SKEW GROUP RINGS

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Contents

1. Introduction

Skew group rings are a natural generalization of group rings, where one does not require that the ground ring to commute with the group elements. This construction is analogous to that of a semidirect product of groups (see Example 2.6).

Besides generalizing group rings, skew group rings also appear in many areas of mathematics. For instance, the category of representations of certain wreath product algebras (see Example 2.8) are related to the geometry of Hilbert schemes [Wan02]. Similarly, they have also appeared via twisted group algebras in relation to the geometry of flag varieties [HMLSZ14, KK86]. More recently, they have also appeared in the field of categorification, where one develops a graphical categorification of the Heisenberg algebra that depends on a Frobenius algebra [RS15]. The resulting graphical category then acts naturally on modules over wreath product algebras.

There are many questions regarding group rings and skew group rings which are difficult to answer. One such problem is giving a precise description of the center of an arbitrary group ring or skew group ring. The goal of this document is not to answer these questions, but to provide some examples of skew group rings and to explore some properties of skew group rings using the language of categories. In particular, we will explore some interesting examples of adjunctions and equivalences of categories that are induced by group rings and skew group rings.

Prerequisites. This document was written as an Undergraduate Honours Project at the University of Ottawa. For the most part, it should be accessible to students with a basic understanding of algebra. However, some knowledge of category theory will be assumed for the latter parts of the document.

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2. Skew group rings

Definition 2.1 (Skew group ring)**.** Let R be a ring, G a finite group and $\varphi\colon G \to \mathrm{Aut}(R)$ a group homomorphism. The *skew group ring* of G over R induced by φ is the ring of formal sums

$$R \rtimes_\varphi G = \left\{ \sum_{g \in g} a_g g \colon a_g \in R \right\}$$

where the addition operation is component-wise and multiplication is given by $ag \cdot bh = a\varphi(g)(b)gh$ and then extending linearly.

Remark 2.2. To reduce the number of parentheses, we will sometimes write $g(b)$ or $\varphi_g(b)$ to denote $\varphi(g)(b)$.

Proposition 2.3. *The skew group ring $R \rtimes_\varphi G$ is a ring.*

Proof. By construction, the underlying abelian group is the free R-module generated by G. Thus it remains to verify the necessary axioms for multiplication as well the compatability with the addition structure.

Indeed, for $a, b, c \in R$ and $g, h, k \in G$, we have that

$$(ag \cdot bh) \cdot ck = (a\varphi_g(b)gh) \cdot ck = a\varphi_g(b)\varphi_{gh}(c)ghk = a\varphi_g(b\varphi_h(c))agk = ag \cdot (b\varphi_h(c)hk) = ag \cdot (bh \cdot ck),$$

where the third equality follows from the fact that φ is a group homomorphism. That is, the multiplication is associative. Moreover, we have that

$$ag \cdot 1_R 1_G = a\varphi_g(1_R)g1_G = ag = 1_R\varphi_{1_G}(a)1_G g = 1_R 1_G \cdot ag.$$

Thus, $R \rtimes_\varphi G$ contains a multiplicative identity.

We will proceed to prove that multiplication is distributive over addition. For all $a, b, c \in R$ and $g, h, k \in G$, we have the following set of equalities:

$$ag \cdot (bh + ck) = a(\varphi_g(b)gh + \varphi_g(c)gk) = a\varphi_g(b)gh + \varphi_g(c)gk = ag \cdot bh + ag \cdot ck,$$
$$(bh + ck) \cdot ag = (b\varphi_h(a)h + c\varphi_k(a)k)g = b\varphi_h(a)hg + c\varphi_k(a)kg = bh \cdot ag + ck \cdot ag.$$

The proof then follows by extending linearly. □

Let A be a ring and B a subring of A. Recall that A is a Frobenius extension of B if A is finitely generated and projective as a right B-module and there exists a homomorphism of (B, B)-bimodules $\mathrm{tr}\colon {}_BA_B \to {}_BB_B$ such that

- if $\mathrm{tr}(aA) = 0$ for some $a \in A$, then $a = 0$,
- for every $\varphi \in \mathrm{Hom}_B^R(A_B, B_B)$, there exists an $a \in A$ such that $\varphi = \mathrm{tr} \circ^\ell a$.

Proposition 2.4. *Let R be a ring and G be a finite group equipped with a group homomorphism $\varphi\colon G \to \mathrm{Aut}(R)$. The skew group ring $R \rtimes_\varphi G$ is a Frobenius extension of R.*

Proof. By construction, $R \rtimes_\varphi G$ is free as a R-module. Now consider the map

$$\mathrm{tr}\colon R \rtimes_\varphi G \to R, \quad ag \mapsto \begin{cases} a & \text{if } g = 1_G, \\ 0 & \text{if } g \neq 1_G. \end{cases}$$

We claim that tr is a trace map. Let $r \in R$ and $\sum_{g \in G} a_g g, \sum_{g \in G} b_g g \in R \rtimes_\varphi G$. Then

$$\mathrm{tr}\left(a \sum_{g \in G} a_g g + \sum_{g \in G} b_g g \right) = \mathrm{tr}\left(\sum_{g \in G} (aa_g + b_g)g \right)$$

$$= \sum_{g\in G} \delta_{g,1_G}(aa_g + b_g)$$

$$= a\mathrm{tr}\left(\sum_{g\in G} a_g g\right) + \mathrm{tr}\left(\sum_{g\in G} b_g g\right).$$

That is, tr is a homomorphism of left R-modules. It remains to show that tr is nondegenerate. Suppose that $a_1g_1 + \cdots + a_ng_n$ is an element in $R \rtimes_\varphi G$ such that $\mathrm{tr}\left(\sum_{i=1}^n a_ig_i(R \rtimes_\varphi G)\right) = 0$. Then for each g_i, we have that

$$0 = \mathrm{tr}((a_1g_1 + \cdots + a_ng_n)g_i^{-1}) = \mathrm{tr}(a_1g_1g_i^{-1} + \cdots + a_i1_G + \cdots + a_ng_n) = a_i.$$

□

Example 2.5 (Group rings)**.** Let R be a ring and G be a finite group and let φ be the trivial group homomorphism

$$\varphi\colon G \to \mathrm{Aut}(R), \quad g \mapsto \mathrm{id}_R \quad \text{for all } g \in G.$$

Then $R \rtimes_\varphi G$ is the usual group ring $R[G]$.

Example 2.6 (Group rings arising from the semidirect product of groups)**.** Let R be a ring and G be a finite group with a subgroup H and a normal subgroup N. Suppose that G is the semidirect product of N and H, denoted as $G = N \rtimes H$. There is a group homomorphism

$$\varphi\colon H \to \mathrm{Aut}(R[N]), \quad h \mapsto (an \mapsto ahnh^{-1}) \quad a \in R, n \in N.$$

We claim that the function

$$U\colon R[N] \rtimes_\varphi H \to R[N \rtimes H], \quad (an)h \mapsto anh \quad a \in R, n \in N, h \in H.$$

yields an isomorphism of rings. Let $(an)h$ and $(bm)k \in R[N] \rtimes_\varphi H$. The function U is clearly a homomorphism of groups with respect to addition since

$$U((an)h + (bm)k) = (a)nh + (bm)k = U((an)h) + U((bm)k) \text{ and } U((0_R1_G)1_G) = 0_R1_G.$$

Moreover, observe that

$$U((an)h \cdot (bm)k) = U((abnhmh^{-1})hk) = abnhmh^{-1}hk = abnhmk = U((an)h)U((bm)k)$$

and

$$U((1_R1_G)1_G) = 1_R1_G1_G = 1_R1_G.$$

That is, U is a homomorphism of monoids with respect to the multiplication. Lastly, it is clear that ψ is both injective and surjective, whence ψ is an isomorphism of rings.

Example 2.7 (Skew Laurent polynomial rings)**.** Let R be a ring. Recall that the ring of Laurent polynomials with coefficients in R is given by

$$R[x, x^{-1}] = \{a_{-m}x^{-m} + \cdots + a_{-1}x^{-1} + a_0 + a_1x + \cdots + a_1x^n : n, m \in \mathbb{N}, a_i \in R\}$$

where the addition is defined component wise and the multiplication is given by $a_mx^m \cdot b_nx^n = a_mb_nx^{m+n}$.

Now suppose that there is a homomorphism of groups $\varphi\colon \mathbb{Z} \to \mathrm{Aut}(R)$, say $n \mapsto (a \mapsto na)$. The skew Laurent polynomial ring induced by φ is the ring $R[x, x^{-1}, \varphi]$ whose underlying abelian group is $R[x, x^{-1}]$ but the multiplication is defined to be $a_mx^m \cdot b_nx^n = a_m\varphi(m)(b)x^{m+n}$. As the name suggests, this is an example of a skew group ring.

Example 2.8. Let $\mathbb{K}$ be a field, A a $\mathbb{K}$-algebra and let S_n denote the symmetric group on n elements. Moreover, let G be a subgroup of S_n. The algebra $A_n = A^{\otimes n} \otimes_{\mathbb{K}} \mathbb{K}[G]$ is a skew group ring where an element $\sigma \in G$ acts on $A^{\otimes n}$ via a permutation of the factors. When $G = S_n$, then the algebra A_n is called a *wreath product algebra*.

Example 2.9. Let R be a commutative ring and let $G = \langle g \rangle$ be a cyclic group of order n. Then we have a group homomorphism

$$\varphi \colon G \to \mathrm{Aut}(R^n) \quad g \mapsto ((a_1, a_2, \cdots, a_n) \mapsto (a_n, a_1, a_2, \cdots a_{n-1})).$$

We claim that $R^n \rtimes_\varphi G \cong \mathrm{Mat}_n(R)$.

Let $\mathrm{diag}(a_1, a_2, \cdots, a_n)$ denote the matrix whose diagonal is given by $(a_1, a_2, \cdots, a_n)$ and with zeroes elsewhere. Furthermore, let σ denote the permutation $(1\ 2\ \cdots\ n)$ and I_σ the matrix obtained by permuting the row vectors of the identity matrix by σ. That is, the i^{th} row of I_σ is given by the $\sigma(i)^{th}$ row of the identity matrix.

Now consider the map $\psi \colon R^n \rtimes_\varphi G \to \mathrm{Mat}_n(R)$ given by

$$(a_1, a_2, \cdots, a_n)g^\ell \mapsto \mathrm{diag}(a_1, a_2, \cdots, a_n)I_{\sigma^\ell}.$$

and then extending linearly.

We will proceed to verify that ψ is a homomorphism of rings. By construction, the map ψ preserves the addition. Furthermore, ψ also preserves the multiplicative and additive identities since

$$\psi((0_R, 0_R, \cdots 0_R)1_G) = \mathrm{diag}(0, 0, \cdots, 0) \cdot I_{\sigma^0} = \mathrm{diag}(0_R, 0_R, \cdots 0_R),$$
$$\psi((1_R, 1_R, \cdots 1_R)1_G) = \mathrm{diag}(1_R, 1_R, \cdots 1_R) \cdot I_{\sigma^0} = \mathrm{diag}(1_R, 1_R, \cdots, 1_R).$$

o Before proving that ψ preserves the multiplication, first observe that for all $(a_1, a_2, \cdots, a_n)g^\ell$ and $(b_1, b_2, \cdots, b_n)g^k \in R \rtimes_\varphi G$, we have that

$$\begin{aligned}(a_1, a_2, \cdots, a_n)g^\ell \cdot (b_1, b_2, \cdots b_n)g^k &= (a_1, a_2, \cdots, a_n)g^{\ell-1} \cdot ((1, 1, \cdots, 1)g \cdot (b_1, b_2, \cdots, b_n)g^k) \\ &= (a_1, a_2, \cdots, a_n)g^{\ell-1} \cdot (b_n, b_1, \cdots, b_{n-1})g^{k+1} \\ &= (a_1, a_2, \cdots, a_n)g^{\ell-1} \cdot (b_{\sigma^{-1}(1)}, b_{\sigma^{-1}(2)}, \cdots, b_{\sigma^{-1}(n)})g^{k+1} \\ &= \cdots = (a_1 b_{\sigma^{-\ell}(1)}, a_2 b_{\sigma^{-\ell}(2)}, \cdots, a_n b_{\sigma^{-1}(n)})g^{k+\ell}.\end{aligned}$$

Furthermore, for any matrix $A \in \mathrm{Mat}_n(R)$, the matrix $I_\sigma \cdot A$ is the matrix obtained by permutating the row vectors by σ. In particular, if we take $A = I_\sigma$, we have that $I_\sigma \cdot I_\sigma = I_{\sigma^2}$. It follows by an induction argument that $I_{\sigma^\ell} \cdot I_{\sigma^k} = I_{\sigma^{\ell+k}}$ for all $\ell, k \in \mathbb{Z}$. Using the two facts above, we obtain the equalities:

$$\begin{aligned}\psi((a_1, a_2, \cdots, a_n)g^\ell \cdot (b_1, b_2, \cdots, b_n)g^k) &= \psi((a_1(b_{\sigma^{-\ell}(1)}, a_2 b_{\sigma^{-\ell}(2)}, \cdots, a_n b_{\sigma^{-\ell}(n)})g^{\ell+k}) \\ &= \mathrm{diag}(a_1 b_{\sigma^{-\ell}(1)}, a_2 b_{\sigma^{-\ell}(2)}, \cdots, a_n b_{\sigma^{-\ell}(n)})I_{\sigma^{l+k}} \\ &= \mathrm{diag}(a_1, a_2, \cdots, a_n)I_{\sigma^\ell} \cdot \mathrm{diag}(b_{\sigma^\ell(1)}, b_{\sigma^\ell(2)}, \cdots b_{\sigma^\ell(n)})I_{\sigma^k} \\ &= \psi((a_1, a_2, \cdots, a_n)g^\ell) \cdot \psi((b_1, b_2, \cdots, b_n)g^k).\end{aligned}$$

Thus, ψ is a ring homomorphism. It remains to show that ψ is a bijection. Indeed, any matrix in $\mathrm{Mat}_n(R)$ can be obtained via

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} = \psi\left(\sum_{\ell=1}^n (a_{1,\sigma^{-\ell}(1)}, a_{2,\sigma^{-\ell}(2)}, \cdots, a_{n,\sigma^{-\ell}(n)})g^\ell\right).$$

Thus ψ is surjective. Lastly, we have that

$$0_{\mathrm{Mat}_n(R)} = \psi\left(\sum_{i=1}^n a_g g^i\right) \iff a_g = (0, 0, \cdots, 0) \text{ for all } g \in G.$$

So ψ is injective.

3. Categorical relationships from group rings

Let R be a ring and let R-Mod denote the category of R-modules whose objects are R-modules and whose morphism are R-module homomorphisms. We will also denote Grp and R-Alg as the category of groups and the category of unital associative R-algebras respectively, with the usual morphisms. The ring R induces the functor

$$R[-]\colon \text{Grp} \to R\text{-Alg}, \quad G \mapsto R[G] \text{ and } (G \xrightarrow{f} K) \mapsto (R[G] \xrightarrow{R[f]} R[K]),$$

where $R[G]$ is the usual group ring and $R[f]$ is the ring homomorphism defined to be

$$R[f](ag) = af(g) \text{ for } a \in R, g \in G,$$

and then extending linearly.

Proposition 3.1. *The functor $R[-]$ is right adjoint to the functor*

$$-^{\times}\colon R\text{-Alg} \to \text{Grp} \quad A \mapsto A^{\times} \textit{ and } f\colon A \to B \mapsto f|_{A^{\times}} : A^{\times} \to B^{\times}.$$

That is, the functor $-^{\times}$ sends a ring to its group of units and restricts the domain and codomain of a ring homomorphism to the corresponding group of units.

Proof. Let G and H be groups and $f\colon G \to H$ a group homomorphism. The map $R[f]\colon R[G] \to R[H]$ is clearly a ring homomorphism since for all $a, b \in R$ and $g, h \in G$, we have

$$R[f](ag + bh) = af(g) + bf(h) = aR[f](g) + bR[f](h),$$
$$R[f](1_R 1_G) = 1_R f(1_G) = 1_R 1_H,$$
$$R[f](ag \cdot bh) = R[f](abgh) = abf(gh) = abf(g)f(h) = af(g) \cdot bf(h) = R[f](ag) \cdot R[f](bh),$$
$$R[f](0_R 1_G) = 0_R f(1_G) = 0_R 1_H.$$

We claim that unit of the adjunction is given by the map $\eta\colon \text{id}_{\text{Grp}} \Rightarrow -^{\times} \circ R[-]$, where it is defined on an object $G \in \text{Ob}\,\text{Grp}$ by

$$\eta_G\colon G \to R[G]^{\times}, \quad g \mapsto 1_R g \text{ for } g \in G.$$

We proceed by verifying that η is a natural transformation. Let $G \in \text{Ob}\,\text{Grp}$. The map η_G is a group homomorphism because for all $g, h \in G$,

$$\eta_G(gh) = 1_R gh = 1_R g \cdot 1_R h = \eta_G(g) \cdot \eta_G(h) \text{ and } \eta_G(1_G) = 1_R 1_G.$$

Naturality of η then follows from the following commutative diagrams:

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle \eta_G}\downarrow & & \downarrow{\scriptstyle \eta_H} \\ R[G]^{\times} & \xrightarrow[R[f]|_{R[G]^{\times}}]{} & R[H]^{\times} \end{array} \qquad \begin{array}{ccc} g & \longmapsto & f(g) \\ \downarrow & & \downarrow \\ 1_R g & \longmapsto & 1_R f(g) \end{array}$$

Moreover, we claim that the counit of the adjunction is given by the map $\epsilon\colon R[-] \circ -^{\times} \Rightarrow \text{id}_{R\text{-Alg}}$, where for $A \in \text{Ob}\,R\text{-Alg}$, the map is defined by

$$\epsilon_A\colon R[A^{\times}] \to A, \quad cu \mapsto c \cdot u \quad c \in R, u \in A^{\times}.$$

and then extending linearly. By construction, ϵ_A is a homomorphism of abelian groups with respect to the addition. In fact, the map ϵ_A is a homomomorphism of R-algebras since for all $c, d \in R$ and $u, v \in A^{\times}$, we have that

$$\epsilon_A(cu \cdot dv) = \epsilon_A(cduv) = cd \cdot uv = (c \cdot u) \cdot (d \cdot v) = \epsilon_A(cu) \cdot \epsilon_A(dv),$$

$$\epsilon_A(c(du)) = \epsilon_A(cdu) = cd \cdot u = c(d \cdot u) = c\epsilon_A(du).$$

Naturality of ϵ follows from the following commutative diagrams:

$$\begin{CD} R[A^\times] @>{R[f|_{A^\times}]}>> R[B^\times] \\ @V{\epsilon_A}VV @VV{\epsilon_B}V \\ A @>>{f}> B \end{CD} \qquad \begin{CD} au @>>> af(u) \\ @VVV @VVV \\ a \cdot u @>>> f(a \cdot u) = a \cdot f(u) \end{CD}$$

Lastly, the zig-zag equations are satisfied by the commutative diagrams below for arbitrary $G \in \operatorname{Ob}\mathrm{Grp}$ and $A \in \operatorname{Ob} R\text{-Alg}$:

$$\begin{array}{ccc} R[G] & \xrightarrow{R[\eta_G]} & R[R[G]^\times] \\ & {\scriptstyle \mathrm{id}_{R[G]}} \searrow & \downarrow {\scriptstyle \epsilon_{R[G]}} \\ & & R[G] \end{array} \qquad \begin{array}{ccc} ag & \mapsto & a(1_R g) \\ & \searrow & \downarrow \\ & & ag \end{array} \qquad \begin{array}{ccc} A^\times & \xrightarrow{\eta_{A^\times}} & R[A^\times]^\times \\ & {\scriptstyle \mathrm{id}_{A^\times}} \searrow & \downarrow {\scriptstyle \epsilon_A|_{R[A^\times]^\times}} \\ & & A^\times \end{array} \qquad \begin{array}{ccc} u & \mapsto & 1_R u \\ & \searrow & \downarrow \\ & & 1_R \cdot u = u \end{array}$$

□

Recall that any group H gives rise to the groupoid $B(H)$ whose object is a formal object $*$ and $\operatorname{Mor}_{B(H)}(*, *) = H$. Let $\operatorname{Fun}(B(H), R\text{-Mod})$ denote the functor category whose objects are functors from $B(H)$ to R-Mod and whose morphisms are natural transformations.

Proposition 3.2. *There is an equivalence of categories between* $R[H]\text{-Mod}$ *and* $\operatorname{Fun}(B(H), R\text{-Mod})$.

Proof. Recall that if M is an $R[H]$-module, there exists a ring homomorphism

$$\varphi\colon R[H] \to \operatorname{End}_R(M), \quad ah \mapsto (m \mapsto ahm), \quad a \in R, h \in H, m \in M.$$

By restricting the domain of φ to H, we have a group homomorphism from H to $\operatorname{Aut}_R(M)$.

Now consider the functor $F\colon R[H]\text{-Mod} \to \operatorname{Fun}(B(H), R\text{-Mod})$ defined as follows:

$$M \mapsto (F(M)\colon B(H) \to R\text{-Mod}), \quad \begin{cases} F(M)(*) = M, & * \in \operatorname{Ob} B(H), \\ F(M)(h) = \varphi(h), & h \in H, \end{cases}$$

$$(f\colon M \to N) \mapsto (F(f)\colon F(M) \to F(N)), \quad F(f)(m) = f(m), \quad m \in M.$$

The map $F(M)$ is truly a functor since for all $h, k \in H$, we have that

$$F(M)(h \circ k) = \varphi(h \cdot k) = \varphi(h) \circ \varphi(k) = F(M)(h) \circ F(M)(k) \text{ and } F(M)(1_H) = \varphi(1_H) = \mathrm{id}_M.$$

Furthermore, for $f \in \operatorname{Mor}_{R[H]\text{-Mod}}(M, N)$, the map $F(f)$ is a natural transformation because the following diagrams commute for all $h \in \operatorname{Mor}_{B(H)}(*, *)$:

$$\begin{CD} M = F(M)(*) @>{F(M)(h)}>> F(M)(*) = M \\ @V{F(f)}VV @VV{F(f)}V \\ N = F(N)(*) @>>{F(N)(h)}> F(N)(*) = N \end{CD} \qquad \begin{CD} m @>>> hm \\ @VVV @VVV \\ f(m) @>>> f(hm) = hf(m) \end{CD}$$

To prove the proposed equivalence of categories, we will use the characterization that a functor yields an equivalence of categories if and only if it is essentially surjective on objects and fully faithful on morphisms. Let $G \in \operatorname{Ob}\operatorname{Fun}(B(H), R\text{-Mod})$. The R-module $G(*)$ can be viewed as a $R[H]$-module where the action of $R[H]$ on $G(*)$ is given by

$$ah \cdot m = aG(h)(m) \quad a \in R, h \in H, m \in G(*),$$

and then extending linearly. It follows that $G = F(G(*))$, that is, F is essentially surjective on objects. Now let $f, g \in \operatorname{Mor}_{R[H]\text{-Mod}}(M, N)$ and suppose that $F(f) = F(g)$. Then for all $m \in M$,

$$F(f)(m) = F(g)(m) \iff f(m) = g(m) \iff f = g.$$

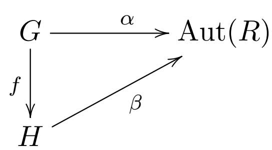
Thus, F is faithful on morphisms. Now let $G, G' \in \operatorname{Ob}\operatorname{Fun}(B(H), R\text{-Mod})$ and $\alpha\colon G \Rightarrow G'$. As before, we can view $G(*)$ and $G'(*)$ as $R[H]$-modules and α_* as a $R[H]$-module homomorphism since

$$\alpha_*(ah \cdot m) = \alpha_*(aG(h)(m)) = aG'(h)(\alpha_*(m)) = ah \cdot \alpha_*(m), \quad a \in R, h \in H, m \in G(*),$$

where the second equality follows from the naturality of α. That is, F is full. □

4. Categorical relationships from skew group rings

Let $\operatorname{Hom}(\mathrm{Grp}, \operatorname{Aut}(R))$ denote the category whose objects consist of group homomorphisms $\alpha\colon G \to \operatorname{Aut}(R)$. A morphism between two objects $\alpha\colon G \to \operatorname{Aut}(R)$ and $\beta\colon H \to \operatorname{Aut}(R)$ is a group homomorphism $f\colon G \to H$ such that $\alpha = \beta \circ f$. That is, the following diagram commutes:

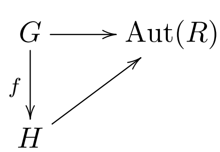


We define the composition of morphisms in the category to be the usual composition of group homomorphisms. It is an easy exercise to show that the composition of two morphisms will satisfy the commutative diagram above and that the identity morphisms are precisely the identity group homomorphisms. Lastly, composition of morphisms is associatve since the composition of group homomorphisms is associative. We will sometimes refer to objects of $\operatorname{Hom}(\mathrm{Grp}, \operatorname{Aut}(R))$ as pairs (G, α), where G is a group and $\alpha\colon G \to \operatorname{Aut}(R)$ is a homomorphism of groups.

Proposition 4.1. *Let* $\operatorname{Hom}_{\mathrm{triv}}(\mathrm{Grp}, \operatorname{Aut}(R))$ *denote the full subcategory of* $\operatorname{Hom}(\mathrm{Grp}, \operatorname{Aut}(R))$ *where the objects of* $\operatorname{Hom}_{\mathrm{triv}}(\mathrm{Grp}, \operatorname{Aut}(R))$ *are trivial group homomorphisms with codomain* $\operatorname{Aut}(R)$*. There is an equivalence of categories between* $\operatorname{Hom}_{\mathrm{triv}}(\mathrm{Grp}, \operatorname{Aut}(R))$ *and* Grp*.*

Proof. Consider the functor $F\colon \operatorname{Hom}_{\mathrm{triv}}(\mathrm{Grp}, \operatorname{Aut}(R))$ which sends $\alpha\colon G \to \operatorname{Aut}(R)$ to G and leaves $f\colon G \to H$ unaltered. Since every group G is equipped with a group homomorphism to $\operatorname{Aut}(R)$ whose image is $\{\mathrm{id}_R\}$, it is clear that F is essentially surjective on objects.

It is also immediately obvious that F is faithful on morphisms. Moreover, every group homomorphism $f\colon G \to H$ satisfies the following commutative diagram:



where the arrows without labels are the trivial group homomorphisms. Thus, F is full. □

The following proposition illustrates how the construction of skew group rings depends on the input data of a group homomorphism with codomain $\operatorname{Aut}(R)$.

Proposition 4.2. *The map* $R \rtimes -$ *that sends a group homomorphism* $\alpha\colon G \to \operatorname{Aut}(R)$ *to the skew group ring* $R \rtimes_\alpha G$ *and a group homomorphism* $f\colon G \to H$ *to the map*

$$(R \rtimes f)(ag) = af(g), \quad a \in R, g \in G$$

and then extending linearly, is a functor from $\operatorname{Hom}(\mathrm{Grp}, \operatorname{Aut}(R)) \to R\text{-Alg}$*.*

Proof. Let (G,φ) and (K,ψ) be objects in $\mathrm{Hom}(\mathrm{Grp},\mathrm{Aut}(R))$ and let $f\in\mathrm{Hom}(\mathrm{Grp},\mathrm{Aut}(R))((G,\varphi),(K,\psi))$. We claim that the map $(R\rtimes f)$ is a homomorphism of rings.

The map $(R\rtimes f)$ clearly preserves the multiplicative and additive identities since

$$(R\rtimes f)(1_R1_G)=1_Rf(1_G)=1_R1_K \text{ and } (R\rtimes f)(0_R1_G)=0_Rf(1_G)=0_R1_K.$$

Moreover, by construction, $(R\rtimes f)$ is linear. It remains to show that $(R\rtimes f)$ preserves the multiplication. Let $a,b\in R$ and $g,h\in G$. Then we have the following equalities:

$$\begin{aligned}(R\rtimes f)(ag\cdot bh)&=(R\rtimes f)(a\varphi_g(b)gh)\\&=a\varphi_g(b)f(gh)\\&=a\varphi_g(b)f(g)f(h)\\&=a\psi_{f(g)}(b)f(g)f(h)\\&=af(g)\cdot bf(h)\\&=(R\rtimes f)(ag)\cdot(R\rtimes f)(bh),\end{aligned}$$

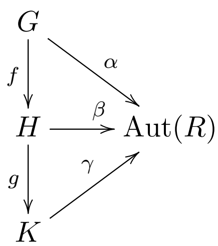
where the fourth equality follows from the property that $\varphi=\psi\circ f$.

Furthermore, for all $a\in R$ and $g\in G$, we have that

$$(R\rtimes \mathrm{id}_G)(ag)=a\mathrm{id}_G(g)=\mathrm{id}_{R\rtimes\mathrm{id}_G}(ag).$$

That is, $(R\rtimes f)$ preserves the identiy maps of $\mathrm{Hom}(\mathrm{Grp},\mathrm{Aut}(R))$.

We proceed to prove that the functor $(R\rtimes -)$ preserves the composition of morphisms. Suppose we have the following data in $\mathrm{Hom}(\mathrm{Grp},\mathrm{Aut}(R))$:



Then for $a\in A$ and $x\in G$, we have

$$(R\rtimes(g\circ f))(ax)=ag\circ f(x)=(R\rtimes g)(af(x))=(R\rtimes g)\circ(R\rtimes f)(ax).\qquad\square$$

Let G be a group. We can view the groupoid $B(G)$ as a 2-category where the 2-morphisms of $B(G)$ are elements in the center of G. For the remainder of the document, we will be viewing $B(G)$ as a 2-category consisting of only one 2-morphism; 1_G. To avoid confusion, we will denote the identity element of G as 1_G when we referring to it as a 1-morphism and as 1_{1_G} when we are referring to it as a 2-morphism.

Definition 4.3 (2-representation of a group in a category)**.** Let $\mathcal{C}$ be a 2-category. A 2-*representation* of G on $\mathcal{C}$ is a pseudo 2-functor ρ from $B(G)$ to $\mathcal{C}$.

Example 4.4 (The trivial 2-representation)**.** Let G be a group and $\mathcal{C}$ a small 2-category. The trivial 2-representation of G on $\mathcal{C}$ is the functor $\underline{1}\colon B(G)\to\mathrm{Cat}$ that sends $*$ to $\mathcal{C}$ and every g in G to the identity functor.

Proposition 4.5. *Let R be a ring, G a group and $\varphi\colon G\to\mathrm{Aut}(R)$ a group homomorphism. Let* Cat *denote the* 2*-category of small categories with the usual morphisms. The map $\rho\colon B(G)\to\mathrm{Cat}$ defined by*

$$\rho(*)=R\text{-Mod} \text{ and } \rho(g)\colon R\text{-Mod}\to R\text{-Mod},$$

$$\rho(g)(M) = {}^{g^{-1}}M \text{ for } M \in R\text{-Mod},$$
$$\rho(g)(f) = f \text{ for } f\colon M \to M,$$
$$\rho(1_{1_G})_M = \mathrm{id}_M.$$

is a 2*-representation of* G *on* Cat*.*

Proof. The map ρ is clearly a 1-functor since $\rho(1_G)(M) = {}^{1_G}M = M$ for $M \in R$-Mod and for any g and $h \in G$, we have that $\rho(gh)(M) = {}^{(gh)^{-1}}M = {}^{h^{-1}g^{-1}}M = \rho(g) \circ \rho(h)(M)$. It follows trivially that ρ preserves the vertical and horizontal composition of 2-morphisms. □

Definition 4.6 (G-action, categorical representation of G)**.** We say that a small category $\mathcal{C}$ has a *G-action* or a *categorical representation of G* if there exists a 2-representation of G on Cat; $\rho\colon B(G) \to \mathcal{C}$ such that $\rho(*) = \mathcal{C}$. We will define the 2-category of categorical representations of G $2\,\mathrm{Rep}(G)$ to be the category whose objects are categories equipped with G-actions.

Definition 4.7 (Category of G-equivariant objects)**.** We define the *category of G-equivariant objects in $\mathcal{V}$*, to be the category whose objects are pairs $(X, (\epsilon_g\colon \rho(g)(X) \to X)_{g\in G})$ where ϵ_g is an isomorphism satisfiying the following two conditions:

(i) For $g = 1_G$, we have that the following equality:
$$\epsilon_{1_G} = \phi_{1,X}\colon \rho(1_G)(X) \mapsto X$$

(ii) For any g and h in G, the following diagram commutes:
$$\begin{array}{ccc} X & \xleftarrow{\ \epsilon_g\ } & \rho(g)(X) \\ {\scriptstyle \epsilon_{gh}}\uparrow & & \uparrow{\scriptstyle \rho(g)(\epsilon_h)} \\ \rho(gh)(X) & \xleftarrow{\ \cong\ } & \rho(g)(\rho(h)(X)) \end{array}$$

A morphism f between objects $(X, (\epsilon_g\colon X \to \rho(g)(X))_{g\in G})$ and $(Y, (\eta_g\colon Y \to \rho(g)(Y))_{g\in G})$ is a morphism $f\colon X \to Y$ in $\mathcal{V}$ that intertwines with ϵ_g and η_g. That is, the following diagram commutes for each $g \in G$:
$$\begin{array}{ccc} X & \xleftarrow{\ \epsilon_g\ } & \rho(g)(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \rho(g)(f)} \\ Y & \xleftarrow{\ \eta_g\ } & \rho(g)(Y) \end{array}$$

Proposition 4.8. *The category* R-Mod^G *is equivalent to the category* $(R \rtimes_\varphi G)$-Mod*.*

Proof. Consider the functor $F\colon R\text{-Mod}^G \to (R \rtimes_\varphi G)\text{-Mod}$ that is constant on morphisms but sends an object $(M, (\epsilon_g\colon {}^{g^{-1}}M \to M)_{g\in G})$ to ${}^{\epsilon}M$, where ${}^{\epsilon}M$ is the $(R \rtimes_\varphi G)$-module defined by $rg \cdot m = r\epsilon_g(m)$ and then extending linearly. The object ${}^{\epsilon}M$ is truly a $(R \rtimes_\varphi G)$-module since we have that
$$(rg \cdot sh) \cdot m = (rg(s)gh) \cdot m = rg(s)\epsilon_{gh}(m) = rg(s)\epsilon_g(\epsilon_h(m)) = rg \cdot s\epsilon_h(m) = rg \cdot (sh \cdot m),$$
where the third equality follows from Definition 4.7 (ii). The proof for the remaining module axioms is straightforward and is left as an exercise for the reader.

We claim that the functor F is essentially surjective. We first observe that an $(R \rtimes_\varphi G)$-module M can be regarded as a R-module by restricting the action to R. Moreover, every $g \in G$ induces an R-linear map
$$\epsilon_g\colon {}^{g^{-1}}M \to M, \quad m \mapsto g \cdot m.$$

The map ϵ_g is indeed linear because for all $r \in R$ and $m \in M$, we have that

$$\epsilon_g(r \cdot m) = \epsilon_g(g^{-1}(r)m) = rg(m) = r\epsilon_g(m).$$

Thus for $M \in (R \rtimes_\varphi G)$-Mod, $F((M, (\epsilon_g \colon {}^{g^{-1}}M \to M)_{g \in G})) = M$. That is, F is essentially surjective.

Finally, we may regard an R-module homomorphism

$$f \colon (M, (\epsilon_g \colon {}^{g^{-1}}M \to M)_{g \in G}) \to (N, (\eta_g \colon {}^{g^{-1}}N \to N)_{g \in G})$$

as an $(R \rtimes_\varphi G)$-module homomorphism from ${}^\epsilon M$ to ${}^\eta N$. Indeed, for $r \in R, g \in G$ and $m \in M$, we have that

$$f(rg \cdot m) = f(r\epsilon_g(m)) = rf(\epsilon_g(m)) = r\eta_g(f(m)) = rg \cdot f(m),$$

where the third equality follows from the fact f intertwines with ϵ_g and η_g. It follows that F is fully faithful on morphisms, thus F induces an equivalence of categories between R-Mod^G and $(R \times_\varphi G)$-Mod. □

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