5-VERTEX MODELS, GELFAND-TSETLIN PATTERNS AND SEMISTANDARD YOUNG TABLEAUX

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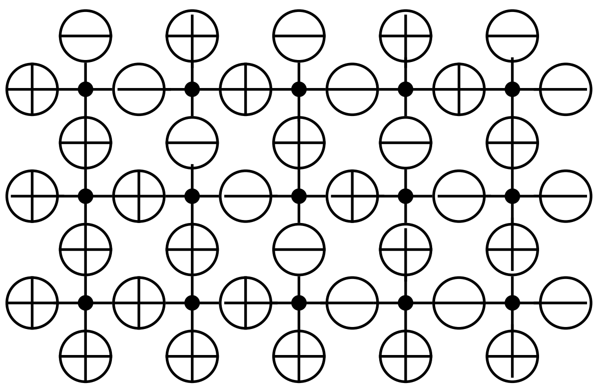
1. Introduction

In statistical mechanics, one studies models based on the interconnections between thermodynamic quantities such as temperature or heat capacity of a macroscopic system, composed usually of a large number of particles, and the quantities related to each of these particles like spins, momenta and velocities. A well-known model is the so-called *ice-type model* or *six-vertex model*, introduced by Linus Pauling and motivated by the study of crystals with hydrogen bonds such as ice or potassium dihydrogen phosphate. The six-vertex model consists generally of a grid graph whose edges are labelled by spins, following a rule called the *ice rule*, representing the state of a given crystal, as illustrated by the following example:

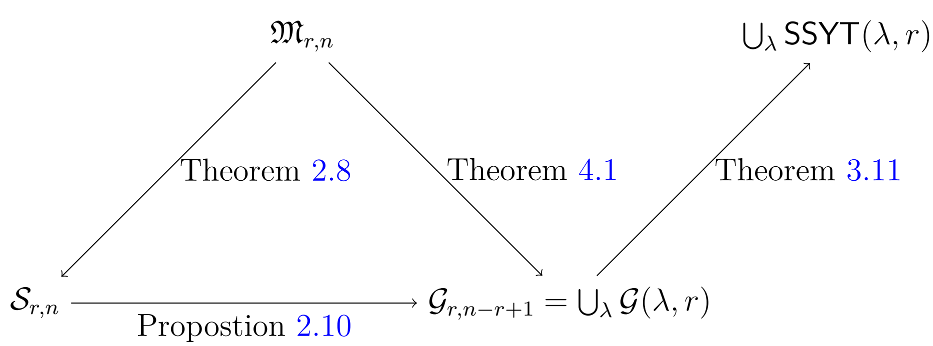


For instance, the vertices can be thought as oxygen atoms and the four edges, with their respective spins, adjacent to each vertex, represent the configuration of the surrounding hydrogen atoms, so that we have a model of an ice crystal. For further examples of models in statistical mechanics, we refer the reader to [1]. This theoretical description and its generalisations permitted the use of mathematical tools from combinatorics, number theory, representation theory, dynamical systems, etc., and produced notable results in each of these areas. In statistical mechanics, one aims to determine the *partition function* of the model, a sum indexed by all the possible states of the model, under specific boundary conditions. In [2], Brubaker, Bump and Friedberg give the partition function of ice-type models with integer partitions as boundary conditions. On the other hand, Tokuyama, in [5], described the partition function of six-vertex models with the aforesaid boundary conditions as a sum over strict Gelfand-Tsetlin patterns with top rows equal to the given

Date: February 24, 2014.
Key words and phrases. Ice models, Gelfand-Tsetlin patterns, Young tableaux.

integer partition. This suggests a bijection between the states of six-vertex models with fixed boundary conditions and strict Gelfand-Tsetlin patterns with fixed top row. Moreover, [2] gives also an expression of the partition function in terms of *Schur polynomials*, which are known to be generating functions of the number of *semi-standard Young tableaux* with a given shape. This fact, together with the one-to-one correspondence between Gelfand-Tsetlin patterns and semi-standard Young tableaux, is a motivation to describe a bijection between the states of six-vertex models and semi-standard Young tableaux. Introduced by I.M. Gelfand and M.L. Tsetlin in representation theory, Gelfand-Tsetlin patterns of top row λ, where λ is a partition of some integer n, parametrise a special basis for the irreducible representation of highest weight λ for $\mathfrak{gl}_n$. These bases are called *Gelfand-Tsetlin bases* and are obtained by applying the *branching rule* to the sequence $\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n$; see, for example, [4]. On the other hand, Young tableaux were introduced by A. Young to study the representation theory of the symmetric group S_n. They parametrise the so-called *Young basis* of an irreducible representation of S_n: if λ is a partition of n, we know there is an unique irreducible representation of S_n associated to λ, the *Specht module*, and applying the *branching rule* to the sequence $S_0 \subset S_1 \cdots \subset S_n$, we obtain the corresponding Young basis indexed by the semi-standard Young tableaux of shape λ with entries in $\{1, \ldots, n\}$. The interested reader can refer to [3].

In this article, we will be concerned with the *five-vertex model*, derived from the six-vertex model by forbidding one vertex configuration. A key result will be the bijection between the set of five-vertex models defined by specified boundary conditions, $\mathfrak{M}_{r,n}$, and a subset of Gelfand-Tsetlin patterns having the same top row expressing these boundary conditions, denoted $\mathcal{S}_{r,n}$. Then we will show that there is in fact a one-to-one correspondence between $\mathcal{S}_{r,n}$ and the set of Gelfand-Tsetlin patterns having the same rank r but of lesser bound $n - r + 1$, $\mathcal{G}_{r,n-r+1}$. We will make use of this bijection to describe a direct one-to-one correspondence between $\mathfrak{M}_{r,n}$ and $\mathcal{G}_{r,n-r+1}$. Finally we will remind the reader of a well-known bijection between Gelfand-Tsetlin patterns of top row λ and rank r, denoted $\mathcal{G}(\lambda, r)$, and the set of semi-standard Young tableaux, $\mathsf{SSYT}(\lambda, r)$, through which it will be possible to describe a direct one-to-one correspondence between five-vertex models and semi-standard Young tableaux. The following diagram, where each arrow represents a bijection, summarizes the results presented in the current paper:



Acknowledgements. I am thankful to Prof. Alistair Savage for providing the theme of this project and for his comments and advice without which I would not be able to carry out the work. His patience and forbearance encouraged me also greatly. Prof. Barry Jessup contributed considerably during times when Alistair was not available to supervise my work and in logistical issues. My thanks go also to Prof. Richard Blute, for giving me the chance to learn about quantum groups and Hopf algebras. The University of Ottawa, the department of mathematics and statistics, together with AIMS-NEI deserve my gratitude for their partnership in bringing forth the AIMS-Headstart program which supported me with the material resources needed. Finally, and not the least, I am indebted to Prof. Benoit Dione who helped me diligently with administrative matters, and whose assistance made possible my coming to Canada.

2. Bijection between semi-strict Gelfand-Tsetlin Patterns of rank r and bound n and $r \times n$ ice models.

Definition 2.1 (Semi-strict Gelfand-Tsetlin pattern)**.** A *semi-strict Gelfand-Tsetlin pattern of rank r and bound n* is a triangular array of positive integers such that each row has one less element than the row above it

$$\mathfrak{G} = \left\{ \begin{array}{ccccccc} a_{1,1} & & a_{1,2} & \cdots & a_{1,r-1} & & a_{1,r} \\ & a_{2,1} & & \cdots & & a_{2,r-1} & \\ & & \ddots & & \iddots & & \\ & & & a_{r,1} & & & \end{array} \right\} \tag{2.1}$$

and satisfying:

$$n \geq a_{i,j} > a_{i+1,j} \geq a_{i,j+1} \quad \text{for all} \quad 1 \leq i \leq r-1 \quad \text{and} \quad 1 \leq j \leq r-i.$$

The set of semi-strict Gelfand-Tsetlin patterns of rank r and bound n will be denoted $\mathcal{S}_{r,n}$.

Let us call the symbols $\ominus$ and $\oplus$ *spins*.

Definition 2.2 (Two-dimensional ice model)**.** For $m, n \in \mathbb{N} \setminus \{0\}$, an $m \times n$ *two-dimensional ice model*, or *ice model* for short, consists of an $m \times n$ rectangular lattice and an assignment of exactly one spin to each of the four edges adjacent to each vertex. The columns are numbered from left to right $n-1, n-2, \ldots, 0$ while the rows, from top to bottom, $1, 2, \ldots, m$.

Example 2.3. An example of a 3×5 ice model is

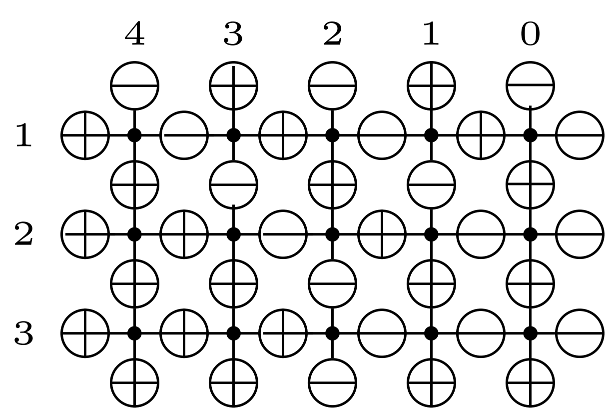
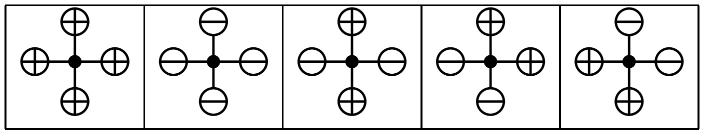


Figure 1. An 3×5 ice model.

Definition 2.4 (Admissible vertex configuration)**.** We call the following 1×1 ice models *admissible vertex configurations*:



Definition 2.5 (5-vertex ice model)**.** An $m \times n$ *5-vertex ice model* is an $m \times n$ ice model M such that the assignment of spins to the four edges adjacent to each vertex of M corresponds to an admissible vertex configuration.

From now on, *'ice model'* will be interpreted as *'5-vertex ice model'*.

Lemma 2.6. *Let us consider a $1 \times (n+1)$ ice model. Let $\alpha_1 > \alpha_2 > \cdots > \alpha_\ell$ be the column indices where the spin of the top vertical edge is $\ominus$ and let $\beta_1 > \beta_2 > \cdots > \beta_{\ell'}$ be the column indices where the spin at the bottom vertical edge is $\ominus$. Furthermore, suppose that the spin at the left boundary horizontal edge is $\oplus$. Then we have:*

(1) *$\ell = \ell'$ or $\ell = \ell' + 1$,*
(2) *$\alpha_1 > \beta_1 \geq \alpha_2 > \beta_2 \geq \cdots$, and*
(3) *if $\ell = \ell'$, then the spin on the right edge is $\oplus$, while if $\ell = \ell' + 1$, it is $\ominus$.*

Proof. For $k \in \{0, \ldots, n+1\}$, let $P(k)$ be the assertion that one of the following statements is true:

$(PC)_k$ There exists an $i_k \in \{0, 1, \ldots, \ell'\}$ such that

$$\alpha_1 > \beta_1 \geq \alpha_2 > \beta_2 \geq \cdots \geq \alpha_{i_k} > \beta_{i_k} \geq k,$$
$$\alpha_{i_k+1}, \ldots, \alpha_\ell, \beta_{i_k+1}, \ldots, \beta_{\ell'} < k,$$

and the spin of the horizontal edge between columns k and $k-1$ is $\oplus$.

$(MC)_k$ There exists an $i_k \in \{0, 1, \ldots, \ell\}$ such that

$$\alpha_1 > \beta_1 \geq \alpha_2 > \beta_2 \geq \cdots \geq \alpha_{i_k-1} > \beta_{i_k-1} \geq \alpha_{i_k} \geq k,$$
$$\alpha_{i_k+1}, \ldots, \alpha_\ell, \beta_{i_k}, \ldots, \beta_{\ell'} < k,$$

and the spin of the horizontal edge between columns k and $k-1$ is $\ominus$.

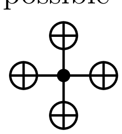
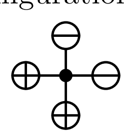
We have that $\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_{\ell'} < n+1$ and the spin of the left edge is $\oplus$. Therefore we have $(PC)_{n+1}$, with $i_{n+1} = 0$. Thus $P(n+1)$ holds.

On the other hand '$P(0)$ is true' is precisely the lemma:

- In the case where $(PC)_0$ is true, we have $i_0 = \ell' = \ell$.
- If $(MC)_0$ is true, we have $i_0 = \ell = \ell' + 1$.

It remains to show that, for $k \in \{1, \ldots, n+1\}$, $P(k)$ implies $P(k-1)$. Let us therefore assume $P(k)$ for some $k \in \{1, \ldots, n+1\}$. We have two cases:

- We have $(PC)_k$. Using the fact that the spin of the horizontal edge between columns k and $k-1$ is $\oplus$, the only possible vertex configurations at column $k-1$ are:

 and 

  - For the leftmost vertex configuration, we have:

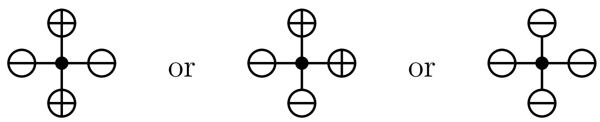
$$\alpha_1 > \beta_1 \geq \alpha_2 > \beta_2 \geq \cdots \geq \alpha_{i_k} > \beta_{i_k} \geq k-1,$$
$$\alpha_{i_k+1}, \ldots, \alpha_\ell, \beta_{i_k+1}, \ldots, \beta_{\ell'} < k-1,$$

    and the spin of the horizontal edge between columns $k-1$ and $k-2$ is $\oplus$. This implies $(PC)_{k-1}$ with $i_{k-1} = i_k$.
  - For the rightmost vertex configuration:

$$\alpha_1 > \beta_1 \geq \alpha_2 > \beta_2 \geq \cdots \geq \alpha_{i_k-1} > \beta_{i_k} \geq \alpha_{i_k+1} \geq k-1,$$
$$\alpha_{i_k+2}, \ldots, \alpha_\ell, \beta_{i_k+1}, \ldots, \beta_{\ell'} < k-1,$$

    and the spin of the horizontal edge between columns $k-1$ and $k-2$ is $\ominus$. We then have $(MC)_{k-1}$ with $i_{k-1} = i_k + 1$.

  We have thus showed that $(PC)_k$ implies $P(k-1)$.
- We have $(MC)_k$. Then, the vertex configurations in column $k-1$ must be:



  - For the first vertex configuration, we have:

$$\alpha_1 > \beta_1 \geq \alpha_2 > \beta_2 \geq \cdots \geq \alpha_{i_k-1} > \beta_{i_k-1} \geq \alpha_{i_k} \geq k-1,$$
$$\alpha_{i_k+1}, \ldots, \alpha_{\ell+1}, \beta_{i_k}, \ldots, \beta_\ell < k-1,$$

    and the spin of the horizontal edge between columns $k-1$ and $k-2$ is $\ominus$. So $(MC)_{k-1}$ holds with $i_{k-1} = i_k$.
  - For the second vertex configuration, we have:

$$\alpha_1 > \beta_1 \geq \alpha_2 > \beta_2 \geq \cdots \geq \alpha_{i_k-1} > \beta_{i_k-1} \geq \alpha_{i_k} > \beta_{i_k} \geq k-1,$$
$$\alpha_{i_k+1}, \ldots, \alpha_\ell, \beta_{i_k+1}, \ldots, \beta_{\ell'} < k-1,$$

    and the spin of the horizontal edge between columns $k-1$ and $k-2$ is $\oplus$. So we have $(PC)_{k-1}$ with $i_{k-1} = i_k$.
  - For the third vertex configuration, we have

$$\alpha_1 > \beta_1 \geq \alpha_2 > \beta_2 \geq \cdots \geq \alpha_{i_k-1} > \beta_{i_k-1} \geq \alpha_{i_k} > \beta_{i_k} = \alpha_{i_k+1} \geq k-1,$$
$$\alpha_{i_k+2}, \ldots, \alpha_\ell, \beta_{i_k+1}, \ldots, \beta_{\ell'} < k-1,$$

    and the spin of the horizontal edge between columns $k-1$ and $k-2$ is $\ominus$. So we have $(MC)_{k-1}$ with $i_{k-1} = i_k + 1$.

  We have thus proved that $(MC)_k$ also implies $P(k-1)$.

As a result $P(k)$ implies $P(k-1)$ for all $k \in \{1, \ldots, n+1\}$. Since $P(n+1)$ is true, we deduce that so is $P(0)$. □

Lemma 2.7. *Let $\alpha_1, \alpha_2, \cdots, \alpha_{\ell+1}$ and $\beta_1, \beta_2, \cdots, \beta_\ell$ be two sequences of semi-strict interleaving positive integers, i.e.:*

$$\alpha_1 > \beta_1 \geq \alpha_2 > \cdots \geq \alpha_\ell > \beta_\ell \geq \alpha_{\ell+1} \geq 0.$$

Furthermore, suppose $n \in \mathbb{N}$ satisfies $n \geq \alpha_1$. Then there exists a unique $1 \times (n+1)$ ice model with spin $\oplus$ at the leftmost edge, spin $\ominus$ at the rightmost edge and spin $\ominus$ at the top *edges at the columns numbered by the α_i and at the bottom edges at the columns numbered by the β_i.*

Proof. For $k \in \{n, n-1, \cdots, -1\}$, let $P(k)$ be the assertion that one of the following two statements are true:

$(PC)_k$ There exists $i_k \in \{0, \ldots, \ell\}$ such that:

- $\alpha_1 > \beta_1 \geq \cdots \geq \alpha_{i_k} > \beta_{i_k} > k \geq \alpha_{i_k+1} > \beta_{i_k+1} \geq \cdots \geq \alpha_\ell > \beta_\ell \geq \alpha_\ell + 1$.
- There exists a unique $1 \times (n+1)$ ice model such that amongst columns $k+1, \ldots, n$, the columns with spin $\ominus$ at their top (resp. at their bottom) edge are labelled by $\alpha_1, \ldots, \alpha_{i_k}$ (resp. $\beta_1, \ldots, \beta_{i_k}$).
- The spin of the horizontal edge between columns $k+1$ and k is $\oplus$.

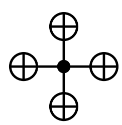
$(MC)_k$ There exists $i_k \in \{0, \ldots, \ell+1\}$ such that:

- $\alpha_1 > \beta_1 \geq \alpha_2 > \cdots > \beta_{i_k-1} \geq \alpha_{i_k} > k \geq \beta_{i_k} \geq \alpha_{i_k+1} > \cdots \geq \alpha_\ell$.
- There exists a unique $1 \times (n+1)$ ice model such that amongst columns $k+1, \ldots, n$, the columns having $\ominus$ at their top (resp. at their bottom) are labelled by $\alpha_1, \ldots, \alpha_{i_k}$ (resp. $\beta_1, \ldots, \beta_{i_k-1}$).
- The spin of the horizontal edge between columns $k+1$ and k is $\ominus$.

We can interpret i_k as the total number of top edges in columns n through $k+1$ having spin $\ominus$. Since the spin at the left edge is $\oplus$, $(PC)_n$ is true with $i_n = 0$, so $P(n)$ is true. Let $k \in \{n, n-1, \ldots, 0\}$ and suppose that $P(k)$ is true. We have two cases:

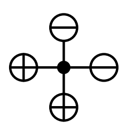
- $(PC)_k$ is true.

  If $k > \alpha_{i_k+1}$ then we have $\alpha_{i_k} > \beta_{i_k} > k > \alpha_{i_k+1} > \beta_{i_k+1}$. Furthermore, since the spin at the horizontal edge between columns $k+1$ and k is $\oplus$, we deduce that there is a unique choice of vertex configuration at column k:

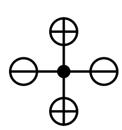
  Thus $(PC)_{k-1}$ is true with $i_{k-1} = i_k$.

  If $\alpha_{i_k+1} = k$, and since $\beta_{i_k} > k > \beta_{i_k+1}$ and the spin at the edge between columns $k+1$ and k is $\oplus$, the unique choice of vertex configuration at column k is:

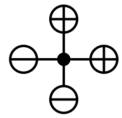
  Therefore $(MC)_{k-1}$ is true with $i_{k-1} = i_k + 1$.
- $(MC)_k$ is true.

  If $k > \beta_{i_k}$, then $\beta_{i_k-1} \geq \alpha_{i_k} > k > \beta_{i_k} \geq \alpha_{i_k+1}$ and since the spin at the horizontal edge between columns $k+1$ and k is $\ominus$, we deduce that the unique choice of vertex configuration at column k is:

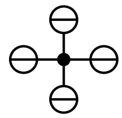
  

  Therefore $(MC)_{k-1}$ is true with $i_{k-1} = i_k$.

If $k = \beta_{i_k} > \alpha_{i_k+1}$, then $\alpha_{i_k} > k$ and the spin at the horizontal edge between columns $k+1$ and k is $\ominus$. Thus the unique choice of vertex configuration at column k is:



We deduce that $(PC)_{k-1}$ is true with $i_{k-1} = i_k + 1$.
If $k = \beta_{i_k} = \alpha_{i_k+1}$, then the unique choice of vertex configuration at column k is:



Therefore $P(k)$ implies $P(k-1)$. Since $P(n+1)$ is true, we deduce that $P(-1)$ is true.

Let us prove finally that $(MC)_{-1}$ is true. Since $\alpha_i \geq 0$ and $\beta_i \geq 0$, there is no $i_{-1} \in \{0, \dots, \ell\}$ such that

$$\alpha_1 > \beta_1 \geq \cdots > \beta_{i_{-1}} > -1 \geq \alpha_{i_{-1}+1} > \beta_{i_{-1}+1} \geq \cdots > \beta_\ell \geq \alpha_{\ell+1}.$$

Hence, $(PC)_{-1}$ is false and $(MC)_{-1}$ must be true. Therefore, the spin at the rightmost of the unique ice model we have constructed is $\ominus$. □

Now, for strictly positive integers r and n, let us denote by $\mathfrak{M}_{r,n}$ the set of $r \times (n+1)$ ice models whose leftmost and bottom edges have spin $\oplus$ and whose rightmost edges have spins $\ominus$ and let $M \in \mathfrak{M}_{r,n}$. Let us consider the map $\psi_{r,n} : \mathfrak{M}_{r,n} \to \mathcal{S}_{r,n}$ defined as follows: we set $\psi_{r,n}(M) = (a_{i,j})$ where, for $1 \leq i \leq r$ and $1 \leq j \leq r+1-i$, $a_{i,j}$ is the column number of the j-th $\ominus$ (from left to right) on the vertical edges between rows $i-1$ and i.

Theorem 2.8. *The map $\psi_{r,n} : \mathfrak{M}_{r,n} \mapsto \mathcal{S}_{r,n}$ constructed above is a bijection.*

Proof. By applying Lemma 2.6 to each successive row of M, we see that $\psi_{r,n}$ is well defined.

Let $\mathfrak{G}$ in $\mathcal{S}_{r,n}$. For $i \in \{1, \dots, r-1\}$, we consider the two consecutive rows i and $i+1$ of $\mathfrak{G}$ whose elements are respectively $a_{i,1}, a_{i,2}, \dots, a_{i,r-i+1}$ and $a_{i+1,1}, a_{i+1,2}, \dots, a_{i+1,r-i}$. By Lemma 2.7, there exists a unique $1 \times (n+1)$ ice model M_i, with a spin $\oplus$ at the leftmost edge, a spin $\ominus$ at the rightmost edge and spins $\ominus$ at the top edges at the columns numbered by $a_{i,1}, a_{i,2}, \dots, a_{i,r-i+1}$ and spins $\ominus$ at the bottom edges at the columns numbered by $a_{i+1,1}, a_{i+1,2}, \dots, a_{i+1,r-i}$. Then we define $\phi_{r,n} : \mathcal{S}_{r,n} \mapsto \mathfrak{M}_{r,n}$ so that $\phi_{r,n}(\mathfrak{G}) = M$ is the element of $\mathfrak{M}_{r,n}$ whose i-th row is M_i for $i \in \{1, \dots, r-1\}$.

On one hand, $\phi_{r,n} \circ \psi_{r,n} = \mathrm{id}_{\mathfrak{M}_{r,n}}$ by the uniqueness guaranteed by Lemma 2.7. At the other hand, $\psi_{r,n} \circ \phi_{r,n} = \mathrm{id}_{\mathcal{S}_{r,n}}$ by construction. We deduce that $\psi_{r,n} = \phi_{r,n}^{-1}$ and thus $\phi_{r,n}$ is a bijection.

□

Definition 2.9 (Gelfand-Tsetlin pattern). A *Gelfand-Tsetlin pattern of rank r and bound n* is a triangular array of positive integers such that each row has one less element than the row above it:

$$\mathfrak{G} = \left\{ \begin{array}{ccccccccc} a_{1,1} & & a_{1,2} & \cdots & a_{1,r-1} & & a_{1,r} \\ & a_{2,1} & & \cdots & & a_{2,r-1} & \\ & & \ddots & & \iddots & & \\ & & & a_{r,1} & & & \end{array} \right\}, \tag{2.2}$$

and satisfying:

$$n \geq a_{i,j} \geq a_{i+1,j} \geq a_{i,j+1} \quad \text{for all} \quad 1 \leq i \leq r-1 \quad \text{and} \quad 1 \leq j \leq r-i+1. \tag{2.3}$$

We let $\mathcal{G}_{r,n}$ denote the set of Gelfand-Tsetlin patterns of rank r and bound n. For $\mathfrak{A} = (a_{i,j})$ in $\mathcal{G}_{r,n}$ and $\mathfrak{B} = (b_{i,j})$ in $\mathcal{G}_{r,n'}$, we define $\mathfrak{A} + \mathfrak{B}$ to be the element $\mathfrak{C} = (a_{i,j} + b_{i,j})$ of $\mathcal{G}_{r,n+n'}$. However, the triangular array obtained by subtracting the corresponding entries of two elements $\mathfrak{A} = (a_{i,j})$ and $\mathfrak{B} = (b_{i.j})$ of $\mathcal{S}_{r,n}$, denoted $\mathfrak{A} - \mathfrak{B}$, is not, in general, a Gelfand-Tsetlin pattern, even if $a_{i,j} \geq b_{i,j}$ for all i, j: For instance, consider

$$\mathfrak{A} = \left\{ \begin{array}{ccccc} 4 & & 3 & & 1 \\ & 3 & & 2 & \\ & & 2 & & \end{array} \right\} \quad \text{and} \quad \mathfrak{B} = \left\{ \begin{array}{ccccc} 4 & & 2 & & 1 \\ & 2 & & 1 & \\ & & 1 & & \end{array} \right\}.$$

We have

$$\mathfrak{A} - \mathfrak{B} = \left\{ \begin{array}{ccccc} 0 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} \right\}.$$

However, for $r \geq 1$ and $n \geq 1$, if we consider $\mathfrak{W} = (w_{i,j}) \in S_{r,n}$ where $w_{i,j} = r - (i+j) + 1$ then we have the following result:

Proposition 2.10. *Define $\mathcal{T}_{r,n} : \mathcal{S}_{r,n} \to \mathcal{G}_{r,n-r+1}$ by $S \mapsto S - \mathfrak{W}$. Then $\mathcal{T}_{r,n}$ is well-defined and is a bijection.*

Proof. We first show that we have, for $S = (s_{i,j})$ in $\mathcal{S}_{r,n}$, $s_{i,j} \geq w_{i,j}$ for all i and j. Indeed, $s_{i,j} > s_{i+1,j} \geq s_{i,j+1}$ for all i and j. Thus $s_{i,j} \geq s_{i,j+1} + 1$ for all i, j and so

$$s_{i,j} \geq s_{i,j+1} + 1 \geq s_{i,j+2} + 2 \geq \cdots \geq s_{i,r-i+1} + r - (i+j) + 1 \geq r - (i+j) + 1 = w_{i,j},$$

where the last inequality follows from the fact that $s_{i,r-i+1} \geq 0$.

We also have

$$n \geq s_{1,1} \geq s_{2,1} + 1 \geq \cdots \geq s_{i,1} + (i-1) \geq s_{i,2} + (i-1) + 1 \geq \cdots \geq s_{i,j} + (i-1) + (j-1).$$

Therefore, for all i, j,

$$s_{i,j} - w_{i,j} = s_{i,j} - r + (i+j) - 1 \leq n - (i-1) - (j-1) - r + (i+j) - 1 = n - r + 1.$$

On the other hand,

$$\begin{gathered} s_{i,j} \geq s_{i+1,j} + 1 \geq s_{i,j+1} + 1 \\ w_{i,j} = w_{i+1,j} + 1 = w_{i,j+1} + 1. \end{gathered}$$

Thus

$$s_{i,j} - w_{i,j} \geq s_{i+1,j} - w_{i+1,j} \geq s_{i,j+1} - w_{i,j+1}.$$

Therefore $\mathcal{T}_{r,n}$ is well defined.

Now we consider an element $\mathfrak{G} = (g_{i,j})$ in $\mathcal{G}_{r,n-r+1}$. Since we have $r - 1 \geq w_{i,j}$ for all i, j, it follows that $\mathfrak{W}$ is an element of $\mathcal{S}_{r,r-1}$. We deduce that $\mathfrak{G} + \mathfrak{W}$ is an element of $\mathcal{G}_{r,n}$. Furthermore,

$$g_{i,j} \geq g_{i+1,j} \geq g_{i,j+1}$$
$$w_{i,j} = w_{i+1,j} + 1 = w_{i,j+1} + 1$$

for all i, j. Hence

$$g_{i,j} + w_{i,j} \geq g_{i+1,j} + w_{i+1,j} + 1 \geq g_{i,j+1} + w_{i,j+1} + 1.$$

Thus

$$g_{i,j} + w_{i,j} > g_{i+1,j} + w_{i+1,j} \geq g_{i,j+1} + w_{i,j+1}.$$

Therefore $\mathfrak{G} + \mathfrak{W}$ is an element of $\mathcal{S}_{r,n}$. Hence the map $\mathcal{P}_{r,n-r+1} : \mathcal{G}_{r,n-r+1} \to \mathcal{S}_{r,n}$, $\mathfrak{G} \mapsto \mathfrak{G} + \mathfrak{W}$ is well defined. Finally, it is easy to check that $\mathcal{T}_{r,n} \circ \mathcal{P}_{r,n-r+1} = \mathrm{id}_{\mathcal{G}_{r,n-r+1}}$ and $\mathcal{P}_{r,n-r+1} \circ \mathcal{T}_{r,n} = \mathrm{id}_{\mathcal{S}_{r,n}}$. We deduce that $\mathcal{T}_{r,n}$ is a bijection. □

3. Bijection between Gelfand-Tsetlin patterns with top row λ and rank r and semi-standard Young tableaux of shape λ and elements $\{1, \ldots, r\}$.

Definition 3.1 (Partition)**.** A *partition* of a positive integer n is a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with the condition

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0 \quad \text{and} \quad n = \lambda_1 + \lambda_2 + \cdots + \lambda_k.$$

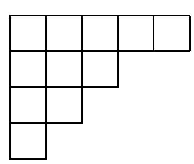
We denote $|\lambda| = n$.

Remark 3.2. If two partitions λ^1 and λ^2 of n have the same nonzero terms then we will consider them as being equal.

Example 3.3. The partitions $\lambda^1 = (5, 3, 2)$ and $\lambda^2 = (5, 3, 2, 0, 0)$ of 10 are equal.

Definition 3.4 (Young diagram)**.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n. The *Young diagram of shape* λ, which will be denoted also λ, is a left justified array of k rows of boxes, such that, from top to bottom, the i-th row has exactly λ_i boxes.

Example 3.5. The Young diagram associated to the partition $\lambda = (5, 3, 2, 1)$ is

$\lambda =$ 

Definition 3.6 (Young tableau)**.** A *semi-standard Young tableau of shape* λ *with entries*, or *with labels, from the set* $\{1, \ldots, r\}$ is a Young diagram λ to each box of which is assigned an element of $\{1, \ldots, r\}$ in such a way that the entries are weakly increasing from left to right along rows and strictly increasing from top to bottom in each column.

Example 3.7. A semi-standard Young tableau of shape $(5, 3, 2, 1)$ with entries from $\{1, 2, 3, 4, 5\}$ is given by

1	2	2	4	5
2	3	3		
3	5			
4				

Remark 3.8. In the definition of a semi-standard Young tableau λ with entries from $\{1,\ldots,r\}$, we always assume that r is the greatest entry of λ.

Remark 3.9. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{k'})$ be partitions such that $k \geq k'$ and

$$\lambda_1 \geq \lambda'_1, \ldots, \lambda_{k'} \geq \lambda'_{k'}.$$

Then the Young diagram of shape λ' is included in the Young diagram of shape λ: $\lambda' \subseteq \lambda$.

Definition 3.10 (Skew diagram)**.** Let λ and μ be two partitions such that $\mu \subseteq \lambda$. The *skew diagram of shape* (λ, μ), denoted $\lambda \backslash \mu$, is the diagram obtained from removing the boxes of the Young diagram of shape μ from the Young diagram of shape λ.

We define a *skew tableau of shape* (λ, μ) *with entries in* $\{1,\ldots,r\}$ as in Definition 3.6 by putting 'skew' before every 'Young'.

We denote by $\mathsf{SSYT}(\lambda, r)$ the set of semi-standard Young tableaux of shape λ with entries from the set $\{1,\ldots,r\}$ and by $\mathcal{G}(\lambda, r)$ the set of Gelfand-Tsetlin patterns of rank r and top row λ. We have the following theorem:

Theorem 3.11. *Let r be a nonnegative integer and $\lambda = (\lambda_1, \ldots, \lambda_r)$ a partition where we allow some of the λ_i to be equal to zero. Then there exists a bijection between the sets $\mathsf{SSYT}(\lambda, r)$ and $\mathcal{G}(\lambda, r)$.*

Proof. We first define a map $\theta_1 \colon \mathsf{SSYT}(\lambda, r) \to \mathcal{G}(\lambda, r)$. Consider $\mathcal{T} \in \mathsf{SSYT}(\lambda, r)$. Set $\mathcal{T}^1 = \mathcal{T}$, $\lambda^1 = \lambda$, and for $i \in \{1,\ldots,r-1\}$, let $\mathcal{T}^{i+1}$ be the obtained from $\mathcal{T}^i$ by removing the boxes labelled by $(r-i+1)$. For all $i \in \{1,\ldots,r-1\}$, by construction, the entries of $\mathcal{T}^{i+1}$ are less than or equal to $r-i$ and are weakly increasing from left to right along each row and strictly increasing from top to bottom in each column. We deduce that $\mathcal{T}^{i+1}$ has at most $r_i = r-i$ rows. Let $\lambda^{i+1} = (\lambda_1^{i+1}, \ldots, \lambda_{r_i}^{i+1})$, where λ_k^{i+1} is the number of boxes of $\mathcal{T}^{i+1}$ at row k and we allow some of the λ_k^{i+1} to be equal to zero. We have $\mathcal{T}^{i+1} \subseteq \mathcal{T}^i$. Furthermore, on one hand, we remove at each row of $\mathcal{T}^i$ the boxes labelled by $r-i+1$, and on the other hand, theses boxes are at the bottom of their respective columns. We deduce that $\lambda_k^i \geq \lambda_k^{i+1} \geq \lambda_{k+1}^i$ for all $i \in \{1,\ldots,r-1\}$ and $k \in \{1,\ldots,r-i+1\}$. We can thus define $\theta_1(\mathcal{T}) = \mathfrak{G} \in \mathcal{G}(\lambda, r)$, where $\mathfrak{G}$ is the Gelfand-Tsetlin pattern of rank r whose i-th row is λ^i for all $i \in \{1,\ldots,r\}$.

Conversely, for $\mathfrak{G} \in \mathcal{G}(\lambda, r)$ we let λ^j be the j-th row of $\mathfrak{G}$ for all $j \in \{1,\ldots,r\}$ and define a map $\theta_2 : \mathcal{G}(\lambda, r) \to \mathsf{SSYT}(\lambda, r)$ as follows: let $\mathcal{T}^r$ be the semi-standard Young tableau of shape λ^r with entries equal to 1 and, for $j = r-1, r-2, \ldots, 2, 1$, let $\mathcal{T}^j$ be the obtained from $\mathcal{T}^{j+1}$ by filling with $r-j+1$ the skew diagram of shape $(\lambda^j, \lambda^{j+1})$. Since λ^j and λ^{j+1} are interleaving sequences for all $j \in \{1,\ldots,r-1\}$, we deduce that the entries of $\mathcal{T}^j$ are weakly increasing from left to right along each row and strictly increasing from top to bottom along each column, thus $\mathcal{T}^j$ is a semi-standard Young tableau of shape λ^j and entries from $\{1,\ldots,r-j+1\}$ for all $j \in \{1,\ldots,r\}$. We then set $\theta_2(\mathfrak{G}) = \mathcal{T}^1$. It is clear that $\theta_2 \circ \theta_1 = \mathrm{id}_{\mathsf{SSYT}(\lambda,r)}$ and $\theta_1 \circ \theta_2 = \mathrm{id}_{\mathcal{G}(\lambda,r)}$. ∎

Example 3.12. Consider $\mathcal{T} = \mathcal{T}^1 = \begin{array}{cccc} 1 & 2 & 2 & 5 \\ 2 & 3 & 3 & \\ 3 & 5 & & \\ 5 & & & \end{array}$. Here $r_1 = 5$ and $\lambda^1 = (4, 3, 2, 1, 0)$. Then

$$\mathcal{T}^2 = \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & & \end{array} \quad \text{with} \quad r_2 = 4 \quad \text{and} \quad \lambda^2 = (3, 3, 1, 0),$$

$$\mathcal{T}^3 = \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & & \end{array} \quad \text{with} \quad r_3 = 3 \quad \text{and} \quad \lambda^3 = (3, 3, 1),$$

$$\mathcal{T}^4 = \begin{array}{ccc} 1 & 2 & 2 \\ 2 & & \end{array} \quad \text{with} \quad r_4 = 2 \quad \text{and} \quad \lambda^4 = (3, 1),$$

$$\mathcal{T}^5 = \begin{array}{c} 1 \end{array} \quad \text{with} \quad r_5 = 1 \quad \text{and} \quad \lambda^5 = (1).$$

Therefore

$$\theta_1(\mathcal{T}) = \left\{ \begin{array}{ccccccccc} 4 & & 3 & & 2 & & 1 & & 0 \\ & 3 & & 3 & & 1 & & 0 & \\ & & 3 & & 3 & & 1 & & \\ & & & 3 & & 1 & & & \\ & & & & 1 & & & & \end{array} \right\}.$$

4. A bijection between $\mathfrak{M}_{r,n}$ and $\mathcal{G}_{r,n-r+1}$.

From Theorem 2.8 and Proposition 2.10, we deduce that there is a one-to-one correspondence between the sets $\mathfrak{M}_{r,n}$ and $\mathcal{G}_{r,n-r+1}$, namely $\mathcal{T}_{r,n} \circ \psi_{r,n}$. In this section, we are going to give a direct description of this bijection.

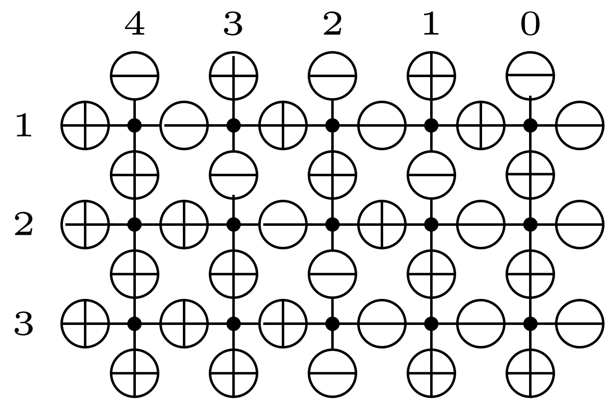
Theorem 4.1. *Let r, n be nonnegative integers such that $n \geq r - 1$. Define $\phi_{r,n} \colon \mathfrak{M}_{r,n} \to \mathcal{G}_{r,n-r+1}$ as follows: if $M \in \mathfrak{M}_{r,n}$, we set $\phi_{r,n}(M) = (b_{i,j})$, where for $1 \leq i \leq r$ and $1 \leq j \leq r + 1 - i$, $b_{i,j}$ is the number of spins $\oplus$ to the right of the j-th $\ominus$ (from the left), on the vertical edges between rows $i - 1$ and i. Then $\phi_{r,n}$ is a bijection.*

Proof. We want to show that $\mathcal{T}_{r,n} \circ \psi_{r,n} = \phi_{r,n}$. In other terms we have to prove that the following equality is true for all $1 \leq i \leq r$ and $1 \leq j \leq r - i + 1$:

$$b_{i,j} = a_{i,j} - r + (i + j) - 1,$$

where $a_{i,j}$ is the column number of the j-th $\ominus$ (from the left) on the vertical edges between rows $i - 1$ and i. Lemma 2.6 (or Theorem 2.8) implies that there are exactly $r - i + 1$ spins $\ominus$ on the vertical edges between rows $i - 1$ and i. Since $a_{i,j}$ is the number of $\oplus$ spins to the right of the j-th $\ominus$, there are $r - (i + j) + 1$ spins $\ominus$ to the right of column $a_{i,j}$, between rows $i - 1$ and i. Moreover, the columns of M are labelled from left to right $n, n - 1, \ldots, 0$. Thus the label of any column is equal to the number of columns to its right. Therefore we have $a_{i,j} - r + (i + j) - 1$ spins $\oplus$ to the right of column $a_{i,j}$ between rows $i - 1$ and i. We deduce that $b_{i,j} = a_{i,j} - r + (i + j) - 1$ for all i, j. □

Example 4.2. Let M be the following ice-model:



At row 1, we consider the spins at the top vertical edges:

- There are two $\oplus$ to the right of the first $\ominus$ (in column 4) thus $b_{1,1} = 2$.
- There is one $\oplus$ to the right of the second $\ominus$ (in column 2) thus $b_{1,2} = 1$.
- The third $\ominus$ is in column 0 thus $b_{1,3} = 0$.

Now we consider the spins of the vertical edges between rows 1 and 2:

- There are two $\oplus$ to the right of the first $\ominus$ (in column 3) so $b_{2,1} = 2$.
- There is one $\oplus$ to the right of the second $\ominus$ (in column 1) thus $b_{2,2} = 1$.

Finally, we consider the spins of the vertical edges between rows 2 and 3: there are two $\oplus$ to the right of the first $\ominus$ (in column 2) thus $b_{3,1} = 2$.

Therefore

$$\phi_{3,4}(M) = \left\{ \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \right\}.$$

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