrary meets and joins of subobjects.

Definition 1.3.27 (Union and Intersection of subobjects)**.** Let $\{A_i\}_{i\in I}$ be a collection of subobjects of A, then we define the union and intersection of $\{A_i\}_{i\in I}$ to be their join and meet in $\operatorname{sub} A$:

$$\bigcup_{i\in I} A_i = \bigvee_{i\in I} A_i,$$
$$\bigcap_{i\in I} A_i = \bigwedge_{i\in I} A_i.$$

Example 1.3.28. In the category of sets the subobject lattice $\operatorname{sub} A$ of A is the usual powerset of A, with unions and intersections of subsets being the usual set theoretic union and intersection.

Example 1.3.29. In the category of groups, the intersection of two subgroups is the set theoretic intersection of groups; and the union of $A, B \subseteq G$ is the group generated by the set theoretic union of A and B, $\langle A \cup B\rangle$.

Example 1.3.30. In the category of R-modules, the intersection of $W, V \subseteq M$ is the set theoretic intersection; and the union of W and V is their sum:

$$W + V = \{w + v \mid w \in W, \quad v \in V\}.$$

Definition 1.3.31 (Simple Objects)**.** Let $\mathscr{C}$ be a category that has a zero object 0, and A be an object in that category that is not 0. We call A a simple object if its only subobjects are 0 or A up to an isomorphism. Additionally, we call a subobject simple if its domain is simple, and a quotient object simple if its codomain is.

The complexity of simple objects can vary wildly depending on the category in question. In the category of vector spaces the simple objects are exactly the one dimensional spaces, while the classification of finite simple groups, the simple objects in the category of finite groups, is a major achievement in the theory of finite groups whose proof spans several hundred individual research papers [2].

Example 1.3.32. In the category FinAb of finite abelian groups, the simple objects are exactly the cyclic groups of prime order, $\mathbb{Z}/p\mathbb{Z}$ for some prime p.

Example 1.3.33. In the category $\mathrm{Vect}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$, the simple objects are the one dimensional spaces.

Definition 1.3.34 (Composition Series)**.** For an object A in a category $\mathscr{C}$ with a zero object, a composition series of A is a finite sequence of subobjects:

$$A = A_n \supsetneq \cdots \supsetneq A_1 = 0.$$

such that $\operatorname{coker}(A_i \hookrightarrow A_{i+1}) = A_{i+1}/A_i$, called a *composition factor*, is simple. If an object A has a composition series we say it is of *finite length*.

Definition 1.3.35 (Multiplicity)**.** If S is a simple object and A is of finite length, we define the *composition multiplicity*, or simply *multiplicity*, $[A : S]$ to be the number of composition factors of B that are isomorphic to S. If A is not of finite length we define $[A : S]$ to be the supremum of the multiplicities of all the finite length subobjects of A.

Definition 1.3.36 ((Increasing) Filtration)**.** Let $\mathscr{C}$ be a category with arbitrary unions of subobjects and A be in $\mathscr{C}$. Then a *filtration* of A is a finite or infinite series of subobjects of A, $F_0 \subseteq F_1 \subseteq \dots$.

Definition 1.3.37 (Locally Artinian)**.** We say a category $\mathscr{C}$ is *locally artinian* if it has arbitrary direct limits of subobjects, and every object is a union of its subobjects of finite length.

Definition 1.3.38 (Grothendieck's Condition [8, axiom AB5])**.** we say X satisfies Grothendieck's condition if, for any subobject B and family of subobjects $\{A_\alpha\}$ of X:

$$B \cap (\bigcup A_\alpha) = \bigcup (B \cap A_\alpha).$$

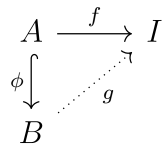
We say a category satisfies Grothendieck's condition if all its objects do.

Example 1.3.39. The category of abelian groups satisfies Grothendieck's condition.

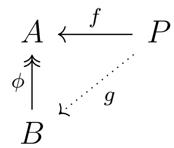
Many times when talking about morphisms between objects it is important to consider how, and when, one can extend the domain or codomain of a morphism to an object that is intuitively "larger". The notion of *injective* and *projective* objects are meant to generalize this to arbitrary morphisms.

Definition 1.3.40. For a given category $\mathscr{C}$:

- **Injective Object**: We call $I \in \mathscr{C}$ an *injective object* if:
  For every monomorphism $\phi\colon A \to B$, and morphism $f\colon A \to I$ there exists $g\colon B \to I$ such that $g \circ \phi = f$, shown in the following commutative diagram:



- **Projective Object**: We call $P \in \mathscr{C}$ a *projective object* if:
  For every epimorphism $\phi\colon B \to A$, and morphism $f\colon P \to A$ there exists $g\colon P \to B$ such that $\phi \circ g = f$, shown in the following commutative diagram:



Let's go through a few examples of injective and projective objects.

Example 1.3.41. In the category $\mathrm{Vect}_{\mathbb{K}}$ of vector spaces over the field $\mathbb{K}$, all objects are injective. Let A, B, V be vector spaces, $f\colon A \to I$ be linear and $\phi\colon A \to B$ be injective and linear.

Proof. Since ϕ is injective, we can restrict its codomain to get the isomorphism $\theta\colon A \to \operatorname{Im}\phi$. Define: $g^*\colon \operatorname{Im}\phi \to V$ by $g^* = f \circ \theta^{-1}$, then the space B can be written as a direct sum of $\operatorname{Im}\phi$ and some other space B', $B = \operatorname{Im}\phi \oplus B'$. Any $b \in B$ can be uniquely written as $b = p + b'$ where $p \in \operatorname{Im}\phi$ and $b' \in B'$. Hence we define $g\colon B \to V$ by $g(x) = g(p+b') = g^*(p)$, and we then see that:

$$g \circ \phi = g^* \circ \phi = (f \circ \theta^{-1}) \circ \phi = f \circ (\theta^{-1} \circ \phi) = f.$$

All vector spaces are projective objects as well (prove this).

■

Example 1.3.42. The objects in the category *Set* of sets are all both injective and projective.

Proof.
(a) All objects in Set are injective. This proof is very similar to the proof for $\mathrm{Vect}_{\mathbb{K}}$.
(b) All objects in Set are projective.
Let $f\colon P \to A$ be any function and let $\phi\colon B \to A$ be any surjection. Define the function $g^*\colon A \to \mathcal{P}(B)$ into the power set of B by $g^*(x) = \phi^{-1}(f(x))$. Then by the axiom of choice there exists a $g_*\colon \operatorname{Im}\theta \to B$ such that $g_*(b) \in B\ \forall b \in \operatorname{Im}\theta$. If we define $g = g_* \circ g^*\colon P \to B$ we can see that $\phi \circ g = f$ as $g(x) \in \phi^{-1}(f(x))$.

■

Definition 1.3.43 (Essential Monomorphism)**.** We call a monomorphism $f\colon A \to B$ *essential* if, for any $g\colon B \to C$, $g \circ f$ is a monomorphism only if g is.

Essential monomorphisms formalize the idea of a smallest injective extension, a critical notion when one needs to work with injective objects.

Definition 1.3.44 (Injective Hull)**.** Let $f\colon A \to H$ be an essential monomorphism, with H an injective object. Then H is unique up to isomorphism, and we call it the *injective hull* of A.

Definition 1.3.45 (Enough injectives)**.** We say a category $\mathscr{C}$ has *enough injectives* if for every object $A \in \mathscr{C}$ there exists a monomorphism $f\colon A \to I$ into an injective object I.

A nice example of a category with enough injectives is once again $Vect_{\mathbb{K}}$.

Lemma 1.3.46. *Any category $\mathscr{C}$ of objects that are all injective, has enough injectives.*

Proof. Let A be any object in $\mathscr{C}$. Then the identity map $1_A\colon A \to A$ is a monomorphism. Therefore $\mathscr{C}$ has enough injectives. ∎

Corollary 1.3.47. $\mathrm{Vect}_{\mathbb{K}}$ *has enough injectives.*

Corollary 1.3.48. Set *has enough injectives.*

Example 1.3.49 (See [11, Lemma 3.29])**.** For any ring R, the category R-mod of modules over R has enough injectives.

There are also course many examples of categories that have enough injectives, but have non-injective objects. For example: the category of abelian groups has enough injectives, as it can be interpreted as the category of $\mathbb{Z}$ modules. but the group $\mathbb{Z}/n\mathbb{Z}$ fails to be injective.

Example 1.3.50. The abelian group $\mathbb{Z}/n\mathbb{Z}$ is not injective.

Proof. Let $f_m\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/nm\mathbb{Z}$ be the injection $f_m(k) = mk$. Say there is a $g\colon \mathbb{Z}/nm\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ such that the following commutes:

$$\begin{array}{ccc} \mathbb{Z}/n\mathbb{Z} & \xrightarrow{1_{\mathbb{Z}/n\mathbb{Z}}} & \mathbb{Z}/n\mathbb{Z} \\ {\scriptstyle f_m}\big\downarrow & \nearrow{\scriptstyle g} & \\ \mathbb{Z}/mn\mathbb{Z} & & \end{array}$$

Then for any $k \in \mathbb{Z}/n\mathbb{Z}$, $g(f_m(k)) = g(mk) = mg(k) = k$. In particular if $k = 1$ we get $mg(1) = 1$, which implies that $g(1) = m^{-1} \in \mathbb{Z}/mn\mathbb{Z}$. But m has no multiplicative inverse in $\mathbb{Z}/mn\mathbb{Z}$. Therefore such a g cannot exist. ∎

Definition 1.3.51 ($\mathbb{K}$-linear categories)**.** We say a category $\mathscr{C}$ is $\mathbb{K}$-linear for a field $\mathbb{K}$ if: For every $A, B \in \mathscr{C}$, $\mathrm{Hom}(A, B)$ has the structure of a vector space over $\mathbb{K}$; and if composition of morphisms $\circ\colon \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ is a bilinear mapping.

A standard example of a $\mathbb{K}$-linear category is the category $\mathrm{Vect}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$. Other examples include the category of representations of a fixed Lie algebra L, and the category of topological vector spaces with continuous maps as morphisms.

Definition 1.3.52 (Highest Weight Category [3])**.** Let $\mathscr{C}$ be a locally Artinian, Abelian, $\mathbb{K}$-linear category with enough injectives that satisfies Grothendieck's condition. Then we call $\mathscr{C}$ a *highest weight category* if there exists a locally finite poset Λ, whose elements are called *weights*, such that:

- Λ exhaustively indexes a set of non isomorphic simple objects $\{S(\lambda)\}$.
- Λ also indexes a collection $A(\lambda)$ of objects in $\mathscr{C}$ such that for each λ there is an embedding $S(\lambda) \to A(\lambda)$, and such that all composition factors $S(\mu)$ of $A(\lambda)/S(\lambda)$ satisfy $\mu < \lambda$.

- For each $\mu, \lambda \in \Lambda$ we have $\dim \mathrm{Hom}(A(\mu), A(\lambda))$ and $[A(\lambda) : S(\mu)]$ are finite.
- Every simple object $S(\lambda)$ has an injective hull $I(\lambda)$ equipped with an increasing filtration $0 = F_0(\lambda) \subseteq F_1(\lambda) \subseteq \dots$ such that:
  (H1) $F_1(\lambda) \cong A(\lambda)$
  (H2) For $n > 1$ $F_n(\lambda)/F_{n-1}(\lambda) \cong A(\mu)$ for some $\mu = \mu(n) > \lambda$.
  (H3) For a fixed $\mu \in \Lambda$, $\mu = \mu(n)$ for finitely many n.
  (H4) $\bigcup F_i(\lambda) = I(\lambda)$.

Remark 1.3.53. The objects $A(\lambda)$ are referred to as the *standard objects* in a highest weight category. It is often useful to dualize the definition of standard objects and consider *costandard* objects. Objects that are both standard and costandard are called *tilting* objects [10].

2. Associative Algebras

2.1. Algebras, and Modules over Rings.

Definition 2.1.1 (Algebra)**.** An algebra consists of a vector space A along with a bilinear map $\cdot \colon A \times A \to A$. Usually one uses multiplicative notation, $x \cdot y = xy$, for the map $\cdot$ unless otherwise specified.

Definition 2.1.2. A map $f \colon A \to B$ between two algebras is called an algebra homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in A$.

The definition above is more general than what we will cover, here we will restrict ourselves to associative algebras.

Definition 2.1.3 (Associative Algebra)**.** We call an algebra A an associative algebra if:

$$x(yz) = (xy)z$$

Definition 2.1.4 (Unital Algebra)**.** We call an associative algebra A *unital* if it has an element 1_A, called the identity, such that for all $x \in A$:

$$1_A x = x 1_A = x.$$

When considering homomorphisms between unital algebras one usually requires that the identity be preserved, $f \colon A \to B$ satisfies $f(1_A) = 1_B$.

It is usual to assume that all associative algebras are unital unless otherwise stated. One can note that any unital algebra is a ring, so all theorems and constructions involving rings apply. Most notably, it makes sense to talk about modules over unital algebras.

Example 2.1.5. Let V be a vector space over a field $\mathbb{K}$. Define the general linear group $\mathfrak{gl}(V) = \mathrm{Hom}(V, V)$ to be the group of linear maps from V to V. Then $\mathfrak{gl}(V)$ is an unital algebra with multiplication as composition of maps.

Definition 2.1.6 (Ideals)**.** Let A be any algebra and $I \subset A$ be a subalgebra.
We call I a *left ideal* if for any $x \in A$

$$xI = \{xi \mid i \in I\} \subseteq I.$$

We call I a *right ideal* if for any $x \in A$

$$Ix = \{ix \mid i \in I\} \subseteq I.$$

We call I an *ideal* if it is both a right and a left ideal, often we also call it a two sided ideal for clarity.

Definition 2.1.7 (Maximal Ideal)**.** Let A be a $\mathbb{K}$-algebra. The ideals of A can be given a partial order via inclusion, we define a *maximal ideal* of A to be a proper (not equal to A) ideal that is not contained in any other ideal.

Definition 2.1.8 (Module over an Associative Algebra)**.** Let A be an associative $\mathbb{K}$-algebra. We define a (left) A-module to be an abelian group M with a map $A \times M \to M$, $(x, m) \mapsto x \cdot m$, such that for any $m, n \in M$ and $x, y \in A$:

(M1) $x \cdot (m + n) = x \cdot m + x \cdot n$
(M2) $(x + y) \cdot m = x \cdot m + y \cdot m$
(M3) $(xy) \cdot m = x \cdot (y \cdot m)$
(M4) $1_A \cdot m = m$

We can define a right A-module by taking the map to be $\cdot \colon M \times A \to M$ so that multiplication acts on the right, with the axioms defined analogously.

The above definition also works if we replace the algebra with a ring R. In fact, associative $\mathbb{K}$-algebras are always rings with an additional vector space structure, and it is often useful to treat them as such.

Definition 2.1.9 (Upper Triangular Matrices)**.** We define the $n \times n$ *upper triangular matrices* over the field $\mathbb{K}$, $U_n(\mathbb{K})$, to be the subalgebra of the $n \times n$ matrices over $\mathbb{K}$, $\mathrm{Mat}_n(\mathbb{K})$, such that for any matrix $(a_{ij}) \in U_n(\mathbb{K})$, $j < i$ implies that $a_{ij} = 0$.

Now we will attempt to provide an explicit classification of Mod-$U_n(\mathbb{K})$, the category of right modules over the algebra $U_n(\mathbb{K})$, by classifying the simple objects in Mod-$U_n(\mathbb{K})$, the projective covers and the injective hulls of these simple objects, and the ideals of the algebra $U_n(\mathbb{K})$.

Definition 2.1.10 (Jacobson Radical)**.** Let R be a ring. Define the *Jacobson radical* of R, $J(R)$, to be the intersection of all maximal left ideals of R.

Example 2.1.11. The Jacobson radical of $U_n(\mathbb{K})$ is $J(U_n(\mathbb{K})) = \bigcap_{i=1}^{n} M_{ii} = \{A \in U_n(\mathbb{K}) \mid a_{ii} = 0, i = 1 \dots n\}$, the ideal of strictly upper triangular matrices.

2.2. **Rings and Modules.**

Lemma 2.2.1. *The Jacobson radical $J(R)$ is the set of $x \in R$ such that $1 - rx$ has a left inverse for all $r \in R$.*

Proof. Suppose x is not in some maximal ideal M. Then $M + Rx = R$, in particular there must be an $r \in R$ such that $1 - rx \in M$. Hence $1 - rx$ cannot have a left inverse.

Now suppose we have an $x, r \in R$ such that $1 - rx$ has no left inverse. then $1 - rx$ must be contained in some maximal ideal M, which cannot contain x. ■

Definition 2.2.2. We say a left ideal N is *nil* if each of its elements is nilpotent.

Lemma 2.2.3. *Let R be a ring, and let N be a left nil ideal of R. Then $N \subseteq J(R)$.*

Proof. Let $x \in N$. Then for $r \in R$, $rx \in N$ is nilpotent as well. Therefore $1 - rx$ is a unit. ■

Corollary 2.2.4. *Let R be a ring, N be a left nil ideal of R and M be a maximal left ideal of R. Then $N \subseteq M$.*

Lemma 2.2.5. *The Jacobson radical $J(R)$ annihilates any simple R-module.*

Proof. Let S be a simple left module and let $s \in S$ be nonzero. Then we can define $q\colon R \to S$ by $r \mapsto r \cdot s$. Then $\operatorname{Im} q$ is not zero, hence f is surjective by the simplicity of S. We know that $\ker q$ is a left ideal, S has no nonzero subobjects, and q is surjective. Therefore $\ker q$ must not be contained in any other proper left ideal, and it is maximal. Therefore $J(R) \subseteq \ker q$. ■

Theorem 2.2.6. *Let R be a ring. Then the simple right R-modules are naturally identified with the simple $R/J(R)$-modules.*

Proof. Let S be a simple left R-module, and let $s \neq 0$ be in S. Then we define multiplication by $r + J(R)$ in $R/J(R)$ by $(r + J(R), s) \mapsto r \cdot s$. If we define $f(r + J(R)) = r \cdot s$ then we see that $\operatorname{Im} f$ is S and S is a simple $R/J(R)$-module. If P is a simple $R/J(R)$-module and $p \neq 0$ is in P then we define multiplication by r in R by $(r, p) \mapsto (r + J(R) \cdot p)$. ■

Lemma 2.2.7. *A left ideal M of a ring R is maximal if and only if the quotient R/M is a simple R-module.*

Proof. Let M be a maximal ideal. Then there are no left ideals I such that $M \subsetneq I \subsetneq R$; hence by the fourth isomorphism theorem there are no proper, nontrivial submodules of R/M. Therefore R/M is simple

Now say R/M is simple. There are no proper, nontrivial submodules of R/M, hence there cannot be any ideal I such that $M \subsetneq I \subsetneq R$. Therefore M is maximal. ■

Theorem 2.2.8. *Let R be a ring. The isomorphism classes of simple left R-modules are in one-to-one correspondence with the maximal left ideals of R.*

Proof. Let $\mathscr{M}$ be the set of maximal left ideals of R and let $\mathscr{S}$ be the collection of isomorphism classes of simple left R-modules. Define the map $f\colon \mathscr{M} \to \mathscr{S}$ by $M \mapsto [R/M]$, where M is a maximal ideal of R and $[R/M]$ is the equivalence class of R/M. We can see that $f(M) = R/M$ is simple by Lemma 2.2.7.

Let M and I be maximal left ideals of R such that $R/M \cong R/I$, and let $\phi\colon R/M \to R/I$ be an isomorphism. Let $r \in I$. Then $\phi(r + M) = \phi(r \cdot (1 + M)) = r \cdot \phi(1 + M) = 0 + I$ as elements of I annihilate elements of R/I. Hence $r + M \in \ker \phi = \{0 + M\}$. Therefore by injectivity of ϕ we can see that $r + M = M$ and hence that $r \in M$. The proof for inclusion in the other direction is identical, therefore $M = I$ and f is injective.

Now if S is a simple left module, pick an $s \in S - \{0\}$ and define $\hat{s}\colon R \to S$ by $\hat{s}(r) = r \cdot s$. By the simplicity of S, $\hat{s}$ is surjective. Since $R/\ker \hat{s} \cong S$ by the first isomorphism theorem, and $\ker \hat{s}$ is maximal by Lemma 2.2.7; the map f is surjective.

■

Example 2.2.9. By Theorem 2.5.2 we know the maximal ideals of $U_n(\mathbb{K})$ are I_i for $1 \leq i \leq n$. Therefore by Theorem 2.2.7 the simple $U_n(\mathbb{K})$-modules are exactly the quotient modules $U_n(\mathbb{K})/I_i$ for $1 \leq i \leq n$.

In particular, if we define $\mathbb{K}_i$ to be $\mathbb{K}$ with multiplication by $U_n(\mathbb{K})$ given by $A \cdot k = a_{ii}k$ for $A = (a_{ij}) \in U_n(\mathbb{K})$ and $k \in \mathbb{K}_i$; then $U_n(\mathbb{K})/I_i \cong \mathbb{K}_i$.

Definition 2.2.10. We call a left R-module F *free* if it is isomorphic to a direct sum of copies of R, i.e. $F \cong \bigoplus_{i\in I} R$, for some indexing set I.

Lemma 2.2.11 (See [11, Corollary 2.6])**.** *An R-module P is projective if and only if there is another module N such that their direct sum $P \oplus N$ is free.*

Lemma 2.2.12 (Baer's Criterion [11, Theorem 3.7])**.** *Let R be a ring. A left R-module M is injective if and only if any homomorphisms $f\colon I \to M$ from a left ideal I of R can be extended to all of R.*

Lemma 2.2.13. *For any ring R, R-mod has all direct limits.*

Proof. We know that R-mod has all small coproducts and cokernels. Therefore R-mod has all small colimits by the dual of [1, Theorem 12.3]. ■

Lemma 2.2.14. *The category of left modules over $U_n(\mathbb{K})$ is locally artinian.*

Proof. By the previous lemma we know that $U_n(\mathbb{K})$-mod has all direct limits. Let M be any $U_n(\mathbb{K})$-module. Pick an element $m \in M$ and define $S_m = U_n(\mathbb{K})m$ to be the submodule generated by m. Clearly S_m is finite dimensional, so it must have finite length. We see that $M = \sum_{m\in M} S_m$ is the sum of subobjects of finite length. Since M is arbitrary we can say that $U_n(\mathbb{K})$-mod is locally artinian. ■

Lemma 2.2.15. *For any ring R, R-mod satisfies Grothendieck's condition.*

Proof. Let M be an R-module, let $\{A_i\}$ be a family of submodules of M and let B be another submodule of M. Pick any $r \in B \cap (\bigcup A_i)$. We can see that $r \in B$ and that $r \in A_j$ for some A_j. Therefore $r \in B \cap A_j$, and $r \in \bigcup (B \cap A_i)$. If $r \in \bigcup (B \cap A_i)$, we can see that $r \in B$ and that $r \in A_j$ for some A_j. Hence $r \in B \cap (\bigcup A_i)$.

Therefore $B \cap (\bigcup A_i) = \bigcup (B \cap A_i)$. ■

2.3. **Semisimple Algebras.**

Definition 2.3.1. Let R be a ring. We say an R-module M is *semisimple* if it is the direct sum of simple R-modules.

Definition 2.3.2. We say a ring R is semisimple if it is a semisimple left module over itself.

Theorem 2.3.3. *Let A be a semisimple associative algebra with unity. The category of left modules over A is a highest weight category.*

Proof. We have by assumption that A is an associative algebra with unity, so A-mod is automatically locally Artinian (by Lemma 2.2.14) with enough injectives (by Lemma 2.2.13), and satisfies Grothendieck's condition (by Lemma 2.2.15). Let I be any set that indexes the simple A-modules $S(i)$, set $A(i) = S(i)$, and say $i \leq j$ if and only if $i = j$. Clearly $S(i) \hookrightarrow A(i)$, and since $A(i)/S(i) = 0$ it has no composition factors. We can see that $\dim \operatorname{Hom}(A(i), A(j)) = 1$ if $i = j$ and 0 if $i \neq j$ by Schur's lemma, and that $[A(i)\colon S(i)] = 1$. Since A is semisimple, any A-module is both injective and projective. Hence the injective hull of $S(i)$ is itself. If we set $F_0(i) = 0$ and $F_1(i) = S(i)$, we can see that $0 = F_0(i) \subsetneq F_1(i) = S(i)$ is an increasing filtration of $S(i)$. In addition we can see that the following axioms for a highest weight category hold:

(H1) $F_1(i) = A(i) = S(i)$.

(H2) Since there is no $n > 0$ this holds trivially.
(H3) For a fixed μ this holds for exactly zero n.
(H4) $\bigcup_p F_p(i) = S(i)$.

Therefore A-mod is a highest weight category. ■

2.4. **Quasi Hereditary Algebras.** In the following section we will give a brief overview of quasi heriditary algebras. For a more in depth treatment of them the reader is referred to [4–6, 10].

Definition 2.4.1. We say a ring R is *semiprimary* if $J(R)$ is nilpotent and $R/J(R)$ is semisimple and artinian. We say an ideal $I \subseteq R$ is *hereditary* if $I^2 = I$, $I \cdot J(R) \cdot I = 0$ and I is projective when viewed as a left A-module.

Let A be an associative unital algebra over a fixed field $\mathbb{K}$. We say A is *quasi-hereditary* if there exists an increasing chain of idempotent ($I^2 = I$) ideals $0 = J_0 \subsetneq J_1 \subseteq \cdots \subsetneq J_{m-1} \subsetneq J_m = A$ such that for $1 \leq i \leq m$, J_i/J_{i-1} is a hereditary ideal of A/J_{i-1}.

Theorem 2.4.2 (See [3, Theorem 3.6])**.** *Let A be an associative $\mathbb{K}$-algebra. The category of modules over A is a highest weight category if and only if A is a quasi hereditary algebra.*

2.5. **Upper Triangular Matrices.**

Lemma 2.5.1. *The algebra of strictly upper triangular $n \times n$ matrices over $\mathbb{K}$ is a left nil ideal of $U_n(\mathbb{K})$.*

Proof. Denote the strictly upper triangular $n \times n$ matrices over $\mathbb{K}$ by $S_n(\mathbb{K})$ We will prove this by induction on n. For $n = 2$: all strictly upper triangular 2×2 matrices are of the form $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ for $a \in \mathbb{K}$. Hence for $a \in \mathbb{K}$:

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Suppose for any $A \in S_n(\mathbb{K})$, $A^n = 0$. Any element B of $S_{n+1}(\mathbb{K})$ can be written as:

$$B = \begin{bmatrix} b & \vec{v} \\ \vec{0}^T & 0 \end{bmatrix}$$

Where $b \in S_n(\mathbb{K})$, $\vec{v}, \vec{0} \in \mathbb{K}^n$ and $0 \in \mathbb{K}$. Then:

$$B^{n+1} = \begin{bmatrix} b & \vec{v} \\ \vec{0}^T & 0 \end{bmatrix}^{n+1} = \begin{bmatrix} b^{n+1} & b^n\vec{v} \\ \vec{0}^T & 0 \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & \vec{0} \\ \vec{0}^T & 0 \end{bmatrix}$$

Therefore all $B \in S_n(\mathbb{K})$ are nilpotent.

Now let $A = (a_{ij})$ be in $U_n(\mathbb{K})$ and let $B = (b_{ij})$ be in $S_n(\mathbb{K})$. Then for any $1 \leq i \leq n$, the diagonal entry of AB is:

$$(AB)_{ii} = \sum_{k=i}^{i} a_{ik}b_{ki} = a_{ii}b_{ii} = 0$$

Since $b_{ii} = 0$. Therefore $S_n(\mathbb{K})$ is a left nil ideal of $U_n(\mathbb{K})$. ■

Lemma 2.5.2. *The maximal left ideals of $U_n(\mathbb{K})$ are exactly the ideals $I_i = \{M \in U_n(\mathbb{K}) \mid m_{ii} = 0\}$, for $1 \leq i \leq n$.*

Proof. We can see that I_i is an ideal for any $1 \leq i \leq n$ since if we take any $A \in U_n(\mathbb{K})$ and $B \in I_i$:

$$(AB)_{ii} = \sum_{k=i}^{i} a_{ik}b_{ki} = a_{ii}b_{ii} = 0.$$

hence $AB \in I_i$. Since I_i have dimension one less than $U_n(\mathbb{K})$ they must also be maximal.

Let I be a maximal ideal of $U_n(\mathbb{K})$. Since it is maximal it contains all the nilpotent elements of $U_n(\mathbb{K})$, which form the nil ideal of strictly upper triangular matrices $S_n(\mathbb{K})$. Let (m_{ij}) be in I. Since I contains $S_n(\mathbb{K})$ we can subtract the non-diagonal entries from (m_{ij}) to get $\sum_{k=1}^{n} m_{kk}E_{kk} \in I$. Since I is an ideal we know that $E_{ii}\sum_{k=1}^{n} m_{kk}E_{kk} = m_{ii}E_{ii} \in I$, hence $E_{ii} \in I$ if and only if there is a matrix (m_{ij}) in I such that $m_{ii} \neq 0$. We know I is not $U_n(\mathbb{K})$, so there must be some E_{ii} not in I. Therefore $I \subseteq I_i$ and by maximality $I = I_i$. ∎

Next we will classify the projective covers of the simple $U_n(\mathbb{K})$-modules.

Lemma 2.5.3. *Consider $\mathbb{K}^n$ as a left $U_n(\mathbb{K})$-module, and consider its standard basis as a $\mathbb{K}$ vector space $B = \{e_1, \dots, e_n\}$. Define V_i to be the span of $e_i \in B$, i.e. $V_i = U_n(\mathbb{K})e_i = \{Ae_i \mid A \in U_n(\mathbb{K})\}$. Then V_i is projective.*

Proof. Consider the map $\phi_i \colon U_n(\mathbb{K}) \to V_i$ that sends an $A \in U_n(\mathbb{K})$ to $\phi_i(A) = Ae_i$. The map ϕ_i is clearly surjective. Now Consider the map $\phi \colon U_n \to \bigoplus_{i=1}^{n} V_i$ defined by $\phi(A) = (\phi_i(A), \dots, \phi_n(A)) = (Ae_1, \dots, Ae_n)$, we can see that ϕ is linear and that $\phi(MA) = (MAe_1, \dots, MAe_n) = M(Ae_1, \dots, Ae_n) = M\phi(A)$. The map ϕ is surjective since each of its component functions is surjective. Notice that $\phi(A)$ is the list of the columns of A; since a matrix A is uniquely determined by its columns the map ϕ must be bijective. Therefore $\bigoplus_{i=1}^{n} V_i \cong U_n(\mathbb{K})$ as a left $U_n(\mathbb{K})$-modules, which is clearly free. ∎

Lemma 2.5.4. *$V_i = U_n(\mathbb{K})e_i$, as defined above, is the projective cover of V_i/V_{i-1}.*

Proof. We know that V_i is projective by Lemma 2.5.3, so we only need to show that there is a $U_n(\mathbb{K})$-module homomorphism $V_i \to V_i/V_{i-1}$ whose kernel is superfluous. Consider the quotient map $f \colon V_i \to V_i/V_{i-1}$. The kernel of f is V_{i-1}; let H be a submodule of V_i such that $H + V_{i-1} = V_i$. There must be an $h \in H$ such that the i entry h_i of h is nonzero. Hence $h_i^{-1}E_{ii}h = e_i$, which generates V_i. Therefore $H = V_i$. ∎

Definition 2.5.5. Let R be a ring. Say for $r, s \in R$, $r * s = sr$. Then we define $R^{\text{op}} = (R, +, *)$ to be the *opposite ring* to R.

Proposition 2.5.6. *Let $L_n(\mathbb{K})$ be the associative algebra of lower triangular matrices, $U_n(\mathbb{K})^{\text{op}} \cong L_n(\mathbb{K})$.*

Proof. Let $T \colon U_n(\mathbb{K})^{\text{op}} \to L_n(\mathbb{K})$ be the transpose map. We know that T is bijective and linear, so we have to show that in addition it preserves the ring multiplication. Let $a, b \in U_n(\mathbb{K})^{\text{op}}$. Then:

$$T(a * b) = T(ba) = T(a)T(b).$$

Therefore $U_n(\mathbb{K})^{\text{op}} \cong L_n(\mathbb{K})$. ∎

Definition 2.5.7. Let R be a finite dimensional associative algebra over a field $\mathbb{K}$. Recall the definition of the dual of a vector space over $\mathbb{K}$, M, $M^* = \operatorname{Hom}(V, \mathbb{K})$. If M is additionally a left R-module we can give M^* a left R^{op}-module structure by defining $(a \cdot f)(m) = f(am)$ for $a \in R^{\text{op}}$, $f \in M^*$ and $m \in M$.

With that in mind we define a contravariant functor $D = \operatorname{Hom}(-, \mathbb{K})\colon R\text{-mod} \to R^{\text{op}}\text{-mod}$ that sends R-modules to R^{op}-modules in the way defined above, and sends a map $f\colon M \to N$ to the transpose map $Df\colon N^* \to M^*$ by $(Df)(n) = n \circ f$.

Lemma 2.5.8. *The functor* $D = \operatorname{Hom}(-, \mathbb{K})\colon R\text{-mod} \to R^{\text{op}}\text{-mod}$ *is well defined.*

Proof. Let M be an R-module. Pick $f, g \in DM$ and $r, s \in R^{\text{op}}$, then for all $m \in M$:

- (M1): $(r \cdot (f+g))(m) = f(rm) + g(rm) = (r \cdot f)(m) + (r \cdot g)(m)$.
- (M2): $((r+s) \cdot f)(m) = f((r+s)m) = f(rm) + f(sm) = (r \cdot f)(m) + (s \cdot f)(m)$.
- (M3): $(r \cdot (s \cdot f))(m) = (s \cdot f)(rm) = f(s * rm) = ((sr) \cdot f)(m) = ((r * s) \cdot f)(m)$.
- (M4): $(1_R \cdot f)(m) = f(1_R m) = f(m)$.

Therefore DM is a module over R^{op}.

Now let $f\colon M \to N$ be an R-module homomorphism. We already know that the map Df is linear, so pick $r \in R^{\text{op}}$, $n \in N^*$ and $m \in M$. Then we see that the following holds:

$$(Df(r \cdot n))(m) = ((r \cdot n) \circ f)(m) = n(rf(m)) = (n \circ (rf))(m) = (rDf(n))(m).$$

Therefore Df is an R^{op}-module homomorphism, and since $(D(f \circ g)(n) = n \circ f \circ g = (Dg \circ Df)(n)$ we can conclude that the functor D is well defined and contravariant. ■

Lemma 2.5.9. *If R is a finite dimensional algebra, and M is a finitely generated R-module; then $M \cong D^2 M$.*

Proof. Define the map $\phi\colon M \to D^2M$ by $\phi(m) = \hat{m}\colon DM \to \mathbb{K}$, $\hat{m}(f) = f(m)$. Anyone who has done linear algebra will know that this is an isomorphism between M and D^2M as vector spaces, so we only need show that ϕ is an R-module homomorphism. Pick any $r \in R$, $m \in M$, and $f \in DM$. Then we can see that the following holds:

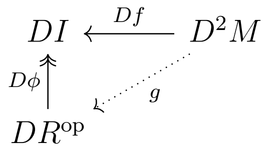
$$(r \cdot \phi(m))(f) = (r \cdot \hat{m})(f) = \hat{m}(r \cdot f) = (\phi(m))(r \cdot f) = (r \cdot \phi(m))(f).$$

Therefore ϕ is not only an isomorphism of vector spaces, but an isomorphism of R-modules. ■

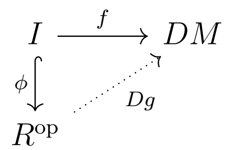
The important thing about the above lemma is that it shows that the finitely generated R-modules and the finitely generated R^{op}-modules form categories dual to each other exactly when R is a finite dimensional algebra.

Lemma 2.5.10. *Let M be a finitely generated module over a finite dimensional algebra R. Then if M is projective, DM is injective.*

Proof. Let M be finitely generated and projective , let I be a left ideal of R^{op}, let $f\colon I \to DM$ be an R^{op}-module homomorphism and $\phi\colon I \to R^{\text{op}}$ be the inclusion map. Then $D\phi\colon DR^{\text{op}} \to DI$ is surjective and by the projectivity of $D^2M \cong M$ there exists a map $g\colon D^2M \to DR^{\text{op}}$ such that $Df = D\phi \circ g$. In other words the following diagram commutes:



If we take the image of the above diagram under D, and notice that all the modules in it are finitely generated; we can see that Lemma 2.5.9 applies. Hence we can see that the following diagram commutes (noting that D^2 is the identity functor on finitely generated modules, up to isomorphism):



■

Hence the module DM satisfies Baer's criterion.

Lemma 2.5.11. *The maximal ideals of $L_n(\mathbb{K})$ are $J_i = \{(m_{pq}) \in L_n(\mathbb{K}) \mid m_{ii} = 0\}$, for $1 \leq i \leq n$.*

Proof. The proof is identical in nature to the proof of Lemma 2.5.2. ■

Corollary 2.5.12. *The simple $L_n(\mathbb{K})$-modules are $T_i = L_n(\mathbb{K})/J_i$, for $1 \leq i \leq n$.*

Lemma 2.5.13. *Let $\mathbb{K}^n$ be a left $L_n(\mathbb{K})$-module in the natural way with the canonical basis $B = \{e_1, \ldots, e_n\}$. Define $W_i = L_n(\mathbb{K})e_i$. The modules W_i for $1 \leq i \leq n$ are projective.*

Proof. The proof that $\bigoplus_{i=1}^n W_i \cong L_n(\mathbb{K})$ is identical to that of Lemma 2.5.3. Therefore W_i is projective. ■

Lemma 2.5.14. *The projective hull of T_i is W_i.*

Proof. Analogously to the simple $U_n(\mathbb{K})$-modules, the simple modules over $L_n(\mathbb{K})$ are $T_i \cong W_i/W_{i-1}$ for $1 \leq i \leq n$. Let $f\colon W_i \to W_i/W_{i-1}$ be the quotient map. We can see that $\ker f = W_{i-1}$. If there is an $H \subseteq W_i$ such that $H + W_{i-1} = W_i$, then $h \in H$ such that $h_i \neq 0$. Hence $e_i = h_i^{-1}E_{ii}h \in H$, and $H = W_i$. Since W_i is projective we can say that W_i is the projective hull of T_i. ■

The above lemma is important as it allows us to find the injective hulls of the S_i, which are exactly DW_i. Of course we want a more concrete description of DW_i, so we want to find an easier to work with $U_n(\mathbb{K})$-module that is isomorphic to DW_i.

Lemma 2.5.15. *The modules DW_i and V_n/V_{i-1} are isomorphic as $U_n(\mathbb{K})$-modules.*

Proof. Define $f_t \in DW_i$ for $1 \leq t \leq n$ as: $f_t(e_t) = 1$, $f_t(e_p) = 0$ for $p \neq t$. We want to show that the unique linear map ϕ that sends $f_k \mapsto [e_k]$ is an isomorphism. It is clearly a bijective linear map, so we must only show that for all $A \in U_n(\mathbb{K})$ and $g \in DW_i$, $\phi(Ag) = A\phi(b)$. Pick $b = \sum_{k=i}^n b_k e_k$, and am $A = (a_{ij}) \in U_n(\mathbb{K})$. We can see that:

$$(A \cdot f_t)(b) = f_t(A^T b) = f_t\left(\sum_{k=i}^n b_k A^T e_k\right) = \sum_{k=i}^n b_k \sum_{p=1}^p a_{kp} f_t(e_p) = \sum_{k=i}^t b_k a_{kt} = \sum_{k=i}^t a_{kt} f_k(b).$$

Therefore $A \cdot f_t = \sum_{k=i}^{t} a_{kt} f_k$.

Since the following holds as well:

$$A[e_t] = \sum_{k=i}^{n} a_{kt} e_k,$$

We can see that $\phi(A \cdot f_t) = \phi(\sum_{k=i}^{n} a_{kt} f_k) = \sum_{k=i}^{n} a_{kk}[e_k] = A[e_t] = A \cdot \phi(f_t)$. Therefore, since ϕ preserves the action on a generating set, $\phi(Aw) = A\phi(w)$ for all w in DW_i and all A in $U_n(\mathbb{K})$. ∎

Theorem 2.5.16. *The category of $U_n(\mathbb{K})$-modules is a highest weight category.*

Proof. For any ring R, R-mod is abelian, has enough injectives 2.2.13, and satisfies Grothendieck's condition 2.2.15. The category of modules over $U_n(\mathbb{K})$ is locally artinian 2.2.14. If A, B are $U_n(\mathbb{K})$-modules then $\operatorname{Hom}(A, B)$ is a vector space over $\mathbb{K}$, and hence $U_n(\mathbb{K})$ a $\mathbb{K}$-linear category.

Let $\Lambda = \{1, \dots, n\}$, and $S(i) = A(i) = V_i/V_{i-1}$. Clearly $1_{S(i)} \colon S(i) \to S(i) = A(i)$ is an injection, and $A(i)/S(i) \cong 0$ has no simple composition factors. Since $\dim S(i)$ is finite for any $i \in \Lambda$, $\dim \operatorname{Hom}(A(i), S(i))$ is finite; and $[A(i) \colon S(j)]$ is either 0 or 1. The injective hull of $S(i) = V_i/V_{i-1}$ is $I(i) = V_n/V_{i-1}$. Define $F_p(i) = V_{i-1+p}/V_{i-1}$ for $0 \leq p \leq n-i+1$. Then the following is an increasing filtration of $I(i)$

$$0 \cong F_1(i) \subsetneq \cdots \subsetneq F_{n-i+1}(i) = I(i).$$

(H1) $F_1(i) \cong A(i) = V_i/V_{i-1}$.
(H2) For $n-i+1 > p > 1$, $F_p(i)/F_{p-1}(i) = (V_{i-1+p}/V_{i-1})/(V_{i-2+p}/V_{i-1}) \cong V_{i-1+p}/V_{i-2+p} = A(i-1+p)$, $i-1+p > i$.
(H3) For a fixed $i \in \Lambda$, $i-1+p$ is the only value $\mu \in \Lambda$ such that $F_p(i)/F_{p-1}(i) \cong A(\mu)$.
(H4) It is clear that $\bigcup_{p=0}^{n-i+1} F_p(i) = I(i)$, since $F_{n-i+1}(i) = I(i)$.

Therefore we can see that the category of modules over $U_n(\mathbb{K})$ is a highest weight category. ∎

3. Lie Algebras and Lie Modules

3.1. Lie Algebras.

Definition 3.1.1 (Lie algebra)**.** Let $\mathbb{K}$ be a field. Then a Lie algebra is a $\mathbb{K}$-vector space V along with a bilinear map $[\cdot,\cdot] \colon V \times V \to V$ called a Lie bracket that satisfies:

(i) $[x, x] = 0 \; \forall x \in V$.
(ii) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$, $\forall x, y, z \in V$ (The Jacobi identity).

One of the places Lie algebras appear is in the study of endomorphism rings of vector spaces, one usually only considers finite dimensional vector spaces but in general the endomorphism ring with the commutator forms a Lie algebra. The following is then stated without proof, as the proof is simply checking the definitions.

Lemma 3.1.2. *Let V be a vector space over a fixed field $\mathbb{K}$. Let $\operatorname{End} V$ be the endomorphism ring of V. Then $\operatorname{End} V$, with $[T, S] = TS - ST$ for $S, T in \operatorname{End} V$ as the Lie bracket, is a Lie algebra.*

Definition 3.1.3 (Lie algebra homomorphism)**.** For Lie algebras L_1 and L_2, we call $f\colon L_1 \to L_2$ a Lie algebra homomorphism if it is both a linear map and preserves the Lie bracket:

$$f([x,y]_{L_1}) = [f(x), f(y)]_{L_2} \quad \text{for all } x, y \in L$$

The Lie algebra homomorphisms are the morphisms in the category of Lie algebras.

A Lie algebra is a type of algebra as defined in the previous section, and a Lie algebra homomorphism is simply an algebra homomorphism where the Lie bracket is the algebra's multiplication.

Remark 3.1.4. It is worth noting that any associative algebra A can be made into a Lie algebra by taking the Lie bracket to be the commutator $[x, y] = x \cdot y - y \cdot x$.

Definition 3.1.5 (Representation of a Lie algebra)**.** If L is a Lie algebra and V is a vector space over the field $\mathbb{K}$, then we call any Lie homomorphism $\phi\colon L \to \mathfrak{gl}(V)$ a *representation* of L.

Definition 3.1.6 (Lie module)**.** Given a Lie algebra L over a field $\mathbb{K}$, we define a Lie module over L to be a vector space V over $\mathbb{K}$ along with bilinear map $\cdot\colon L \times V \to V$ such that for every $v \in V$:

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

3.2. **Category $\mathscr{O}$.** The following section will give a brief overview of the Berstein-Gelfand-Gelfand category $\mathscr{O}$, which is a particular category of modules over a semisimple Lie algebra that serves as a motivating example for highest weight categories. In this subsection, we assume some familiarity with the basic theory of Lie algebras. We refer the reader to [9] for further details.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$ and let $\Lambda \subseteq \mathfrak{h}^*$ be the lattice of weights of $\mathfrak{g}$. Let Φ be a root system of $\mathfrak{g}$ and let Φ^+ be a system of positive roots. For $\alpha \in \Phi$ let $\mathfrak{g}_\alpha$ be the root space of the root α, and let $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$, a nilpotent subalgebra of $\mathfrak{g}$.

If M is a $\mathfrak{g}$-module and $\lambda \in \mathfrak{h}^*$, then we define $M_\lambda = \{v \in M \mid \forall h \in \mathfrak{h}, h \cdot v = \lambda(h)v\}$, which we call the *weight space* of the weight λ.

Definition 3.2.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. The objects of category $\mathscr{O}$ are the $\mathfrak{g}$-modules M such that the following hold:

- (O1) M is finitely generated.
- (O2) $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$
- (O3) For each $v \in M$, the $\mathfrak{n}$-module generated by v is finite dimensional.

The morphisms of $\mathscr{O}$ are the $\mathfrak{g}$-module homomorphisms.

Theorem 3.2.2 (See [3, Example 3.3(b)])**.** *The category $\mathscr{O}$ is a highest weight category.*

Example 3.2.3. As a concrete example, let $\mathfrak{g} = \mathfrak{sl}_n$ be the Lie algebra of trace zero complex $n \times n$ matrices. The diagonal matrices in $\mathfrak{sl}_n$ form a Cartan subalgebra $\mathfrak{h}$. Define $\epsilon_i(E_{jj}) = \delta_{ij}$ for $1 \leq i, j \leq n$, and $h_i = E_{ii} - E_{i+1,i+1}$ for $1 \leq i \leq n-1$. Then $\{h_1, \ldots, h_{n-1}\}$ forms a basis for $\mathfrak{h}$, and $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}$ is the root system of $\mathfrak{sl}_n$ with $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \operatorname{span}(E_{ij})$. It can be shown that $\Phi^+ = \{\epsilon_i - \epsilon_j \in \Phi \mid j > i\}$, hence we can see that $\mathfrak{n} = \bigoplus_{\epsilon_i - \epsilon_j \in \Phi^+} \operatorname{span}(E_{ij})$ is the Lie algebra of strictly upper triangular matrices.

Consider the subcategory of $\mathfrak{sl}_n$-mod consisting of finitely generated M that satisfy the BGG axioms (O1), (O2) and (O3). This subcatigory is the BGG category $\mathscr{O}$ over $\mathfrak{sl}_n$.

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