PROJECTIVE REPRESENTATIONS OF GROUPS

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Abstract. We present an introduction to the basic concepts of projective representations of groups and representation groups, and discuss their relations with group cohomology. We conclude the text by discussing the projective representation theory of symmetric groups and its relation to Sergeev and Hecke-Clifford Superalgebras.

Contents

Introduction

The theory of group representations emerged as a tool for investigating the structure of a finite group and became one of the central areas of algebra, with important connections to several areas of study such as topology, Lie theory, and mathematical physics. Schur was

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the first to realize that, for many of these applications, a new kind of representation had to be introduced, namely, projective representations. The theory of projective representations involves homomorphisms into projective linear groups. Not only do such representations appear naturally in the study of representations of groups, their study showed to be of great importance in the study of quantum mechanics.

Schur [Sch04] laid the foundations for the general theory of projective representations, showing the existence of a certain finite central extension $\tilde{G}$ of a group G, which is called a representation group of G. Such a central extension reduces the problem of determining all projective representations of G to the determination of all linear representations of $\tilde{G}$. In [Sch07, Sch11], Schur gave an estimation for the number of non-isomorphic representation groups and determined all irreducible projective representations of the symmetric and alternating groups.

The representation theory of symmetric group is a special case of the representation theory of finite groups that provides a vast range of applications, ranging from theoretical physics, through geometry and combinatorics. Recently, new approaches to the study of projective representations of the symmetric group have been born, including the study of Sergeev and Hecke-Clifford superalgebras.

The goal of this paper is to give an introduction to the theory of projective representations of groups accessible to undergraduates. We assume only basic knowledge of group theory and linear algebra.

In the first section, we introduce the basic concepts of group cohomology, and we give some important properties of the 2^{nd}-cohomology group.

In the second section, we define projective representations and the concept of equivalence. Then we define Schur multipliers and show their relation with 2^{nd}-cohomology groups. We finish the section by showing that the cohomology class associated with a projective representation depends only on the equivalence class of the projective representation.

We define central extensions of a group in Section 3, and and show the bijection between the set of equivalence classes of central extensions and the 2^{nd}-cohomology group.

In Section 4, we show the existence of representation groups and the equivalence of two possible definitions. At the end of the section, we discuss the uniqueness of representation groups in the specific case of perfect groups.

We finish the text by discussing projective representations of symmetric groups in the last section. We discuss two representation groups for S_n (for $n \geqslant 4$), that are isomorphic only for $n = 6$, developing the discussion to show how the study of Sergeev and Hecke-Clifford superalgebras is equivalent to the study of spin representations of symmetric groups.

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1. Group cohomology

When we study projective representations, the cohomology of groups naturally appears. So we will introduce in the first subsection the basic concepts of group cohomology, following the approach of MacLane and Eilenberg in [EM47], and in the second subsection some important properties of the 2^{nd}-cohomology group.

1.1. **Cohomology groups.**

Throughout this subsection, let G be an arbitrary group and M an abelian group. In groups cohomology studies, there exists a more general approach where G acts on M. But as it is not necessary for our study, we will not consider such action in the following definitions. Since in the next sections M will be a multiplicative abelian group, we will use the multiplicative notation in this document.

Definition 1.1.1 (n-cochain)**.** Let n be a positive integer. A *n-cochain* of G in M is a set map $f\colon G^n \to M$. We define $C^n(G, M)$ to be the abelian group of all n-cochains of G in M, where the multiplication, identity and inverse, are given, respectively, by:

(a) $fg\colon (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)g(x_1, \dots, x_n)$, for all $f, g \in C^n(G, M)$;
(b) $1\colon (x_1, \dots, x_n) \mapsto 1_M$;
(c) $f^{-1}\colon (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)^{-1}$, for all $f \in C^n(G, M)$;

A 0-*cochain* is defined to be an element of M.

Remark 1.1.2. The commutativity of $C^n(G, M)$ follows from the fact that its elements take values in M, which is an abelian group.

Definition 1.1.3 (Coboundary)**.** The *coboundary* of an n-cochain f is the $(n+1)$-cochain $\delta^n f$, defined by:

$$(\delta^n f)(x_1, \dots, x_{n+1}) = f(x_2, \dots, x_{n+1}) \left(\prod_{i=1}^{n} f(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1})^{(-1)^i} \right) f(x_1, \dots, x_n)^{(-1)^{n+1}},$$

for all $(x_1, \dots, x_{n+1}) \in G^{n+1}$.

The coboundary has two properties that can be verified directly:

Lemma 1.1.4. *For all n-cochains f and g, we have:*

(a) $\delta^n(fg) = (\delta^n f)(\delta^n g)$;
(b) $\delta^{n+1}(\delta^n f) = 1$.

Proof. Part (a) follows by the definition of δ and the commutativity of $C^n(G, M)$.

For any $f \in C^n(G, M)$ define the set map $\hat{f}\colon G^{n+1} \to M$ by setting

$$\hat{f}(x_1, \dots, x_{n+1}) = f(x_1^{-1}x_2, x_2^{-1}x_3, \dots, x_n^{-1}x_{n+1}),$$

for all $(x_1, \dots, x_{n+1}) \in G^{n+1}$. Notice that $\hat{f}$ satisfies

$$\hat{f}(yx_1, \dots, yx_{n+1}) = \hat{f}(x_1, \dots, x_{n+1}),$$

for all $(x_1, \dots, x_{n+1}) \in G^{n+1}$ and $y \in G$. For any $g \in C^{n+1}(G, M)$ define the $(n+2)$-cochain $\partial^n g\colon G^{n+2} \to M$ by setting

$$\partial^n g(x_1, \dots, x_{n+2}) = \prod_{i=1}^{n+2} g(x_1, \dots, \hat{x}_i, \dots, x_{n+2})^{(-1)^{i+1}},$$

for all $(x_1, \dots, x_{n+2}) \in G^{n+2}$, where $\hat{x}_i$ indicates that the variable x_i has been omitted. It can be verified, after some combinatorial calculations, that $\partial^{n+1}\partial^n \hat{f} = 1$, for all $f \in C^n(G, M)$.

By [EM47, Section 2], for all $(x_1, \ldots, x_{n+1}) \in G^{n+1}$, we have that

$$\delta^n f(x_1, \ldots, x_{n+1}) = \partial^n \hat{f}(1, x_1, x_1x_2, \ldots, x_1 \cdots x_{n+1}). \tag{1.1}$$

Therefore, by (1.1) and the fact that $\partial^{n+1}\partial^n \hat{f} = 1$, for all $f \in C^n(G, M)$, part (b) holds. □

It follows from Lemma 1.1.4(a) that the coboundary map $\delta^n \colon C^n(G, M) \to C^{n+1}(G, M)$ is a group homomorphism, for all non-negative integers n.

Definition 1.1.5 (The n-cocycles and n-coboundaries sets)**.** Let n be a positive integer. Then we define $Z^n(G, M) = \operatorname{Ker} \delta^n$ and $B^n(G, M) = \operatorname{Im} \delta^{n-1}$. We call the elements of $Z^n(G, M)$ *n-cocycle.* The elements of $B^n(G, M)$ are called *n-coboundaries*;

It follows from Lemma 1.1.4(b) that $\operatorname{Im} \delta^{n-1} \subseteq \operatorname{Ker} \delta^n$ and therefore $B^n(G, M)$ is a subgroup of $Z^n(G, M)$. Thus, we can define:

Definition 1.1.6 (n^{th}-Cohomology group)**.** Let n be a non-negative integer. Then the *n^{th}-cohomology group* is defined to be the quotient group:

$$H^n(G, M) = \frac{Z^n(G, M)}{B^n(G, M)},$$

and its elements are called *cohomology classes*.

Two cocycles contained in the same cohomology class are called to be *cohomologous*.

We denote by $\bar{\cdot} \colon Z^n(G, M) \to H^n(G, M)$ the canonical projection that takes any n-cocycle ρ to its cohomology class $\overline{\rho}$.

Example 1.1.7 (2^{nd}-Cohomology group)**.** Let us describe the 2-cocycles and 2-coboundaries more explicitly.

We have that $\rho \in Z^2(G, M)$, if and only if $\delta\rho = 1$. Therefore a set map $\rho \colon G \times G \to \mathbb{K}^\times$ is a 2-cocycle if and only if:

$$\rho(h, k)\rho(gh, k)^{-1}\rho(g, hk)\rho(g, h)^{-1} = 1, \quad \text{for all } g, h, k \in G. \tag{1.2}$$

Now let $\rho \in B^2(G, M)$. Then there exist a 1-cochain $f \colon G \to M$ such that:

$$\rho(g, h) = f(h)f(gh)^{-1}f(g), \quad \text{for all } g, h \in G.$$

Then two 2-cocycles ρ and ρ' are cohomologous if, and only if, there is a 1-cochain $f \colon G \to M$ such that:

$$\rho'(g, h) = f(g)f(gh)^{-1}f(h)\rho(g, h), \quad \text{for all } g, h \in G. \tag{1.3}$$

1.2. **2^{nd}-Cohomology group.** From now, let $\mathbb{K}$ be a field and $\mathbb{K}^\times$ be its multiplicative group.

Lemma 1.2.1. *Any 2-cocycle $\rho \in Z^2(G, M)$ satisfies, for all $g, h \in G$:*

$$\rho(g, 1) = \rho(1, 1) = \rho(1, h) \tag{1.4}$$

and

$$\rho(g, g^{-1}) = \rho(g^{-1}, g). \tag{1.5}$$

In particular, for all $\alpha \in H^2(G, M)$, there is a 2-cocycle ω representative of α such that $\omega(1, 1) = 1$ and satisfy (1.4) and (1.5).

Proof. Let ρ be a 2-cocycle. We have that ρ satisfies (1.2) for all $g, h, k \in G$. Replacing g by 1 and k by $h^{-1}g$ in (1.2), we have $\rho(h, h^{-1}g)\rho(h, h^{-1}g)^{-1}\rho(1, g)\rho(1, h)^{-1} = 1$, for all $g, h \in G$. Thus,

$$\rho(1, h) = \rho(1, g), \quad \text{for all } g, h \in G. \tag{1.6}$$

Now, replacing k by 1 and g by gh^{-1} in (1.2), we have $\rho(h, 1)\rho(g, 1)^{-1}\rho(gh^{-1}, h)\rho(gh^{-1}, h)^{-1} = 1$, for all $g, h \in G$. Thus,

$$\rho(g, 1) = \rho(h, 1), \quad \text{for all } g, h \in G. \tag{1.7}$$

Therefore, by (1.6) and (1.7), we have:

$$\rho(g, 1) = \rho(1, 1) = \rho(1, h),$$

for all $g, h \in G$. Therefore ρ satisfies (1.4).

Replacing k by g and h by g^{-1} in (1.2), we have, for all $g \in G$:

$$\rho(g^{-1}, g)\rho(1, g)^{-1}\rho(g, 1)\rho(g, g^{-1})^{-1} = 1.$$

Therefore, by (1.4):

$$\rho(g, g^{-1}) = \rho(g^{-1}, g),$$

for all $g \in G$. Hence ρ satisfies (1.5).

Now, let $\alpha \in H^2(G, M)$ and ρ be any 2-cocyle representative of α. Define $\omega\colon G^2 \to M$ by setting $\omega(g, h) = a\rho(g, h)$, for all $g, h \in G$, where $a = \rho(1, 1)^{-1} \in M$. Then, considering $f\colon G \to M$ as the constant map a^{-1}, we have:

$$\omega(g, h)\omega(gh, k)^{-1}\omega(g, hk)\omega(g, h)^{-1} = a\rho(g, h)a^{-1}\rho(gh, k)^{-1}a\rho(g, hk)a^{-1}\rho(g, h)^{-1} = 1,$$
$$\rho(g, h)\omega(g, h)^{-1} = a^{-1} = f(g)f(gh)^{-1}f(h),$$

for all $g, h \in G$. Therefore ω is a 2-cocycle cohomologous to ρ such that $\omega(1, 1) = 1$. ☐

Theorem 1.2.2 ([CR06, Theorem 53.3])**.** *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p \in \mathbb{N}$ and $H = H^2(G, \mathbb{K}^\times)$. Then the following statements are true:*

(a) The order of every element of H divides the order of G.
(b) Every element α in H can be represented by a 2-cocycle ρ such that $\rho(1, 1) = 1$ and $\rho(g, h)$ is an e-th root of $1 \in \mathbb{K}$, for all $g, h \in G$, where e is the order of α.
(c) H has finite order not divisible by p.

Proof. Let ρ be a 2-cocycle and let n be the order of G. By Lemma 1.2.1, we can assume $\rho(1, 1) = 1$.

Define the set map $f\colon G \to \mathbb{K}^\times$ by setting $f(g) = \prod_{h \in G} \rho(g, h)$, for all $g \in G$. Then, we have:

$$\begin{aligned}
\frac{f(g)f(h)}{f(gh)} &= \frac{\prod_{r \in G} \rho(g, r) \prod_{s \in G} \rho(h, s)}{\prod_{t \in G} \rho(gh, t)} \\
&= \prod_{r \in G} \left(\frac{\rho(g, hr)\rho(h, r)}{\rho(gh, r)} \right) \\
&= \rho(g, h)^n \prod_{r \in G} \left(\frac{\rho(g, hr)\rho(h, r)}{\rho(gh, r)\rho(g, h)} \right) \\
&= \rho(g, h)^n, \qquad \text{(by (1.2))}, \qquad (1.8)
\end{aligned}$$

for all $g, h \in G$. Therefore, since $\bar{\cdot}$ is a group homomorphism, it follows from (1.8) that $\overline{\rho}^n = 1 \in H$, and that proves (a). Furthermore, since $\rho(1,1) = 1$, notice that we have $f(1) = 1$.

Now, let e be the order of $\overline{\rho}$, and if $p > 0$, write $e = p^a q$, where $a, q \in \mathbb{N}$ and $p \nmid q$. Then, since $\overline{\rho^e} = \overline{\rho}^e = 1 \in H$, there is a set map $f' \colon G \to \mathbb{K}^\times$ such that $\rho(g,h)^e = f'(g)f'(gh)^{-1}f'(h)$, for all $g, h \in G$.

Since $\mathbb{K}$ is algebraically closed, there exist a set map $f'' \colon G \to \mathbb{K}^\times$ such that $f''(1) = 1$ and $f''(g)^{p^a} = f'(g)$, for all $g \in G$, satisfying:

$$\rho(g,h)^q = f''(g)f''(gh)^{-1}f''(h),$$

for all $g, h \in G$. Thus, $\overline{\rho^q} = 1$. Since $e = p^a q$ is the order of $\overline{\rho}$, follows from $\overline{\rho}^q = 1$ that $p^a = 1$, hence $p \nmid e$.

Now, for each $g \in G$, take $\alpha(g) \in \mathbb{K}^\times$ such that $\alpha(g)^e = f'(g)^{-1}$, imposing $\alpha(1) = 1$, since $f'(1) = 1$, and define the map $\rho' \colon G \times G \to \mathbb{K}^\times$ by setting

$$\rho'(g,h) = \alpha(g)\alpha(gh)^{-1}\alpha(h)\rho(g,h),$$

for all $g, h \in G$. Notice that $\rho'(1,1) = 1$. It is easy to see that ρ' is a 2-cocycle cohomologous to ρ satisfying:

$$\begin{aligned}
\rho'(g,h)^e &= \alpha(g)^e\alpha(gh)^{-e}\alpha(h)^e\rho(g,h)^e \\
&= f'(g)^{-1}f'(gh)f'(h)^{-1}f'(g)f'(gh)^{-1}f'(h)^{-1} \\
&= 1,
\end{aligned}$$

for all $g, h \in G$. Therefore, there is a 2-cocycle ρ' representative of $\overline{\rho}$, such that $\rho'(1,1) = 1$ and $\rho'(g,h)$ is a e-th root of $1 \in \mathbb{K}$, for all $g, h \in G$. This proves (b).

Now, since G is finite and for any $e|n$, the number of e-th roots of 1 are finite, there are at most a finite number of 2-cocycles ρ whose values $\rho(g,h)$ are an e-th root of $1 \in K$, for all $g, h \in G$. Therefore, since all cohomology class, whose order is e, can be represented by a 2-cocycle as above, there are at most a finite number of cohomology classes in H of order e. Since $e|n$, it follows that there are at most a finite number of cohomology classes in H, i.e, H is a finite group. Furthermore, because no elements of H are divisible by the characteristic of $\mathbb{K}$, it follows that $p \nmid |H|$. And this concludes the proof of (c). □

By Lemma 1.2.1 and Theorem 1.2.2, from now on we will assume that all 2-cocycles ρ satisfy $\rho(1,1) = 1$.

To finish the section, we define a group homomorphism that will be useful for our studies.

Definition 1.2.3. Let $\rho \in Z^2(G, M)$ be a 2-cocycle. Then we define the group homomorphism.

$$\hat{\rho} \colon \operatorname{Hom}(M, N) \to H^2(G, N)$$

by setting, for all $\alpha \in \operatorname{Hom}(M, N)$, $\hat{\rho}(\alpha) = \overline{\alpha \circ \rho}$, the cohomology class of $\alpha \circ \rho$.

2. Projective Representations

Throughout this section, let V a $\mathbb{K}$-vector space and $\mathrm{GL}(V)$ the *general linear group* of V. We will identify $\mathbb{K}^\times$ with $\mathbb{K}^\times \operatorname{Id}_V$. Thus, the *projective general linear group* is defined to be the quotient $\mathrm{PGL}(V) = \dfrac{\mathrm{GL}(V)}{\mathbb{K}^\times}$ and we denote the *canonical projection* by $\pi \colon \mathrm{GL}(V) \to$

PGL(V). Sometimes, when it is necessary to distinguish the vector space of V, we will denote this projection by π_V.

2.1. **Projective representation.** We now introduce projective representations. Usually, a projective representation is defined in terms of general linear group and Schur multiplier, and after it is shown the equivalent definition in terms of projective general linear group. In this section we will make the opposite direction: first we define as Yamazaki in [Yam64] and show the equivalence with the usual definition such as Karpilovsky in [Kar87], and Hoffman and Humphreys in [HH92].

After that, we define two concepts of equivalence of projective representations allowing the study of their classifications.

Definition 2.1.1 (Projective representation)**.** A *projective representation* of a group G on a vector space V is a group homomorphism

$$P\colon G \to \mathrm{PGL}(V).$$

Proposition 2.1.2. *Let P be a projective representation of G on V. Then, there are set maps $P'\colon G \to \mathrm{GL}(V)$ and $\rho\colon G \times G \to \mathbb{K}^\times$ such that*

$$P'(g)P'(h) = \rho(g,h)P'(gh), \quad \textit{for all } g,h \in G. \tag{2.1}$$

Conversely, if there are set maps P' and ρ satisfying (2.1), then there exists a unique homomorphism $P\colon G \to \mathrm{PGL}(V)$ such that $P(g) = \pi P'(g)$, for all $g \in G$.

Proof. Let X be a set of coset representatives of GL(V) in PGL(V), and define $P'\colon G \to \mathrm{GL}(V)$ by setting for each $g \in G$, $P'(g)$ as the unique element of X such that $\pi P'(g) = P(g)$.

Now, let $g,h \in G$. Then we have $P'(gh)\mathbb{K}^\times = P'(g)P'(h)\mathbb{K}^\times$, which implies that there exists a unique $\rho(g,h) \in \mathbb{K}^\times$ such that $\rho(g,h)P'(gh) = P'(g)P'(h)$.

Conversely, if we have set maps P' and ρ satisfying (2.1), define $P\colon G \to \mathrm{PGL}(V)$ as $P = \pi P'$. Then, for all $g,h \in G$,

$$\begin{aligned} P(gh) &= \pi(P'(gh)) = \pi(\rho(g,h)^{-1}P'(g)P'(h)) \\ &= \pi(P'(g)P'(h)) = \pi(P'(g))\pi(P'(h)) = P(g)P(h). \end{aligned}$$

Therefore P is an group homomorphism, i.e., a projective representation of G on V. □

Remark 2.1.3. In the proof of Proposition 2.1.2, define $Y = \{\rho(1,1)x \mid x \in X\}$ and let Q' be the section of P corresponding to Y, and define $\rho'(g,h) = \rho(g,h)/\rho(1,1)$, for all $g,h \in G$. Thus we obtain Q' and ρ' satisfying (2.1) and $\rho'(1,1) = 1$. Therefore, we lose no generality in assuming that $\rho(1,1) = 1$, which we will do from now on.

Proposition 2.1.2 gives a new way to see a projective representation which is equivalent. So, if we have two set maps $P\colon G \to \mathrm{GL}(V)$ and $\rho\colon G \times G \to \mathbb{K}^\times$ satisfying equation (2.1), we will call P a *projective representation* or a *ρ-representation*.

Given the set X of representatives of GL(V) in PGL(V) and P' as in Proposition 2.1.2, the choice of ρ is unique, by construction. But there may be more than one definition for P', depending on the choice of the set X. We call the set map P' a *section* of P, and ρ is called a *Schur multiplier* for the section P'. By Remark 2.1.3, we will assume that a Schur multiplier ρ satisfies $\rho(1,1) = 1$.

Let now $\phi\colon V_1 \to V_2$ be an isomorphism between two vector spaces. We have that conjugation by ϕ induces an group isomorphism from $\mathrm{GL}(V_1)$ to $\mathrm{GL}(V_2)$. Since this isomorphism preserves the scalar matrices, it induce an isomorphism from $\mathrm{PGL}(V_1)$ to $PGL(V_2)$.

Now we can define the notion of equivalence of projective representaitons.

Definition 2.1.4 (Projective equivalence)**.** Let $P_1\colon G \to \mathrm{PGL}(V_1)$ and $P_2\colon G \to \mathrm{PGL}(V_2)$ be two projective representations of a group G on $\mathbb{K}$-vector spaces V_1 and V_2 respectively. We say that P_1 and P_2 are *projectively equivalent* if exists isomorphism $\phi\colon V_1 \to V_2$ such that:

$$\phi \circ P_1(g) \circ \phi^{-1} = P_2(g),$$

for all g in G.

The next lemma show how projective equivalence and sections of projective representations are related.

Lemma 2.1.5. *Let $P_1\colon G \to \mathrm{PGL}(V_1)$ and $P_2\colon G \to \mathrm{PGL}(V_2)$ be two projective representations. Let $P_1'\colon G \to \mathrm{GL}(V_1)$ and $P_2'\colon G \to \mathrm{GL}(V_1)$ be two sections for P_1 and P_2, respectively. Then the following statements are equivalent:*

(a) P_1 and P_2 are equivalent projective representations

(b) There is a set map $c\colon G \to \mathbb{K}^\times$ and a linear isomorphism $\phi\colon V_1 \to V_2$ such that

$$\phi P_1'(g) = c(g) P_2'(g)\phi, \quad \text{for all } g \in G. \tag{2.2}$$

Proof. Let P_1 and P_2 be equivalent projective representations. Then there exists an isomorphism $\phi\colon V_1 \to V_2$ satisfying $\phi \circ P_1(g) \circ \phi^{-1} = P_2(g)$, for all $g \in G$. Then, since conjugation by ϕ commutes with canonical projections, we have $\pi_{V_2}(\phi P_1'(g)\phi^{-1}) = \pi_{V_2}(P_2'(g))$, for all $g \in G$. Therefore, for all $g \in G$, there exists $c(g) \in \mathbb{K}^\times$ such that:

$$\phi P_1'(g)\phi^{-1} = c(g) P_2'(g).$$

Then, there is a set map $c\colon G \to \mathbb{K}^\times$ satisfying (2.2).

Conversely, let P_1' and P_2' satisfy (2.2) for some set map $c\colon G \to \mathbb{K}^\times$ and an isomorphism $\phi\colon V_1 \to V_2$. Then, for all $g \in G$ we have:

$$\phi P_1(g)\phi^{-1} = \pi_{V_2}(\phi P_1'(g)\phi^{-1}) = \pi_{V_2}(c(g)P_2'(g)) = P_2(g).$$

Therefore, P_1 and P_2 are equivalent projective representations. □

Lemma 2.1.5 gives a new way to view equivalency of projective representations. So, if we have P_1' and P_2' two sections for two projective representations $P_1\colon G \to \mathrm{PGL}(V_1)$ and $P_2\colon G \to \mathrm{PGL}(V_2)$, respectively, that satisfy the equation (2.2), we will call P_1' and P_2' equivalent projective representations.

In particular, two linear representations $T\colon G \to \mathrm{GL}(V)$ and $U\colon G \to \mathrm{GL}(W)$ satisfying (2.2) will be also called *projecitively equivalent*.

Definition 2.1.6 (Linear equivalence of ρ-representations)**.** Let ρ be a 2-cocycle. Two ρ-representations $P_1\colon G \to \mathrm{GL}(V)$ and $P_2\colon G \to \mathrm{GL}(W)$ are *linearly equivalent* if there is an isomorphism $\phi\colon V \to W$ satisfying:

$$\phi P_1(g)\phi^{-1} = P_2(g),$$

for all $g \in G$.

2.2. **Schur multiplier and cohomology class.** In this subsection we will discuss a little more about the Schur multiplier and show how group cohomology appears naturally.

Let $P\colon G \to \mathrm{GL}(V)$ be a projective reresentations with Schur multiplier ρ. From the associativity of G, we have:

$$\begin{aligned}\rho(g,h)\rho(gh,k)P(ghk) &= \rho(g,h)P(gh)P(k)\\ &= P(g)P(h)P(k)\\ &= \rho(h,k)P(g)P(hk)\\ &= \rho(g,hk)\rho(h,k)P(ghk),\end{aligned}$$

for all $g,h,k \in G$. Thus, for all $g,h,k \in G$, we have $\rho(g,hk)\rho(h,k) = \rho(gh,k)\rho(g,h)$, or equivalently:

$$\rho(h,k)\rho(gh,k)^{-1}\rho(g,hk)\rho(g,h)^{-1} = 1.$$

Therefore, from the equation (1.2), we can conclude that a Schur multiplier ρ is a 2-cocycle in $Z^2(G,\mathbb{K}^\times)$.

Now, let Q and Q' be two sections for a projective representation $P\colon G \to \mathrm{PGL}(V)$, and ρ, ρ' be their respective Schur multipliers. Then, for all $g \in G$, Q and Q' satisfy $\pi(Q(g)) = \pi(Q'(g))$. Therefore, for each $g \in G$ there is $f(g) \in \mathbb{K}^\times$ such that $Q'(g) = f(g)Q(g)$. But for all $g,h \in G$, we have:

$$\begin{aligned}\rho'(g,h)f(gh)Q(gh) &= \rho'(g,h)Q'(gh)\\ &= Q'(g)Q'(h)\\ &= f(g)f(h)Q(g)Q(h)\\ &= f(g)f(h)\rho(g,h)Q(gh),\end{aligned}$$

Thus, for all $g,h \in G$, f, ρ and ρ' satisfy

$$\rho'(g,h) = f(g)f(gh)^{-1}f(h)\rho(g,h),$$

and hence it follows from equation (1.3) that ρ and ρ' are cohomologous 2-cocycles. Therefore, the cohomology class $\overline{\rho}$ of ρ is independent on the choice of the section Q of P.

Definition 2.2.1 (Cohomology class associated)**.** Let $P\colon G \to \mathrm{PGL}(V)$ be a projective representation and ρ a Schur multiplier of a section $P'\colon G \to \mathrm{GL}(V)$ of P. Then the cohomology class $\overline{\rho}$ of ρ is called the *cohomology class associated* to the projective representation P and will be denoted by C_P.

Actually, we can prove that there is a section for any Schur multiplier. Precisely:

Proposition 2.2.2. *Let $P\colon G \to \mathrm{PGL}(V)$ be a projective representation with associated cohomology class $C_P \in H^2(G,\mathbb{K}^\times)$ and let $\rho \in Z^2(G,\mathbb{K}^\times)$ be any 2-cocycle representative of C_P. Then there is a section $P'\colon G \to \mathrm{GL}(V)$ for P such that its Schur multiplier is ρ.*

Proof. Let $P''\colon G \to \mathrm{GL}(V)$ be any section for P and $\omega \in Z^2(G,\mathbb{K}^\times)$ its Schur multiplier. Thus, since the cohomology class associated to P is independent of the section, we have that ρ and ω are cohomologous. Therefore, there is a set map $f\colon G \to \mathbb{K}^\times$ such that $\rho(g,h) = f(g)f(h)f(gh)^{-1}\omega(g,h)$, for all $g,h \in G$. Then, for all $g,h \in G$, we have:

$$P''(g)P''(h) = \omega(g,h)P''(gh) = f(g)^{-1}f(h)^{-1}f(gh)\rho(g,h)P''(gh).$$

Thus, define the set map $P' \colon G \to \mathrm{GL}(V)$ by setting $P'(g) = f(g)P''(g)$, for all $g \in G$. Clearly P' is a section for P with Schur multiplier ρ. □

Having introduced the concept of the cohomology class associated to a projective representation, we can explain how projective representations and twisted group algebras are related.

Definition 2.2.3 (Twisted group algebra)**.** Consider a 2-cocycle $\rho \in Z^2(G, \mathbb{K}^\times)$. We define the group algebra $\mathbb{K}_\rho G$ to be the $\mathbb{K}$-vector space with base $\{g \in G\}$ and multiplication given by:

$$g \cdot h = \alpha(g,h)gh,$$

for all $g, h \in G$, and extending linearly. We call $\mathbb{K}_\rho G$ the *twisted group algebra* of G by ρ.

See [Kar85, Lemma 3.2.1] for a proof that $\mathbb{K}_\rho G$ is well defined. It can be proved that two 2-cocycles ρ, ω are cohomologous if, and only if, their corresponding twisted group algebras $\mathbb{K}_\rho G$ and $\mathbb{K}_\omega G$ are isomorphic algebras (see [Kar85, Lemma 3.2.2]).

We also have the following theorem:

Theorem 2.2.4 ([Kar85, Theorem 3.2.5])**.** *Let ρ be a 2-cocycle. Then there is a bijective correspondence between ρ-representations of G and $\mathbb{K}_\rho G$-modules. This correspondence preserves sums and bijectively maps linearly equivalent (respectively irreducible, completely reducible) representations into isomorphic (respectively irreducible, completely reducible) modules.*

Remark 2.2.5 (Existence of projective representations)**.** Let ρ be a 2-cocycle. Taking $\mathbb{K}_\rho G$ to be the regular $\mathbb{K}_\rho G$-module, Theorem 2.2.4 allows us to conclude that there exists a projective representation of G with associated cohomology class $\bar{\rho}$. Therefore, since $\mathbb{K}_\rho G$ and $\mathbb{K}_\omega G$ are isomorphic for cohomologous 2-cocycle ρ and ω, we can conclude that for any cohomology class $c \in H^2(G, \mathbb{K}^\times)$, there is a projective representation associated to it.

2.3. **Equivalent projective representations.** Now we can prove that the associated cohomology class is invariant up to projective equivalence.

Lemma 2.3.1. *Let $P \colon G \to \mathrm{PGL}(V_1)$ and $Q \colon G \to \mathrm{PGL}(V_2)$ be two equivalent projective representations. Then their associated cohomology classes are equal.*

Proof. Let $P' \colon G \to \mathrm{GL}(V_1)$ and $Q' \colon G \to \mathrm{GL}(V_2)$ be two sections for P and Q, respectively, with respective Schur multipliers ρ and ω. Then, by Lemma 2.1.5, there is an isomorphism $\phi \colon V_1 \to V_2$ and a set map $c \colon G \to \mathbb{K}^\times$ such that:

$$\phi P'(g)\phi^{-1} = c(g)Q'(g),$$

for all $g \in G$. Then we have:

$$\begin{aligned}
c(g)c(h)Q'(g)Q'(h) &= (c(g)Q'(g))(c(h)Q'(h)) \\
&= (\phi P'(g)\phi^{-1})(\phi P'(h)\phi^{-1}) \\
&= \phi P'(g)P'(h)\phi^{-1} \\
&= \phi\rho(g,h)P'(gh)\phi^{-1} \\
&= \rho(g,h)c(gh)Q'(gh) \\
&= c(gh)\omega(g,h)^{-1}\rho(g,h)Q'(g)Q'(h),
\end{aligned}$$

for all $g, h \in G$. Thus:

$$\omega(g,h)^{-1}\rho(g,h) = c(g)c(gh)^{-1}c(h),$$

for all $g, h \in G$, which implies that ρ and ω are cohomologous. Therefore P and Q are associated to the same cohomology class. □

A question that arises naturally is whether classes of equivalence of projective representations are uniquely determined by a class of cohomology. For now we can only answer part of the question:

Proposition 2.3.2 ([Kar87, Lemma 2.3.1(ii)]). *Let $P\colon G \to \mathrm{PGL}(V)$ be a projective representation. Then the following statements are equivalent:*

(a) The cohomology class associated to P is the trivial class;
(b) There is a linear representation $T\colon G \to \mathrm{GL}(V)$ such that P is projective equivalent to $Q = \pi \circ T$, the projective representation induced by T.

Proof. Supose that P is associated to the trivial cohomology class. Let $P'\colon G \to \mathrm{GL}(V)$ be a section for P and $\rho\colon G \times G \to \mathbb{K}^\times$ be its Schur multiplier. Then the cohomology class $\overline{\rho}$ given by the 2-cocycle ρ is trivial, by hypothesis. Therefore, there is a set map $f\colon G \to \mathbb{K}^\times$ such that:

$$\rho(g,h)^{-1} = f(g)f(gh)^{-1}f(h),$$

for all $g, h \in G$. Then, for all $g, h \in G$, we have:

$$\begin{aligned} P'(g)P'(h) &= \rho(g,h)P'(gh) \\ &= f(g)^{-1}f(gh)f(h)^{-1}P'(gh) \\ &\Leftrightarrow \\ (f(g)P'(g))(f(h)P'(h)) &= f(gh)P'(gh) \end{aligned}$$

Thus, defining the set map $Q\colon G \to \mathrm{GL}(V)$ by setting $Q(g) = f(g)P'(g)$, for all $g \in G$, we have, from the equation above:

$$Q(gh) = Q(g)Q(h),$$

for all $g, h \in G$. Therefore Q is a group homomorphism, and hence, a linear representation of G. But, since $Q(g) = f(g)P'(g)$, for all $g \in G$, we have that Q is projective equivalent to P' (with isomorphism given by the identity map of V).

Conversely, let $T\colon G \to \mathrm{GL}(V)$ be a group homomorphism such that P is equivalent to $Q = \pi \circ T$. Then, by Lemma 2.3.1, P and Q are associated to the same cohomology class. But, since T is a section for Q with Schur multiplier equal to the constant map 1, we have that the cohomology class associated to Q is trivial. Therefore P is associated to the trivial cohomology class. □

3. Central Extensions

In this section, we will define what is a central extension of a group and study its relation with cohomology groups.

3.1. **Central extension of a group.**

Definition 3.1.1. An *exact sequence* of groups is a sequence of group homomorphisms

$$1 \longrightarrow G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} G_n \longrightarrow 1,$$

such that $\mathrm{Im}(f_{i-1}) \subseteq \mathrm{Ker}(f_i)$ for $i = 1, \ldots, n$.

When the n above is equal to 2, we call the sequence a *short exact sequence.*

Definition 3.1.2. An *extension* of a group Q by the group N is a short exact sequence

$$1 \longrightarrow N \xrightarrow{f} G \xrightarrow{g} Q \longrightarrow 1.$$

When G is a finite group, then we call the sequence a *finite extension* of the group Q.

When $\mathrm{Im}(f)$ is in the center of G, $Z(G)$, that is, for each n in N, $f(n)$ commutes with all elements of G, then we call the sequence above a *central extension* of the group Q.

Remark 3.1.3. Let $1 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} Q \longrightarrow 1$ be an extension. Because i is an injective group homomorphism, we can assume N is a subgroup of G such that it is the kernel of p. Therefore the main information about an extension are just the group G and the group homomorphism p. That way, we denote this extension by (G, p). When the homomorphisms are not so important to the context that we are discussing, we will also call G an *extension of Q by N.*

Example 3.1.4. Let V be a $\mathbb{K}$-vector space. The exact sequence below is an example of a central extension:

$$1 \longrightarrow \mathbb{K}^\times \xrightarrow{\delta} \mathrm{GL}(V) \xrightarrow{\pi} \mathrm{PGL}(V) \longrightarrow 1,$$

where $\forall k \in \mathbb{K}^\times$, $\delta(k)$ is the dilation $\delta(k) \colon v \mapsto kv$ and π is the canonical projection.

Remark 3.1.5. Let $1 \longrightarrow N \xrightarrow{f} G \xrightarrow{g} Q \longrightarrow 1$ be a central extension of the group Q by the group N. Since $\mathrm{Im}(f) \subseteq Z(G)$, we have that $\mathrm{Im}(f)$ is an abelian group. Since $f \colon N \to \mathrm{Im}(f)$ is an isomorphism, we have that N must be an abelian group.

Definition 3.1.6 (Morphism of exact sequences)**.** A *morphism of exact sequences* is a commutative diagram of group homomorphisms:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "f"] \arrow[d, "\alpha"] & G \arrow[r, "g"] \arrow[d, "\beta"] & Q \arrow[r] \arrow[d, "\gamma"] & 1 \\
1 \arrow[r] & \tilde{N} \arrow[r, "\tilde{f}"] & \tilde{G} \arrow[r, "\tilde{g}"] & \tilde{Q} \arrow[r] & 1,
\end{tikzcd}$$

where each line is an exact sequence of groups. Such morphism is denoted by (α, β, γ).

Definition 3.1.7 (Equivalent extensions)**.** Let $1 \longrightarrow N \xrightarrow{i_1} G_1 \xrightarrow{p_1} Q \longrightarrow 1$ and $1 \longrightarrow N \xrightarrow{i_2} G_2 \xrightarrow{p_2} Q \longrightarrow 1$ be two extensions of the group Q by the group N. We say that (G_1, p_1) and (G_2, p_2) are *equivalent* if there is a morphism $(\mathrm{Id}_N, \beta, \mathrm{Id}_Q)$ of exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "i_1"] \arrow[d, equal] & G \arrow[r, "p_1"] \arrow[d, "\beta"] & Q \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N \arrow[r, "i_2"] & \tilde{G}_2 \arrow[r, "p_2"] & \tilde{Q} \arrow[r] & 1.
\end{tikzcd}$$

Remark 3.1.8. Notice, by the Five Lemma ([Mac67, Lemma 3.3]), the homomorphism β of the definition above is a group isomorphism.

Remark 3.1.9. Its easy to see that the notion of equivalence of extensions is a reflective, symmetric and, by the commutativity of the diagram, transitive relation. Therefore equivalence of extensions is an equivalence relation.

3.2. **Central extensions and $2^{\mathbf{nd}}$-cohomology group.** Now let us start to study the relations between a central extension of a group Q by an abelian group N, and $H^2(Q, N)$. We will show the known fact that, up to equivalence of extensions, central extensions and 2^{nd}-cohomology groups are essentially the same thing. We can find such results in [Kar85, Chapter 2; Section 1].

Proposition 3.2.1. *Let $1 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} Q \longrightarrow 1$ be a central extension of the group Q by the group N. Then the following statements are true:*

(a) For each section $f\colon Q \to G$ of p (i.e, a map such that $p \circ f = \mathrm{Id}_Q$) such that $f(1) = 1$, it is true that $f(q)f(q')f(qq')^{-1} \in N$, for all $q, q' \in Q$. Therefore, the set map $\rho_f\colon Q \times Q \to N$ is well defined by setting $\rho_f(q, q') = i^{-1}(f(q)f(q')f(qq')^{-1})$, where $i^{-1}\colon \mathrm{Im}(i) \to N$ is the inverse of the group isomorphism i, and ρ_f is a 2-cocycle;

(b) Let f, $f'\colon Q \to G$ be two sections of p, satisfying $f(1) = 1 = f'(1)$, and let $\rho_f, \rho_{f'}$ their respective induced 2-cocycles. Then ρ_f and $\rho_{f'}$ are cohomologous.

Proof. Let $\varphi\colon \dfrac{G}{\mathrm{Ker}(p)} \to Q$ be the group isomorphism induced by p (i.e. the function defined by setting $\varphi(g\,\mathrm{Ker}(p)) = p(g)$, for all $g \in G$) and let $f\colon Q \to G$ be a section of p. Then we have $\varphi(f(q)\,\mathrm{Ker}(p)) = p(f(q)) = q$, for all $q \in Q$. Since the extension is an exact sequence, we have $\mathrm{Im}(i) = \mathrm{Ker}(p)$, and therefore we have:

$$\varphi(f(q)\,\mathrm{Im}(i)) = q,$$

for all $q \in Q$.

Now let $q, q' \in Q$. Then we have:

$$\varphi(f(qq')\,\mathrm{Im}(i)) = qq' = \varphi((f(q)\,\mathrm{Im}(i))(f(q')\,\mathrm{Im}(i))) = \varphi(f(q)f(q')\,\mathrm{Im}(i)).$$

Therefore, since φ is a bijection, $f(qq')\,\mathrm{Im}(i) = f(q)f(q')\,\mathrm{Im}(i)$. Thus, for all $q, q' \in Q$, there exist $n \in N$ such that $f(q)f(q')f(qq')^{-1} = i(n)$ and since i is an injection, it follows that ρ_f is well defined.

Now, since i is a homomorphism, $\operatorname{Im}(i) \subseteq Z(G)$ and N is abelian, we have:

$$\begin{aligned}
i(\rho_f(r,s)\rho_f(qr,s)^{-1}\rho_f(q,rs)\rho_f(q,r)^{-1}) &= \\
&= i(\rho_f(qr,s)^{-1}\rho_f(q,r)^{-1}\rho_f(q,rs)\rho_f(r,s)) \\
&= i(\rho_f(qr,s))^{-1}i(\rho_f(q,r))^{-1}i(\rho_f(q,rs))i(\rho_f(r,s)) \\
&= i(\rho_f(qr,s))^{-1}f(qr)f(r)^{-1}f(q)^{-1}f(q)f(rs)f(qrs)^{-1}i(\rho_f(r,s)) \\
&= f(qrs)f(s)^{-1}f(qr)^{-1}f(qr)f(r)^{-1}f(rs)f(qrs)^{-1}i(\rho_f(r,s)) \\
&= f(qrs)f(s)^{-1}f(r)^{-1}f(rs)f(qrs)^{-1}i(\rho_f(r,s)) \\
&= f(qrs)f(s)^{-1}f(r)^{-1}i(\rho_f(r,s))f(rs)f(qrs)^{-1} \\
&= f(qrs)f(s)^{-1}f(r)^{-1}f(r)f(s)f(rs)^{-1}f(rs)f(qrs)^{-1} \\
&= 1_G \in G,
\end{aligned}$$

for all $q, r, s \in Q$. Thus, since N is abelian (by Remark 3.1.5) and i is an injection, we have, for all $q, r, s \in Q$:

$$\rho_f(r,s)\rho_f(qr,s)^{-1}\rho_f(q,rs)\rho_f(q,r)^{-1} = 1_N.$$

And it is easy to see that $\rho_f(1,1) = 1$. Therefore, ρ_f is a 2-cocycle of Q on N. This proves (a).

Let f, $f'\colon Q \to G$ be two sections of p such that $f(1) = 1 = f'(1)$, and let $\rho_f, \rho_{f'}$ their respective induced 2-cocycles. Then, for all $q \in Q$, we have $\varphi(f(q)\operatorname{Im}(i)) = q = \varphi(f'(q)\operatorname{Im}(i))$ and thus, since φ is a group isomorphism, $f(q)\operatorname{Im}(i) = f'(q)\operatorname{Im}(i)$, for all $q \in Q$. Therefore $f(q)f'(q)^{-1} \in \operatorname{Im}(i)$, for all $q \in Q$. Thus, define the set map $\tilde{f}\colon Q \to N$ by setting $\tilde{f}(q) = i^{-1}(f(q)f'(q)^{-1})$, for all $q \in Q$. Then, since i is a homomorphism, $\operatorname{Im}(i) \subseteq Z(G)$ and N is abelian, we have:

$$\begin{aligned}
i(\rho_f(r,s)\rho_{f'}(r,s)^{-1}) &= i(\rho_f(r,s))i(\rho_{f'}(r,s))^{-1} \\
&= f(r)f(s)f(rs)^{-1}f'(rs)f'(s)^{-1}f'(r)^{-1} \\
&= f(r)f(s)i(\tilde{f}(rs))^{-1}f'(s)^{-1}f'(r)^{-1} \\
&= f(r)i(\tilde{f}(s))f'(r)^{-1}i(\tilde{f}(rs)^{-1}) \\
&= f(r)f'(r)^{-1}i(\tilde{f}(s))i(\tilde{f}(rs)^{-1}) \\
&= i(\tilde{f}(r))i(\tilde{f}(s)\tilde{f}(rs)^{-1}) \\
&= i(\tilde{f}(r)\tilde{f}(s)\tilde{f}(rs)^{-1}),
\end{aligned}$$

for all $r, s \in Q$. Then, since i is injective, we have $\rho_f(r,s)\rho_{f'}(r,s)^{-1} = \tilde{f}(r)\tilde{f}(s)\tilde{f}(rs)^{-1}$, for all $r, s \in Q$. Therefore ρ_f and $\rho_{f'}$ are cohomologous. This concludes the proof. □

Proposition 3.2.1 shows us that for each central extension (G, p) of Q by N we have an associated cohomology class that is idependent of the choice of a section for p. In this way, we have the next definition:

Definition 3.2.2 (Cohomology class of central extension)**.** Let (G, p) be a central extension of Q by the normal group N and $c_{(G,p)}$ be the cohomology class of the 2-cocycle $\rho_f\colon Q \times Q \to N$ induced by any section $f\colon Q \to G$ of p. Then we call $c_{(G,p)} \in H^2(Q, N)$ the *cohomology class of the central extension* (G, p).

Proposition 3.2.3. *Let $1 \longrightarrow N \xrightarrow{i_1} G_1 \xrightarrow{p_1} Q \longrightarrow 1$ and $1 \longrightarrow N \xrightarrow{i_2} G_2 \xrightarrow{p_2} Q \longrightarrow 1$ be equivalent central extensions of the group Q by the abelian group N. Then their cohomology classes $c_{(G_1,p_1)}, c_{(G_2,p_2)} \in H^2(Q,N)$ are equal.*

Proof. Since (G_1,p_1) and (G_2,p_2) are equivalent, there is a group isomorphism $\phi\colon G_1 \to G_2$ such that $i_2 = \phi \circ i_1$ and $p_1 = p_2 \circ \phi$.

Let $f_1\colon Q \to G_1$ be a section of p_1, such that $f_1(1) = 1$, and define the set map $f_2\colon Q \to G_2$ by $f_2 = \phi \circ f_1$. Then we have:

$$p_2 \circ f_2 = p_2 \circ \phi \circ f_1 = p_1 \circ f_1 = \mathrm{Id}_Q .$$

Since ϕ is a group homomorphism, we also have $f_2(1) = 1$. Therefore f_2 is a section of p_2. Thus, let ρ_{f_1} and ρ_{f_2} be the 2-cocycles associated to f_1 and f_2 as in Proposition 3.2.1(a). Then, since ϕ, i_1 and i_2 are group homomorphisms and $\phi \circ i_1 = i_2$, we have:

$$\begin{aligned}
\rho_{f_2}(g,h) &= i_2^{-1}(f_2(g)f_2f_2(gh)^{-1}) \\
&= i_2^{-1}(\phi(f_1(g)f_1(h)f_1(gh)^{-1})) \\
&= i_2^{-1}\phi i_1(\rho_{f_1}(g,h)) \\
&= i_2^{-1}i_2(\rho_{f_1}(g,h)) = \rho_{f_1}(g,h),
\end{aligned}$$

for all $r, s \in Q$. Hence $\rho_{f_1} = \rho_{f_2}$ and $c_{(G_1,p_1)} = c_{(G_2,p_2)}$. ☐

By Proposition 3.2.3, the associated cohomology class is invariant up to equivalence of central extensions.

Proposition 3.2.4. *Let $c \in H^2(Q,N)$ and ρ be a 2-cocycle representative of c. Define the set $G_\rho = Q \times N$ and the multiplication $(r,n)(s,m) = (rs, \rho(r,s)nm)$, for all $(r,n),(s,m) \in G_\rho$. Then the following statements are true:*

- *(a) Provided with the multiplication defined above, G_ρ is a group with identity element $(1,1)$ and inverse given by $(g,n)^{-1} = (g^{-1}, \rho(g,g^{-1})^{-1}n^{-1})$, for all $g,n \in G_\rho$.*
- *(b) The set maps $\mu\colon N \to G_\rho$ and $\tau\colon G_\rho \to Q$, respectively defined by setting $\mu(n) = (1,n)$ and $\tau(q,m) = q$, for all $q \in Q$ and $n,m \in N$, are group homomorphisms such that the sequence:*

$$1 \longrightarrow N \xrightarrow{\mu} G_\rho \xrightarrow{\tau} Q \longrightarrow 1,$$

is a central extension of Q by N with associated cohomology class $c_{(G_\rho,\tau)} \in H^2(Q,N)$ equal to c.

Proof. Since ρ is a 2-cocycle, we have:

$$\begin{aligned}
((q,a)(r,b))(s,c) &= (qr, \rho(q,r)ab)(s,c) \\
&= (qrs, \rho(qr,s)\rho(q,r)abc) \\
&= (qrs, \rho(q,rs)\rho(r,s)abc) \\
&= (q,a)(rs, \rho(r,s)bc) \\
&= (q,a)((r,b)(s,c)),
\end{aligned}$$

for all $(q,a),(r,b),(s,c) \in G_\rho$. Therefore the multiplication is associative. By Lemma 1.2.1, we have $\rho(r,1) = \rho(1,1) = \rho(1,s) = 1$, for all $r,s \in Q$. Then we have:

$$\begin{aligned}(q,n)(1,1) &= (q,\rho(q,1)n)\\ &= (q,n)\\ &= (q,\rho(1,q)n)\\ &= (1,1)(q,n),\end{aligned}$$

for all $(q,n) \in G_\rho$. Therefore $(1,1)$ is an identity element of G_ρ. Also by Lemma 1.2.1, we have $\rho(q,q^{-1}) = \rho(q^{-1},q)$, for all $q \in Q$. Then we have:

$$\begin{aligned}(q,n)(q^{-1},\rho(q,q^{-1})^{-1}n^{-1}) &= (qq^{-1},\rho(q,q^{-1})\rho(q,q^{-1})^{-1}nn^{-1})\\ &= (1,1)\\ &= (q^{-1}q,\rho(q^{-1},q)\rho(q,q^{-1})^{-1}n^{-1}n)\\ &= (q^{-1},\rho(q,q^{-1})^{-1}n^{-1})(q,n),\end{aligned}$$

for all $(q,n) \in G_\rho$. Therefore $(g,n)^{-1} = (g^{-1},\rho(g,g^{-1})^{-1}n^{-1})$, for all $g,n \in G_\rho$. Therefore G_ρ is a group, proving (a).

Clearly τ is surjective. Now, for all $(r,n),(s,m) \in G_\rho$ we have

$$\tau((r,n)(s,m)) = \tau(rs,\rho(r,s)nm) = rs = \tau(r,n)\tau(s,m),$$

therefore τ is a surjective group homomorphism. It's easy to see that $\ker(\tau) = \{(1,n) \in G_\rho : n \in N\} = \operatorname{Im}(\mu)$. Futhermore we have:

$$\begin{aligned}\mu(nm) &= (1,nm)\\ &= (1,\rho(1,1)nm)\\ &= (1,n)(1,m)\\ &= \mu(n)\mu(m),\end{aligned}$$

for all $n,m \in N$. And $\mu(n) = (1,1)$ if, and only if, $n = 1_N$. Then μ is an injective group homomorphism. Also, since $\rho(q,1) = \rho(1,q)$, for all $q \in Q$, and N is abelian, we have

$$\begin{aligned}(1,n)(q,m) &= (q,\rho(1,q)nm)\\ &= (q,\rho(q,1)nm)\\ &= (q,m)(1,n),\end{aligned}$$

for all $q \in Q$ and $n,m \in N$. Then $\operatorname{Im}(\mu) \subseteq Z(G_\rho)$. Therefore $1 \longrightarrow N \xrightarrow{\mu} G_\rho \xrightarrow{\tau} Q \longrightarrow 1$ is a central extension of Q by N.

To complete the proposition, let the set map $f\colon Q \to G_\rho$ be defined by setting $f(q) = (q,1)$, for all $q \in Q$. Clearly f is a section for τ, and we have:

$$\begin{aligned}f(r)f(s)f(rs)^{-1} &= (r,1)(s,1)(s^{-1}r^{-1},\rho(rs,s^{-1}r^{-1})^{-1})\\ &= (rs,\rho(r,s)1)(s^{-1}r^{-1},\rho(rs,s^{-1}r^{-1})^{-1})\\ &= (1,\rho(r,s)\rho(rs,s^{-1}r^{-1})\rho(rs,s^{-1}r^{-1})^{-1})\\ &= (1,\rho(r,s)),\end{aligned}$$

for all $r, s \in Q$. Thus $\mu^{-1}(f(r)f(s)f(rs)^{-1}) = \mu^{-1}(1, \rho(r,s)) = \rho(r,s)$. Therefore $c_{(G_\rho,\tau)} = c$. This concludes the proof. □

Definition 3.2.5 (Induced central extension)**.** Let $\rho \in Z^2(Q,N)$. Then the central extension (G_ρ, τ) constructed as in Proposition 3.2.4 is called *central extension induced by* ρ.

Proposition 3.2.4 show us that for any cohomology class there is a central extension associated with it. Now, let us see if that relation is one-to-one.

Lemma 3.2.6. *Let* $1 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} Q \longrightarrow 1$ *be a central extension of the group* Q *by the group* N*,* $f\colon Q \to G$ *be a section for* p*, such that* $f(1) = 1$*, and let* ρ_f *be its associated* 2*-cocycle as in Proposition 3.2.1(a). Let* (G_{ρ_f}, τ) *be the central extension induced by* ρ_f*. Then* (G,p) *and* (G_{ρ_f}, τ) *are equivalent extensions.*

Proof. Let $\varphi\colon \dfrac{G}{\mathrm{Im}(i)} \to Q$ be the group isomorphism induced by p. Notice that, for all $g \in G$, we have $\varphi(f(p(g))\,\mathrm{Im}(i)) = p(g) = \varphi(g\,\mathrm{Im}(i))$. Therefore, since φ is an isomorphism, $gf(p(g))^{-1} \in \mathrm{Im}(i)$ for all $g \in G$. Thus define the set map $\phi\colon G \to G_\rho$ by setting $\phi(g) = (p(g), i^{-1}(gf(p(g))^{-1}))$, for all $g \in G$. Then ϕ is a group homomorphism. In fact, since i and p are group homomorphisms, $\mathrm{Im}(i) \subseteq Z(G)$ and by the definition of ρ_f, we have:

$$\begin{aligned}
\phi(gh) &= (p(gh), i^{-1}(ghf(p(gh))^{-1})) \\
&= (p(g)p(h), i^{-1}(ghf(p(g)p(h))^{-1})) \\
&= (p(g)p(h), i^{-1}(hf(p(h))^{-1}gf(p(g))^{-1}i(\rho_f(p(g),p(h))))) \\
&= (p(g)p(h), \rho_f(p(g),p(h)))i^{-1}(hf(p(h))^{-1})i^{-1}(gf(p(g))^{-1})) \\
&= (p(g), i^{-1}(hf(p(h))^{-1}))(p(h), i^{-1}(gf(p(g))^{-1})) \\
&= \phi(g)\phi(h),
\end{aligned}$$

for all $g, h \in G$. We also have $\tau(\phi(g)) = p(g)$, for all $g \in G$ and, since $\mathrm{Im}(i) = \mathrm{Ker}(p)$ and $f(1) = 1$, we have

$$\begin{aligned}
\phi(i(n)) &= (p(i(n)), i^{-1}(i(n)f(p(i(n)))^{-1})) \\
&= (1, i^{-1}(i(n)f(1)^{-1})) \\
&= (1, i^{-1}(i(n))) \\
&= (1, n) = \mu(n),
\end{aligned}$$

for all $n \in N$. Therefore ϕ is a group homomorphism between G and G_{ρ_f} such that $\mu = \phi \circ i$ and $p = \tau \circ \phi$. We conclude that (G,p) and (G_{ρ_f}, τ) are equivalent extensions. □

Lemma 3.2.7. *Let* $\rho,\ \omega\colon Q \times Q \to N$ *be cohomologous* 2*-cocycles. Then their induced central extensions* $1 \longrightarrow N \xrightarrow{\mu_1} G_\rho \xrightarrow{\tau_1} Q \longrightarrow 1$ *and* $1 \longrightarrow N \xrightarrow{\mu_2} G_\omega \xrightarrow{\tau_2} Q \longrightarrow 1$ *of the group* Q *by the abelian group* N *are equivalent.*

Proof. By Lemma 3.2.6, it is sufficient to find a section $f\colon Q \to G_\omega$ for τ_2, such that $f(1) = (1,1)$ and $\rho_f(r,s) = \mu_2^{-1}(f(r)f(s)f(rs)^{-1}) = \rho(r,s)$, for all $r, s \in Q$.

Since ρ and ω are cohomologous, there is a set mep $\psi\colon Q \to N$ such that $\rho(r,s) = \psi(r)\psi(s)\psi(rs)^{-1}\omega(r,s)$, for all $r, s \in Q$. Thus define $f\colon Q \to G_\omega$ by setting $f(q) = (q, \psi(q))$.

It is clear that f is a section for τ_2. Since $\rho(1,1) = 1 = \omega(1,1)$, we have $\psi(1) = 1$, thus $f(1) = (1,1)$. Since N is an abelian group, we have:

$$\begin{aligned}
f(r)f(s)f(rs)^{-1} &= (r,\psi(r))(s,\psi(s))(s^{-1}r^{-1},\omega(rs,s^{-1}r^{-1})^{-1}\psi(rs)^{-1}) \\
&= (rs,\omega(r,s)\psi(r)\psi(s))(s^{-1}r^{-1},\omega(rs,s^{-1}r^{-1})^{-1}\psi(rs)^{-1}) \\
&= (rss^{-1}r^{-1},\omega(rs,s^{-1}r^{-1})\omega(r,s)\psi(r)\psi(s)\omega(rs,s^{-1}r^{-1})^{-1}\psi(rs)^{-1}) \\
&= (1,\omega(r,s)\psi(r)\psi(s)\psi(rs)^{-1}) \\
&= (1,\rho(r,s)) \\
&= \mu_2(\rho(r,s)),
\end{aligned}$$

for all $r, s \in Q$. Therefore, since μ_2 is injective, we have $\rho_f(r,s) = \mu_2^{-1}(f(r)f(s)f(rs)^{-1}) = \rho(r,s)$, for all $r, s \in Q$. ☐

Theorem 3.2.8. *Two central extensions, (G_1, p_1) and (G_2, p_2), of the group Q by the abelian group N are equivalent if, and only if, their associated cohomology classes $c_{(G_1,p_1)}, c_{(G_2,p_2)} \in H^2(Q,N)$ are equal.*

Proof. The implication follows by Proposition 3.2.3. Conversely, let $\rho, \omega\colon Q \times Q \to N$ be two 2-cocycles induced by some sections of p_1 and p_2, respectively. Then, since they are representatives of the cohomology classes $c_{(G_1,p_1)}, c_{(G_2,p_2)}$, respectively, and since $c_{(G_1,p_1)} = c_{(G_2,p_2)}$, we have that ρ and ω are cohomologous.

Now, let (G_ρ, τ_1) and (G_ω, τ_2) be the two central extensions induced by ρ and ω, respectively. By Lemma 3.2.6, we have (G_1, p_1) and (G_2, p_2) are equivalent to (G_ρ, τ_1) and (G_ω, τ_2), respectively. But, since ρ and ω are cohomologous, by Lemma 3.2.7 we have that (G_ρ, τ_1) is equivalent to (G_ω, τ_2).

Therefore, since equivalence of extensions is an equivalence relation, we conclude that (G_1, p_1) and (G_2, p_2) are equivalent extensions. ☐

Let $\mathrm{CExt}(Q,N)/\sim$ be the set of all equivalence classes of central extensions of the group Q by the abelian group N. Essentially, Proposition 3.2.1 and Theorem 3.2.8 say that there is a bijection

$$\Phi\colon\ \mathrm{CExt}(Q,N)/\sim\ \to H^2(Q,N)$$

Such conclusion can be founded in [Kar85, Theorem 2.1.2].

4. Representation groups

Throughout this section let $\mathbb{K}$ be an algebraically closed field of characteristic zero and G a finite group. For any group Q, from now on we will fix the notation $Q' = [Q,Q]$ for the commutator subgroup of Q.

In this section, we will study the relation between central extensions and projective representations.

4.1. Representation group.

Definition 4.1.1 (Representation group)**.** A *representation group of G* is a finite central extension (G^*, τ) of G such that $\mathrm{Ker}(\tau) \subseteq (G^*)'$ and $\mathrm{Ker}(\tau) \cong H^2(G, \mathbb{K}^\times)$.

Our next result will be the proof that for any group G there exists a representation group. But first we need to prove some lemmas that will be helpful for us.

Lemma 4.1.2. *Let H be the group $H^2(G,\mathbb{K}^\times)$. Then there exists $\Gamma \in Z^2(G,H)$ such that:*

$$\hat{\Gamma}\colon \operatorname{Hom}(H,\mathbb{K}^\times) \to H^2(G,H)$$

is an isomorphism, where $\hat{\Gamma}$ is the function given in Definition 1.2.3.

Proof. By Theorem 1.2.2(c), we have that H is a finite abelian grup. Thus H is a direct product of cyclic groups. Let $\overline{\alpha_1},\dots,\overline{\alpha_d}$ be generators of these groups, represented by the 2-cocycles $\alpha_1,\dots,\alpha_n$, respectively, and $e_1,\dots,e_d \in \mathbb{N}$ their orders.

By Theorem 1.2.2(b), we can assume that for each $i \in \{1,\dots,d\}$, $\alpha_i(g,h)$ is an e_i-th root of $1 \in \mathbb{K}$, for all $g,h \in G$, and $\alpha_i(1,1)=1$. There exists for each i a primitive e_i-th root of 1, that we will call ω_i. Then, for all $i \in \{1,\dots,d\}$ and $g,h \in G$, there is $a_i(g,h) \in \{0,\dots,e_i-1\}$ such that $\alpha_i(g,h) = \omega_i^{a_i(g,h)}$. Since α_i is a 2-cocycle, we have:

$$\begin{aligned} 1 &= \alpha_i(h,k)\alpha_i(gh,k)^{-1}\alpha_i(g,hk)\alpha_i(g,h)^{-1} \\ &= \omega_i^{a_i(h,k)}\omega_i^{-a_i(gh,k)}\omega_i^{a_i(g,hk)}\omega_i^{-a_i(g,h)} \end{aligned}$$

$$\Leftrightarrow$$

$$a_i(h,k) - a_i(gh,k) + a_i(g,hk) - a_i(g,h) = 0 \mod e_i, \tag{4.1}$$

for all $g,h,k \in G$. And since $\alpha_i(1,1)=1$, it follows that $a_i(1,1)=0$, for all i.

Now, define the set map $\Gamma\colon G\times G \to H$ by setting

$$\Gamma(g,h) = \overline{\alpha_1}^{a_1(g,h)} \cdots \overline{\alpha_d}^{a_d(g,h)},$$

for all $g,h \in G$. Clearly, Γ is well defined. By (4.1), and the fact that $\underline{\alpha_i}$ has order e_i for all i, we have:

$$\Gamma(g,h)\Gamma(gh,k)^{-1}\Gamma(g,hk)\Gamma(g,h)^{-1} = 1, \tag{4.2}$$

for all $g,h,k \in G$. And since $a_i(1,1)=0$, for all i, we have $\Gamma(1,1)=1$. Therefore $\Gamma \in Z^2(G,H)$

Let $C \in H$ be an arbitrary cohomology class. Since $H = \langle\overline{\alpha_1}\rangle \times \cdots \times \langle\overline{\alpha_d}\rangle$, there is a 2-cocycle ρ representative of C such that:

$$\rho(g,h) = (\alpha_1(g,h))^{x_1} \cdots (\alpha_d(g,h))^{x_d} = (\omega_1^{x_1})^{a_1(g,h)} \cdots (\omega_d^{x_d})^{a_d(g,h)}, \tag{4.3}$$

for all $g,h \in G$, where $x_i \in \{0,\dots,e_i-1\}$, for each $i \in \{1,\dots,d\}$.

Define a group homomorphism $\alpha_\rho\colon H \to \mathbb{K}^\times$ by setting $\alpha_\rho(\overline{\alpha_i}) = \omega_i^{x_i}$, for all $i \in \{1,\dots,d\}$. Then, for all $g,h \in G$, we have:

$$\begin{aligned} \alpha_\rho(\Gamma(g,h)) &= \alpha_\rho(\overline{\alpha_1}^{a_1(g,h)} \cdots \overline{\alpha_d}^{a_d(g,h)}) \\ &= (\omega_1^{x_1})^{a_1(g,h)} \cdots (\omega_d^{x_d})^{a_d(g,h)} \\ &= \rho(g,h). \end{aligned}$$

Therefore $\alpha_\rho \circ \Gamma = \rho$. This proves that $\hat{\Gamma}$ is surjective. But, by [Kar85, Corollary 2.3.9], $\operatorname{Hom}(H,\mathbb{K}^\times)$ and H are isomorphic, and hence they have the same number of elements. Therefore $\hat{\Gamma}$ is an isomorphism. $\square$

Lemma 4.1.3. *Let G be a finite group and $A, B \subseteq G$ be two abelian subgroups such that B is a proper subgroup of A. Then the following statements are true:*

(a) *Let $f_0 \colon B \to \mathbb{K}^\times$ be a group homomorphism. Then there exists a non-trivial group homormophism $f_1 \colon A \to \mathbb{K}^\times$ such that $f_1|_B = f_0$.*
(b) *Let $f_0 \colon AG' \to \mathbb{K}^\times$ be a group homomorphism such that $G' \subseteq \mathrm{Ker}(f_0)$. Then there exists a group homomorphism $f_1 \colon G \to \mathbb{K}^\times$ such that $f_1|_{AG'} = f_0$.*

Proof. Since $\mathbb{K}$ is an algebraically closed field, then $\mathbb{K}^\times$ is a divisible group. Then, (a) follows from [Kar87, Lemma 2.1.6].

For (b), let $f_0 \colon AG' \to \mathbb{K}^\times$ be a group homomorphism such that $G' \subseteq \mathrm{Ker}(f_0)$. Define the set map $\varphi_0 \colon \dfrac{AG'}{G'} \to \mathbb{K}^\times$ by setting $\varphi_0(xG') = f_0(x)$, for all $x \in AG'$. Since $G' \subseteq \mathrm{Ker}(f_0)$, this function is well defined. Furthermore, since f_0 is a group homomorphism, so is φ_0.

Now, since $\dfrac{AG'}{G'}$ is a subgroup of $\dfrac{G}{G'}$ and both are abelian groups, we can apply the first statement. Then there is a group homomorphism $\varphi_1 \colon \dfrac{G}{G'} \to \mathbb{K}^\times$ that is an extension of φ_0. Therefore, define the set map $f_1 \colon G \to \mathbb{K}^\times$ by setting $f_1(g) = \varphi_1(gG')$. It's easy to see that f_1 is a group homomorphism extending f_0. □

Now, we can prove the main result of this subsection:

Theorem 4.1.4 ([HH92, Theorem 1.2])**.** *Let G be a group. Then there is a representation group G^* of G.*

Proof. Let $H = H^2(G, \mathbb{K}^\times)$. Let $\Gamma \in Z^2(G, H)$ be as in Lemma 4.1.2 and consider the central extension of G by H

$$1 \longrightarrow H \xrightarrow{i} G_\Gamma \xrightarrow{p} G \longrightarrow 1,$$

given in Proposition 3.2.4, i.e., the central extension of G by H induced by the 2-cocycle Γ. Define $G^* = G_\Gamma$ and let's prove that G^* is a representation group of G.

It is sufficient to prove that $i(H) \in (G^*)'$. Let's identify H with $i(H)$. Then, defining $Y = H \cap (G^*)'$, it is sufficient to prove that $Y = H$.

Suppose that Y is a proper subgroup of H. By Lemma 4.1.3(a), there exists a non-trivial group homomorphism $f \colon H \to \mathbb{K}^\times$ such that $Y \subseteq \mathrm{Ker}(f)$. Since $\dfrac{H}{Y} = \dfrac{H}{H \cap (G^*)'} \cong \dfrac{H(G^*)'}{(G^*)'}$ and f is a group homomorphism, the set map $f_0 \colon H(G^*)' \to \mathbb{K}^\times$ defined by setting $f_0(xg') = f(x)$, for all $x \in H$, $g' \in (G^*)'$, is a well defined group homomorphism satisfying $(G^*)' \subseteq \mathrm{Ker}(f_0)$. Then, by Lemma 4.1.3(b), there is a group homomorphism $f_1 \colon G^* \to \mathbb{K}^\times$ such that $f_1|_{H(G^*)'} = f_0$.

Now, for all $g, h \in G$ we have:

$$f_1(g,1)f_1(h,1) = f_1(gh, \Gamma(g,h)) = f_1((1, \Gamma(1,gh)^{-1}\Gamma(g,h))(gh,1)).$$

Since, for all $g, h \in G$, we have $(1, \Gamma(1,gh)^{-1}\Gamma(g,h)) = i(\Gamma(g,h))$ and $f_1|_{H(G^*)'} = f_0$, identifying H with $i(H)$, we have:

$$f_1(g,1)f_1(h,1) = f(\Gamma(g,h))f_1(gh,1),$$

for all $g, h \in G$. Then, define the set map $\phi\colon G \to G^*$ by setting $\phi(g) = (g, 1)$, for all $g \in G$. Let $\varphi = f \circ \phi\colon G \to \mathbb{K}^\times$, then, for all $g, h \in G$ we have:

$$f \circ \Gamma(g, h) = \varphi(g)\varphi(h)\varphi(gh)^{-1}.$$

Therefore $f \circ \Gamma$ is in the trivial cohomology class, i.e, $\hat{\Gamma}(f) = 1$. Since $\hat{\Gamma}$ is a group isomorphism, we have that f is equal to the trivial group homomorphism which is a contradiction.

Therefore $Y = H$, and the central extension is a representation group. □

You can notice that Lemma 4.1.2 is essential to prove the existence of a representation group. But actually we can prove that for any representation group we have an associated 2-cocycle such as in Lemma 4.1.2. Precisely:

Proposition 4.1.5. *Let (G^*, τ) be a representation group of G, $H = \mathrm{Ker}(\tau)$ and let $\Gamma\colon G \times G \to H$ be a 2-cocycle representative of the cohomology class associated to the central extension (G^*, τ). Then the group homomorphism*

$$\hat{\rho}\colon \mathrm{Hom}(H, \mathbb{K}^\times) \to H^2(G, \mathbb{K}^\times)$$

given in Definition 1.2.3, is a bijection.

Proof. At first, without loss of generality consider G^* to be the central extension of G by $H = H^2(G, \mathbb{K}^\times)$ induced by the 2-cocycle Γ:

$$1 \longrightarrow H \xrightarrow{\mu} G_\Gamma \xrightarrow{\tau} G \longrightarrow 1.$$

Let $\alpha \in \mathrm{Ker}(\hat{\Gamma})$. Then, there is a set map $\varphi\colon G \to \mathbb{K}^\times$ such that, for all $g, h \in G$ we have $\alpha \circ \Gamma(g, h) = \varphi(g)\varphi(h)\varphi(g, h)^{-1}$.

Define the function $\beta\colon G^* \to \mathbb{K}^\times$ by setting $\beta(g, \rho) = \varphi(g)\alpha(\rho)$, for all $(g, \rho) \in G^* = G_\Gamma$. Then β is a group homomorphism. In fact, for all $(g, \rho), (g', \rho') \in G^*$ we have:

$$\begin{aligned}
\beta((g, \rho)(g', \rho')) &= \beta(gg', \Gamma(g, g')\rho\rho') \\
&= \varphi(gg')\alpha(\Gamma(g, g')\rho\rho') \\
&= \varphi(gg')\alpha(\Gamma(g, g'))\alpha(\rho)\alpha(\rho') \\
&= \varphi(g)\varphi(g')\alpha(\rho)\alpha(\rho') \\
&= \beta(g, \rho)\beta(g', \rho').
\end{aligned}$$

Furthermore, for all $\rho \in H$, we have $\beta(\mu(\rho)) = \alpha(\rho)$. Since $\mu(H) \subset (G^*)'$ and $\mathbb{K}^\times$ is an abelian group, we can conclude $\beta \circ \mu$ is the trivial homomorphism. Therefore α is the trivial homomorphism and $\hat{\Gamma}$ is injective.

Since $\mathrm{Hom}(H, \mathbb{K}^\times)$ and H have the same quantity of elements, we can conclude that $\hat{\Gamma}$ is a bijection. □

4.2. **Representation groups and projective representations.** Now, we will study the relation between projective representations, linear representations and central extensions. Precisely, we will study conditions on a central extension of a group G that ensure that every projective representation of G corresponds to a linear representation of this central extension.

Lemma 4.2.1. *Consider a commutative diagram of group homomorphisms*

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "f"] \arrow[d, "\alpha"] & G \arrow[r, "g"] \arrow[d, "\beta"] & Q \arrow[r] & 1 \\ 1 \arrow[r] & \tilde{N} \arrow[r, "\tilde{f}"] & \tilde{G} \arrow[r, "\tilde{g}"] & \tilde{Q} \arrow[r] & 1, \end{tikzcd}$$

such that each line is a short exact sequence. Then there is an unique group homomorphism $\gamma\colon Q \to \tilde{Q}$ *such that* $\gamma \circ g = \tilde{g} \circ \beta$.

Proof. Let $g'\colon Q \to G$ be a section for g. Define the set map $\gamma\colon Q \to \tilde{Q}$ by setting $\gamma = \tilde{g} \circ \beta \circ g'$. Then, since it is an exact sequence, it follows that γ is a group homomorphism. The uniqueness follows from commutativity of the diagram. We leave the details for the reader.

□

Corollary 4.2.2. *Let* $1 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} Q \longrightarrow 1$ *be a central extension of groups and let* $1 \longrightarrow \mathbb{K}^\times \xrightarrow{\delta} \mathrm{GL}(V) \xrightarrow{\pi} \mathrm{PGL}(V) \longrightarrow 1$ *be the central extension of Example 3.1.4. Then, for all pairs of group homomorphisms* $T\colon G \to \mathrm{GL}(V)$ *and* $\alpha\colon N \to \mathbb{K}^\times$ *such that* $\delta \circ \alpha = T \circ i$, *there is a projective representation* $P\colon Q \to \mathrm{PGL}(V)$ *such that* $\pi \circ T = P \circ p$, *i.e, the following diagram commutes:*

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "i"] \arrow[d, "\alpha"] & G \arrow[r, "p"] \arrow[d, "T"] & Q \arrow[r] \arrow[d, "P"] & 1 \\ 1 \arrow[r] & \mathbb{K}^\times \arrow[r, "\delta"] & \mathrm{GL}(V) \arrow[r, "\pi"] & \mathrm{PGL}(V) \arrow[r] & 1. \end{tikzcd}$$

Proof. The proof follows directly from Lemma 4.2.1. □

The next step is to study the opposite direction, i.e., study when it is possible to find a central extension (G, p) of Q by an abelian group N, such that, given a projective representation of Q there is a corresponding linear representation of the group G.

Lemma 4.2.3. *Let* $1 \longrightarrow \tilde{N} \xrightarrow{i} \tilde{G} \xrightarrow{p} \tilde{Q} \longrightarrow 1$ *be a central extension and* $P\colon Q \to \tilde{Q}$ *be a group homomorphism. Suppose that there is a section* $P'\colon Q \to \tilde{G}$ *for* P *and a 2-cocycle* $\tilde{\rho} \in Z^2(Q, \tilde{N})$ *satisfying*

$$i(\tilde{\rho}(r,s))P'(rs) = P'(r)P'(s),$$

for all $r, s \in Q$. *If there is a homomorphism* $\alpha\colon N \to \tilde{N}$ *of abelian groups and* $\rho \in Z^2(Q, N)$ *such that* $\alpha \circ \rho = \tilde{\rho}$, *then there is a group homomorphism* $T\colon G_\rho \to \tilde{G}$ *such that the following diagram commutes:*

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "\mu"] \arrow[d, "\alpha"] & G_\rho \arrow[r, "\tau"] \arrow[d, dashed, "T"] & Q \arrow[r] \arrow[d, "P"] \arrow[dl, dotted, "P'"] & 1 \\ 1 \arrow[r] & \tilde{N} \arrow[r, "i"] & \tilde{G} \arrow[r, "p"] & \tilde{Q} \arrow[r] & 1, \end{tikzcd}$$

where (G_ρ, τ) *is the central extension of* Q *by* N *induced by* ρ.

Proof. Define $T\colon G_\rho \to \tilde{G}$ by setting $T(q,n) = P'(q)i(\alpha(n))$, for all $(q,n) \in G_\rho$. Then, since i and α are group homomorphisms, $\alpha \circ \rho = \tilde{\rho}$ and $\mathrm{Im}(i) \in Z(\tilde{G})$, we have:

$$\begin{aligned} T((r,n)(s,m)) &= T(rs, \rho(r,s)nm) \\ &= P'(rs)i(\alpha(\rho(r,s)nm)) \\ &= i(\tilde{\rho}(r,s))^{-1} i(\alpha(\rho(r,s))i(\alpha(n))i(\alpha(m))P'(r)P'(s) \\ &= i(\tilde{\rho}(r,s))^{-1} i(\tilde{\rho}(r,s))P'(r)i(\alpha(n))P'(s)i(\alpha(m)) \\ &= P'(r)i(\alpha(n))P'(s)i(\alpha(m)) \\ &= T(r,n)T(s,m), \end{aligned}$$

for all $(r,n),(s,m) \in G_\rho$. Therefore T is a group homomorphism. And, since $p \circ P' = P$ and $p \circ i$ is the trivial map, we have:

$$p(T(q,n)) = p(P'(q))p(i(\alpha(n))) = P(q) = P(\tau(q,n)),$$

for all $(q,n) \in G_{\rho^{-1}}$. Therefore $p \circ T = P \circ \tau$. □

Remark 4.2.4. Notice that Lemma 4.2.3 gives us a relation between projective representations of G and central extensions of G by $\mathbb{K}^\times$. In particular, the fact that all homomorphism of central extensions of G by $\mathbb{K}^\times$ are isomorphisms (by Remark 3.1.8) corresponds to the fact that we consider only isomorphisms of projective representations, as opposed to arbitrary homomorphisms.

Definition 4.2.5 (Lifting)**.** Let $1 \longrightarrow N \xrightarrow{\mu} G \xrightarrow{\tau} Q \longrightarrow 1$ and $1 \longrightarrow \tilde{N} \xrightarrow{\tilde{\mu}} \tilde{G} \xrightarrow{\tilde{\tau}} \tilde{Q} \longrightarrow 1$ be two central extensions. Let $f\colon Q \to \tilde{Q}$ be a group homomorphism. A *lifting* of f is a morphism of exact sequence $(\alpha, \tilde{f}, f)$:

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "\mu"] \arrow[d, "\alpha"] & G \arrow[r, "\tau"] \arrow[d, "\tilde{f}"] & Q \arrow[r] \arrow[d, "f"] & 1 \\ 1 \arrow[r] & \tilde{N} \arrow[r, "\tilde{\mu}"] & \tilde{G} \arrow[r, "\tilde{\tau}"] & \tilde{Q} \arrow[r] & 1. \end{tikzcd}$$

Theorem 4.2.6 ([HH92, Theorem 1.3, Theorem 1.4])**.** *Let $1 \longrightarrow H \xrightarrow{\mu} G^* \xrightarrow{\tau} G \longrightarrow 1$ be a representation group. Then, for any projective representation $P\colon G \to \mathrm{PGL}(V)$ there is a lifting for P:*

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\mu"] \arrow[d, "\alpha"] & G^* \arrow[r, "\tau"] \arrow[d, "T"] & G \arrow[r] \arrow[d, "P"] & 1 \\ 1 \arrow[r] & \mathbb{K}^\times \arrow[r, "\delta"] & \mathrm{GL}(V) \arrow[r, "\pi"] & \mathrm{PGL}(V) \arrow[r] & 1. \end{tikzcd}$$

Proof. Let Γ be any 2-cocycle representative of the cohomology class associate to the representation group. Then, by Proposition 4.1.5, we have that $\hat{\Gamma}$ is an isomorphism.

Let $1 \longrightarrow H \xrightarrow{\mu'} G_\Gamma \xrightarrow{\tau'} G \longrightarrow 1$ be the central extension induced by Γ. Since $\hat{\Gamma}$ is an isomorphism, there exists a group homomorphism $\alpha \in \mathrm{Hom}(H, \mathbb{K}^\times)$ such that the cohomology class of $\alpha \circ \Gamma$ is the cohomology class associated to P. Then, by Proposition 2.2.2, there exists a section $P'\colon G \to \mathrm{GL}(V)$ for P with Schur multiplier $\alpha \circ \Gamma$. By Lemma 4.2.3, there exists a lifting (α, T', P):

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r, "\mu'"] \arrow[d, "\alpha"] & G_\Gamma \arrow[r, "\tau'"] \arrow[d, "T'"] & G \arrow[r] \arrow[d, "P"] & 1 \\
1 \arrow[r] & \mathbb{K}^\times \arrow[r, "\delta"] & \mathrm{GL}(V) \arrow[r, "\pi"] & \mathrm{PGL}(V) \arrow[r] & 1.
\end{tikzcd}$$

Since two central extensions with the same associated cohomology class are equivalent, there is a group isomorphism $\phi\colon G^* \to G_\Gamma$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r, "\mu"] \arrow[d, equal] & G^* \arrow[r, "\tau"] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H \arrow[r, "\mu'"] & G_\Gamma \arrow[r, "\tau'"] & G \arrow[r] & 1.
\end{tikzcd}$$

Define the group homomorphism $T\colon G^* \to GL(V)$ by $T = T' \circ \phi$. Then it's clear that (α, T, P) is a lifting for P. $\square$

Theorem 4.2.7. *Let* $1 \longrightarrow H \xrightarrow{\mu} G^* \xrightarrow{\tau} G \longrightarrow 1$ *be a representation group of* G*,* $\alpha \in \mathrm{Hom}(H, \mathbb{K}^\times)$ *and* Γ *a 2-cocycle representative of the cohomology class associated to the representation group. Then:*

(a) There is a bijection between projective equivalence classes of linear representations of G^* *that acts on* H *by* α *and projective equivalence classes of projective representations of* G *with associated cohomology class* $\hat{\Gamma}(\alpha)$*.*
(b) There is a bijection between linear equivalence classes of linear representations of G^* *that acts on* H *by* α *and linear equivalence classes of* $\alpha \circ \Gamma$*-representations of* G*.*

Proof. Without loss of generality, consider (G_Γ, τ) to be the central extension induced by Γ in the place of (G^*, τ).

Let $P\colon G \to PGL(V)$ and $Q\colon G \to PGL(W)$ be two projective representations and (α, T, P) and (α, U, Q) their liftings, respectively. Define $P'\colon G \to \mathrm{GL}(V)$ and $Q'\colon G \to \mathrm{GL}(W)$ by setting $P'(g) = T(g, 1)$ and $Q'(g) = U(g, 1)$, for all $g \in G$. By Lemma 4.2.1, it Is easy to see that P' and Q' are sections for P and Q, respectively. Furthermore

$$\begin{aligned}
T(g, \rho) &= T((g, 1)(1, \Gamma(g, 1)^{-1}\rho)) \\
&= T(g, 1)T(\mu(\rho)) \\
&= \alpha(\rho)P'(g),
\end{aligned}$$

for all $(g, \rho) \in G_\Gamma$. The same calculations gives $U(g, \rho) = \alpha(\rho)Q'(g)$, for all $(g, \rho) \in G_\Gamma$.

If P and Q are projectively equivalent, there exists an isomorphism $\phi\colon V \to W$ and a set map $c\colon G \to \mathbb{K}^\times$ satisfying:

$$\phi \circ P'(g) \circ \phi^{-1} = c(g)Q'(g),$$

for all $g \in G$. Then, for all $(g, \rho) \in G_\Gamma$, we have:

$$\begin{aligned}\phi \circ T(g,\rho) \circ \phi^{-1} &= \phi \circ \alpha(\rho) P'(g) \circ \phi^{-1} \\ &= \alpha(\rho)\phi \circ P'(g) \circ \phi^{-1} \\ &= \alpha(\rho) c(g) Q'(g) \\ &= c(g) U(g,\rho).\end{aligned}$$

Therefore T and U are projectively equivalent.

Conversely, suppose T and U are projectively equivalent. Then there exists an isomorphism $\phi \colon V \to W$ and a set map $c \colon G_\Gamma \to \mathbb{K}^\times$ satisfying:

$$\phi \circ T(g,\rho) \circ \phi^{-1} = c(g,\rho) U(g,\rho),$$

for all $(g, \rho) \in G_\Gamma$. Define the set map $d \colon G \to \mathbb{K}^\times$ by setting $d(g) = c(g, 1)$, for all $g \in G$. Then, for all $g \in G$, we have:

$$\begin{aligned}\phi P'(g) \phi^{-1} &= \phi \alpha(1) P'(g) \phi^{-1} \\ &= \phi T(g,1) \phi^{-1} \\ &= c(g,1) U(g,1) \\ &= d(g) Q'(g,1).\end{aligned}$$

Therefore P and Q are projectively equivalent. This completes the proof of (a).

Now, suppose T and U linearly equivalent. Notice that P' and Q' are $(\alpha \circ \Gamma)$-representations. In fact:

$$\begin{aligned}P'(gh) &= T(gh,1) \\ &= T(g,1)T(h,1)T(1, \Gamma(g,h)^{-1}\Gamma(gh,1)^{-1}) \\ &= P'(g)P'(h)T\mu(\Gamma(g,h))^{-1} \\ &= P'(g)P'(h)\alpha(\Gamma(g,h))^{-1},\end{aligned}$$

for all $g, h \in G$. The same calculation give us $Q'(gh) = Q'(g)Q'(h)\alpha(\Gamma(g,h))^{-1}$, for all $g, h \in G$. Then the proof of (a), with c equal the constant map to 1, shows that P' and Q' are linearly equivalent.

The argument above may be reversed finishing the proof of (b). $\square$

Let $1 \longrightarrow H \xrightarrow{\mu} G^* \xrightarrow{\tau} G \longrightarrow 1$ be a finite central extension. Then we will say that such an extension satisfies *the property of lifting of projective representations* if it satisfies the statement: for any projective representation $P \colon G \to \mathrm{PGL}(V)$, there is a lifting for P

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\mu"] \arrow[d, "\alpha"] & G^* \arrow[r, "\tau"] \arrow[d, "T"] & G \arrow[r] \arrow[d, "P"] & 1 \\ 1 \arrow[r] & \mathbb{K}^\times \arrow[r, "\delta"] & \mathrm{GL}(V) \arrow[r, "\pi"] & \mathrm{PGL}(V) \arrow[r] & 1. \end{tikzcd}$$

Lemma 4.2.8. *Let* $1 \longrightarrow H \xrightarrow{\mu} G^* \xrightarrow{\tau} G \longrightarrow 1$ *be a finite central extension that satisfies the property of lifting of projective representations. Then* $|G^*| \geqslant |G||H^2(G, \mathbb{K}^\times)|$.

Proof. To simplify notation, we will identify H with its image $\mu(H)$ and view μ as the inclusion map.

Let $f\colon G \to G^*$ be a section for τ, such that $f(1) = 1$, and define $\Gamma\colon G \times G \to H$ by setting $\Gamma(g,h) = f(g)f(h)f(gh)^{-1}$, for all $g, h \in G$. By Proposition 3.2.1, Γ is a 2-cocycle representative of the cohomology class associated to $1 \longrightarrow H \xrightarrow{\mu} G^* \xrightarrow{\tau} G \longrightarrow 1$. Let $\hat{\Gamma}$ be the function given in Definition 1.2.3. Since, by [Kar85, Corollary 2.3.9], $\mathrm{Hom}(H, \mathbb{K}^\times)$ and H are isomorphic, and hence $|\mathrm{Hom}(H, \mathbb{K}^\times)| = |H|$, to prove the theorem, it is sufficient prove that $\hat{\Gamma}\colon \mathrm{Hom}(H, \mathbb{K}^\times) \to H^2(G, \mathbb{K}^\times)$ is a surjective homomorphism.

Let $c \in H^2(G, \mathbb{K}^\times)$ and $P\colon G \to V$ be a projective representation with associated cohomology class $C_P = c$. There exists such a representation by Remark 2.2.5. Consider a lifting of P:

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r, "\mu"] \arrow[d, "\alpha"] & G^* \arrow[r, "\tau"] \arrow[d, "T"] & G \arrow[r] \arrow[d, "P"] & 1 \\
1 \arrow[r] & \mathbb{K}^\times \arrow[r, "\delta"] & \mathrm{GL}(V) \arrow[r, "\pi"] & \mathrm{PGL}(V) \arrow[r] & 1.
\end{tikzcd}$$

Define $P' = T \circ f$. Then P' is a section for P such that, for all $g, h \in G$:

$$\begin{aligned}
P'(g)P'(h) &= T(f(g))T(f(h)) = T(f(g)f(h)) \\
&= T(\Gamma(g,h)f(gh)) = \alpha(\Gamma(g,h))P'(gh).
\end{aligned}$$

Thus $\alpha \circ \Gamma$ is the Schur multiplier of P'. Therefore, there exists $\alpha \in \mathrm{Hom}(H, \mathbb{K}^\times)$ such that $\hat{\Gamma}(\alpha) = c$. Hence $\hat{\Gamma}$ is surjective. ☐

Remark 4.2.9. Lemma 4.2.8 says that the smallest central extensions with the property of lifting of projective representations are the representation groups.

Now we can prove an important characterization of the representation groups.

Theorem 4.2.10 ([Kar85, Theorem 3.3.7])**.** *Let $E : 1 \longrightarrow H \xrightarrow{\mu} G^* \xrightarrow{\tau} G \longrightarrow 1$ be a finite central extension such that $|G^*| = |G||H^2(G, \mathbb{K}^\times)|$. Then the following statements are equivalent:*

(a) E is a representation group of G.
(b) E has the property of lifting of projective representations.

Proof. The proof that (a) implies (b) follows by Theorem 4.2.6.

Conversely, suppose that $1 \longrightarrow H \xrightarrow{\mu} G^* \xrightarrow{\tau} G \longrightarrow 1$ has the property of lifting of projective representations. The proof of Lemma 4.2.8 shows that there exists a 2-cocycle Γ representative of the cohomology class associated to the central extension (G^*, τ), such that the group homomorphism $\hat{\Gamma}\colon \mathrm{Hom}(H, \mathbb{K}^\times) \to H^2(G, \mathbb{K}^\times)$ is a surjective homomorphism. Since H and $\mathrm{Hom}(H, \mathbb{K}^\times)$ are isomorphic groups and $|H||G| = |G^*| = |G||H^2(G, \mathbb{K}^\times)|$, we conclude that $\hat{\Gamma}$ is a group isomorphism. Thus, the same arguments of the poof of Theorem 4.1.4 show us that $1 \longrightarrow H \xrightarrow{\mu} G^* \xrightarrow{\tau} G \longrightarrow 1$ is a representation group. ☐

Concerning the number of non-equivalent representation groups of G, we have the following theorem:

Theorem 4.2.11. *Let G be a finite group and suppose we have group isomorphisms:*

$$\frac{G}{G'} \cong \mathbb{Z}_{e_1} \times \cdots \times \mathbb{Z}_{e_n} \qquad H^2(G, \mathbb{K}^\times) \cong \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_m}.$$

Then the number of non-equivalent representation groups of G is at most

$$\prod_{\substack{0 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant m}} (e_i, d_j),$$

where (e_i, d_j) is the greatest commom divisor of e_i and d_j.

Proof. See [Kar87, Theorem 2.5.14]. □

4.3. **Perfect groups.** In this section we will define perfect groups and universal central extensions, and prove that there exists, up to equivalence, only one universal central extension for a perfect group, and such an extension is a representation group.

Definition 4.3.1 (Perfect group)**.** A group G is called *perfect group* if $G = G'$.

Definition 4.3.2 (Universal central extension)**.** Let $E : 1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ and $\tilde{E} : 1 \longrightarrow \tilde{N} \longrightarrow \tilde{G} \longrightarrow Q \longrightarrow 1$ be two central extensions of Q. We say that E *covers* (respectively, *uniquely covers*) $\tilde{E}$ if there exists a morphism (respectively, unique morphism) of extensions:

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "f"] \arrow[d, "\alpha"] & G \arrow[r, "g"] \arrow[d, "\beta"] & Q \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \tilde{N} \arrow[r, "\tilde{f}"] & \tilde{G} \arrow[r, "\tilde{g}"] & Q \arrow[r] & 1. \end{tikzcd}$$

If E uniquely covers every central extension of Q, we say that E is a *universal central extension* of Q.

Lemma 4.3.3. *Let $E : (G, \tau)$ and $\tilde{E} : (\tilde{G}, \tilde{\tau})$ be two central extensions of Q. Then the following statements are true:*

(a) Suppose that E and $\tilde{E}$ are universal extension. Then there exists a group isomorphism $\phi \colon G \to \tilde{G}$ such that $\phi(\operatorname{Ker}(\tau)) = \operatorname{Ker}(\tilde{\tau})$.

(b) Suppose G is a perfect group. Then E covers $\tilde{E}$ if and only E uniquely covers $\tilde{E}$.

Proof. Suppose E and $\tilde{E}$ universal central extensions.Then there exist two unique morphisms:

$$\begin{tikzcd} 1 \arrow[r] & \operatorname{Ker}\tau \arrow[r, hook] \arrow[d] & G \arrow[r, "\tau"] \arrow[d, "\phi"] & Q \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \operatorname{Ker}\tilde{\tau} \arrow[r, hook] & \tilde{G} \arrow[r, "\tilde{\tau}"] & Q \arrow[r] & 1, \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[r] & \operatorname{Ker}\tilde{\tau} \arrow[r, hook] \arrow[d] & \tilde{G} \arrow[r, "\tilde{\tau}"] \arrow[d, "\phi'"] & Q \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \operatorname{Ker}\tau \arrow[r, hook] & G \arrow[r, "\tau"] & Q \arrow[r] & 1. \end{tikzcd}$$

(Here all left vertical maps are the restriction of the middle vertical homomorphisms)

By the commutativity of the diagram, we have $\tau = \tilde{\tau} \circ \phi$ and $\tilde{\tau} = \tau \circ \phi'$. Thus we have $\tau \circ \phi' \circ \phi = \tau$. And we have the following morphism of central extension:

$$\begin{tikzcd} 1 \arrow[r] & \operatorname{Ker}\tau \arrow[r, hook] \arrow[d] & G \arrow[r, "\tau"] \arrow[d, "\phi' \circ \phi"] & Q \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \operatorname{Ker}\tau \arrow[r, hook] & G \arrow[r, "\tau"] & Q \arrow[r] & 1. \end{tikzcd}$$

By uniqueness of the definition of universal central extension, we have $\phi' \circ \phi = \mathrm{Id}_G$. A similar argument shows that $\phi \circ \phi' = \mathrm{Id}_{\tilde{G}}$. Therefore ϕ is a group isomorphism such that, by commutativity of the diagram, $\phi(\operatorname{Ker}(\tau)) = \operatorname{Ker}(\tilde{\tau})$. This proves (a).

Now suppose that G is perfect and E covers $\tilde{E}$. Let $\varphi_i \colon G \to \tilde{G}$, $i = 1, 2$, be two group homomorphisms such that $\tilde{\tau} \circ \varphi_1 = \tau = \tilde{\tau} \circ \varphi_2$. Then, for all $g \in G$, $\varphi_1(g)\varphi_2(g)^{-1} \in \operatorname{Ker}\tilde{\tau} \subseteq Z(G)$. Thus we have a group homomorphism $\phi \colon G \to \operatorname{Ker}\tilde{\tau}$ defined by $\phi(g) = \varphi_1(g)\varphi_2(g)^{-1}$, for all $g \in G$. Then, since ϕ is a group homomorphism and $\operatorname{Ker}\tilde{\tau}$ is an abelian group, we have $\phi([x, y]) = [\phi(x), \phi(y)] = 1$, for all $x, y \in G$, and hence $G' \subseteq \operatorname{Ker}\phi$. Since $G' = G$, we conclude that $\varphi_1 = \varphi_2$. $\square$

It is well know that any group G can be written as $G \cong F/R$, where F is a free group and R a normal subgroup of F. Identify G with F/R. Notice that $R/[F, R]$ is a central subgroup of $F/[F, R]$. In fact, for all $r[F, R] \in R/[F, R]$ and $f[F, R] \in F/[F, R]$, we have that $rfr^{-1}f^{-1} \in [F, R]$ and thus $rf[F, R] = fr[F, R]$. By third Theorem of Homomorphism, we have $\dfrac{F/[F, R]}{R/[F, R]} = F/R$. Therefore there exists a natural central extension of G:

$$1 \longrightarrow R/[F, R] \longrightarrow F/[F, R] \longrightarrow G \longrightarrow 1.$$

This extension has the following important property.

Lemma 4.3.4. *Let $G = F/R$, where F is free. Let (B, τ) be a central extension of C, with $A = \operatorname{Ker}\tau$, and $\gamma \colon G \to C$ be a group homomorphism. Then there exists a morphism of extensions:*

$$\begin{tikzcd} 1 \arrow[r] & R/[F, R] \arrow[r] \arrow[d] & F/[R, F] \arrow[r] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, "\gamma"] & 1 \\ 1 \arrow[r] & A \arrow[r] & B \arrow[r, "\tau"] & C \arrow[r] & 1. \end{tikzcd}$$

Here the unmarked vertical map is the restriction homomorphis of β to $R/[F, R]$.

Proof. Since F is free and τ is surjective, there exists a homomorphism $f \colon F \to B$ such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f"] & G \arrow[d, "\gamma"] \\ B \arrow[r, "\tau"] & C, \end{tikzcd}$$

where the map $F \to G$ is the canonical map given by its presentation.

Thus f maps R into $\operatorname{Ker}\tau = A$. We will show that $[F, R] \subseteq \operatorname{Ker} f$. Let $x \in F, r \in R$. Then we have:

$$f([x, r]) = [f(x), f(r)].$$

Since $f(r) \in A$ and A is a central subgroup of B, we conclude that $f([x,r]) = [f(x), f(r)] = 1$. Therefore the generators of $[F,R]$ are mapped to 1 by f, and hence $[F,R] \subseteq \operatorname{Ker} f$.

Therefore f induces a group homomorphism $\beta\colon F/[F,R] \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & R/[F,R] \arrow[r] \arrow[d] & F/[F,R] \arrow[r] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, "\gamma"] & 1 \\
1 \arrow[r] & A \arrow[r] & B \arrow[r, "\tau"] & C \arrow[r] & 1.
\end{tikzcd}$$

□

Before prove the main result of the subsection, we enunciate an important result of Schur.

Theorem 4.3.5. *Let $G = F/R$, where F is free. Then $H^2(G, \mathbb{K}^\times) \cong (F' \cap R)/[F,R]$.*

Proof. See [Kar87, Theorem 2.4.6]. □

Let G be a perfect group and $G = F/R$, where F is free, and consider the central extension $1 \longrightarrow R/[F,R] \longrightarrow F/[F,R] \xrightarrow{\tau} G \longrightarrow 1$. Let $g \in G = G'$. Then there are $h, h' \in G$ such that $g = [h, h']$. Let $x \in \tau^{-1}(h)$ and $y \in \tau^{-1}(h')$. Thus $\tau([x,y]) = [\tau(x), \tau(y)] = [h,h'] = g$. Therefore G is the image of $F'/[F,R]$, the commutator subgroup of $F/[F,R]$. Therefore, the restriction of $F/[F,R] \to G$ to $F'/[F,R]$ induces a central extension

$$1 \longrightarrow (F' \cap R)/[F,R] \longrightarrow F'/[F,R] \longrightarrow G \longrightarrow 1.$$

Naturally, $1 \longrightarrow R/[F,R] \longrightarrow F/[F,R] \xrightarrow{\tau} G \longrightarrow 1$ is covered by $1 \longrightarrow (F' \cap R)/[F,R] \longrightarrow F'/[F,R] \longrightarrow G \longrightarrow 1$.

Theorem 4.3.6. *Let G be a perfect group and let $G = F/R$, where F is free. Then*

(a) $1 \longrightarrow (F' \cap R)/[F,R] \longrightarrow F'/[R,F] \longrightarrow G \longrightarrow 1$ is a representation group of G and a universal central extension.

(b) Let $1 \longrightarrow H \longrightarrow G^* \longrightarrow G \longrightarrow 1$ be universal central extension. Then it is a representation group of G.

Proof. Since $H^2(G, \mathbb{K}^\times) \cong (F' \cap R)/[F,R]$, to show that

$$1 \longrightarrow (F' \cap R)/[F,R] \longrightarrow F'/[F,R] \longrightarrow G \longrightarrow 1$$

is a representation group it is sufficient to prove that $F'/[F,R]$ is perfect group. The inclusion $F''/[F,R] \subseteq F'/[F,R]$ is obvious. Now let $x, y \in F'$. Since $G = F/R$ and G is a perfect group, there exist $f_1, f_2 \in F$ and $r_1, r_2 \in R$ such that $x = f_1 r_1$ and $y = f_2 r_2$. Then, using the identities,

$$[ab, c] = [a,c][[a,c],b][b,c] \text{ and}$$
$$[a, bc] = [a,c][a,b][[a,b],c],$$

we have that:

$$[x,y] = [f_1, y][[f_1,y], r_1][r_1, y] \text{ and}$$
$$[f_1, y] = [f_1, r_2][f_1, f_2][[f_1, f_2], r_2].$$

Therefore $[x,y][F,R] \in F''/[F,R]$, and hence $F'/[F,R]$ is a perfect group.

We will prove now that $1 \longrightarrow (F' \cap R)/[F,R] \longrightarrow F'/[R,F] \longrightarrow G \longrightarrow 1$ is an universal central extension. Let (G^*, τ) be a central extension of G. By Lemma 4.3.4, using the identity map of G instead of γ, we have that $1 \longrightarrow R/[F,R] \longrightarrow F/[F,R] \xrightarrow{\tau} G \longrightarrow 1$ covers (G^*,τ), and hence (G^*,τ) is covered by $1 \longrightarrow (F' \cap R)/[F,R] \longrightarrow F'/[F,R] \longrightarrow G \longrightarrow 1$. By Lemma 4.3.3(b), we conclude that $1 \longrightarrow (F' \cap R)/[F,R] \longrightarrow F'/[F,R] \longrightarrow G \longrightarrow 1$ is a universal central extension. This completes the proof of part (a).

The proof of (b) is a direct consequence of (a) and Lemma 4.3.3(a). ∎

Notice now that Theorem 4.2.11 gives us that there exists a unique, up to equivalence, representation group of a perfect group G, since $G/G' = 1$, and, by Theorem (b), this representation group is equal to the universal central extension of G.

5. Symmetric group

Throughout this section we will work of the field of complex numbers $\mathbb{C}$, instead of an arbitrary field $\mathbb{K}$, and study the projective representation theory of the symmetric group S_n and the Sergeev algebra.

5.1. **Representation groups of symmetric groups.** The purpose of this subsection is to study the 2^{nd}-cohomology group of the symmetric groups on $\mathbb{C}$ making it possible to list all representation groups of the symmetric groups.

Let S_n be the symmetric group of degree n, and denote the transpositions $(i \;\; i+1)$ by t_i, for all $i \in \{1,\dots,n-1\}$. A well known presentation for S_n is:

$$S_n = \langle t_1,\dots,t_{n-1} \mid t_i^2 = 1;\ t_it_j = t_jt_i;\ t_kt_{k+1}t_k = t_{k+1}t_kt_{k+1}; \\ \text{for all } i,j,k \in \{1,\dots,n-1\},\ k \leqslant n-2 \text{ and } |i-j| \geqslant 2\rangle.$$

Using the commutator of elements and the relation $t_i^2 = 1$, we can write the second and the third relation as $[t_i,t_j] = 1$ and $(t_kt_{k+1})^3 = 1$, respectively. Thus, defining F to be the free group generated by $\{t_1,\dots,t_{n-1}\}$ and R to be the normal closure of the set $\{t_i^2;\ [t_i,t_j];\ (t_kt_{k+1})^3 \mid i,j,k \in \{1,\dots,n-1\},\ k \leqslant n-2,\ |i-j| \geqslant 2\}$, we have $S_n = F/R$

Concerning the 2^{nd}-cohomology group of S_n, we have:

Theorem 5.1.1. *The 2^{nd}-cohomology group of S_n, $H^2(S_n,\mathbb{C})$, has order at most 2 and is trivial for $n \leqslant 3$.*

Proof. See [HH92, Theorem 2.7]. ∎

Let $n \geqslant 3$. We will construct two groups of order $2(n!)$, which will subsequently be proved to be representation groups of S_n. With one of these groups we will be able to establish a lower bound for $|H^2(S_n,\mathbb{C})|$.

Definition 5.1.2. Let $n \geqslant 3$. We define $\tilde{S}_n$ to be the group with presentation given by:

$$\tilde{S}_n = \langle z,t_1,\dots,t_{n-1} \mid t_i^2 = (t_kt_{k+1})^3 = (t_it_j)^2 = z;\ z^2 = [z,t_i] = 1 \\ \text{for all } i,j,k \in \{1,\dots,n-1\},\ k \leqslant n-2 \text{ and } |i-j| \geqslant 2\rangle.$$

And we define $\hat{S}_n$ to be the group with presentation:

$$\hat{S}_n = \langle z,t_1,\dots,t_{n-1} \mid t_i^2 = (t_kt_{k+1})^3 = z^2 = [z,t_i] = 1;\ (t_it_j)^2 = z \\ \text{for all } i,j,k \in \{1,\dots,n-1\},\ k \leqslant n-2 \text{ and } |i-j| \geqslant 2\rangle.$$

It is not clear, only by the definition above, that $\tilde{S_n}$ and $\hat{S_n}$ have order $2(n!)$, since the relations given in the presentation could impose $z = 1$, defining S_n instead. In order to prove that $|\tilde{S_n}| = 2(n!) = |\hat{S_n}|$, we begin with the following lemma:

Lemma 5.1.3. *Let $n \geqslant 3$ be a positive integer and let m be the greatest integer less then $(n-1)/2$. Then there exists $n-1$ complex square matrices of order 2^m, $M_1, \dots, M_{n-1}$, satisfying:*

$$M_i^2 = -I \qquad (1 \leqslant i \leqslant n-1), \tag{5.1}$$

$$(M_j M_{j+1})^3 = -I \qquad (1 \leqslant j \leqslant n-2), \tag{5.2}$$

$$(M_k M_l)^2 = -I \qquad (1 \leqslant k, l \leqslant n-1;\ |k-l| \geqslant 2), \tag{5.3}$$

where I is the identity matrix of order 2^m.

Proof. We will show just an outline of the poof, referring the reader to [Kar87, Lemma 2.12.2] and [HH92, Proposition 6.1] for more details. Define the following 2×2 matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Consider the tensor product of matrices $\otimes$. Define $X_0 = I$, $X_1 = A^{\otimes m}$ and, for all $k \in \{1, \dots, m\}$:

$$X_{2k} = A^{\otimes m-k} \otimes B \otimes I^{\otimes k-1} \quad \text{and} \quad X_{2k+1} = A^{\otimes m-k} \otimes C \otimes I^{\otimes k-1},$$

where $M^{\otimes r} = M \otimes M \otimes \cdots \otimes M$, with tensor product applied r times in the right term.

Now, let $x_0 = 0$ and take a family of complex numbers x_i and y_i, with $i \in \{1, \dots, n-1\}$, satisfying $x_{i-1}^2 + y_i^2 = (-1)^{i+1} = 2x_i y_i$. Then, define $M_i = x_{i-1}X_{i-1} + y_i X_i$, for all $i \in \{1, \dots, n-1\}$. Those M_i's will satisfy (5.1), (5.2) and (5.3). □

Let G be the group generated by $M_1, \dots, M_{n-1}$. Defining a group homomorphism $\varphi\colon G \to S_n$ by setting $\varphi(M_i) = t_i$, for all $i \in \{1, \dots, n-1\}$, we will have $\dfrac{G}{\{I, -I\}} \cong S_n$. Therefore $|G| = 2(n!)$, since $|\{I, -I\}| = 2$.

Notice that the generators of G satisfy the same relation given in the presentation of $\tilde{S_n}$. Thus G is a homomorphic image of $\tilde{S_n}$. Therefore, since $|\tilde{S_n}| \leqslant 2(n!) = |G|$, we conclude $|\tilde{S_n}| = 2(n!)$.

Now, define $N_j = iM_j$, for all $j \in \{1, \dots, n-1\}$. Therefore, by (5.1), (5.2) and (5.3), we have:

$$N_i^2 = I \qquad (1 \leqslant i \leqslant n-1),$$

$$(N_j N_{j+1})^3 = I \qquad (1 \leqslant j \leqslant n-2),$$

$$(N_k N_l)^2 = -I \qquad (1 \leqslant k, l \leqslant n-1;\ |k-l| \geqslant 2).$$

Defining H to be the group generated by $N_1, \dots, N_{n-1}$, and repeating the same argument as above, we conclude that H is isomorphic to $\hat{S_n}$ and $|\hat{S_n}| = 2(n!)$.

Also, notice that the presentations of $\tilde{S_n}$ and $\hat{S_n}$ give us that $\{1, z\}$ is a central subgroup of $\tilde{S_n}$ and $\hat{S_n}$. Therefore, the canonical projections

$$\tau\colon \tilde{S_n} \to S_n \quad \text{and} \quad \tau'\colon \hat{S_n} \to S_n$$

give us two central extensions $(\tilde{S_n}, \tau)$ and $(\hat{S_n}, \tau')$.

We are able now to prove the following:

Theorem 5.1.4. *Let $n \geqslant 4$. Then $H^2(S_n, \mathbb{C}) \cong \mathbb{Z}_2$. Furthermore $(\tilde{S_n}, \tau)$ and $(\hat{S_n}, \tau')$ are representation groups of S_n, non-isomorphic if, and only if, $n \neq 6$.*

Proof. Notice that for both, $\tilde{S_n}$ and $\hat{S_n}$, we have $(t_i t_j)^2 = z$, for all $i, j \in \{1, \ldots, n-1\}$ and $|i - j| \geqslant 2$. For $\hat{S_n}$, since we have $t_i^{-1} = t_i$, for all $i \in \{1, \ldots, n-1\}$, it is easy to see that $z = [t_i, t_j]$, for all $i, j \in \{1, \ldots, n-1\}$ and $|i-j| \geqslant 2$. For $\tilde{S_n}$, since $z^2 = 1$ and $t_i^{-1} = t_i z = z t_i$, for all $i \in \{1, \ldots, n-1\}$, then we have

$$z = (t_i t_j)^2 = t_i t_j t_i t_j = t_i t_j t_i t_j z z = t_i t_j t_i z t_j z = [t_i, t_j],$$

for all $i, j \in \{1, \ldots, n-1\}$ and $|i - j| \geqslant 2$. Hence $\{1, z\}$ is subgroup of $(\tilde{S_n})'$ and $(\hat{S_n})'$. Therefore, we just need to prove that $H^2(S_n, \mathbb{C}) \cong \mathbb{Z}_2$, to prove that $(\tilde{S_n}, \tau)$ and $(\hat{S_n}, \tau')$ are representation groups of S_n. Since $|H^2(S_n, \mathbb{C})| \leqslant 2$, it is sufficient to find a non-trivial 2-cocycle.

Fix the natural group homomorphim $\phi \colon \tilde{S_n} \to S_n$ defined by setting $\phi(z) = 1$ and $\phi(t_i) = t_i$, for all $i \in \{1, \ldots, n-1\}$. It is clear that ϕ is surjective with kernel equal to $\{1, z\}$. Let $f \colon S_n \to \tilde{S_n}$ be a section for ϕ, such that $f(1) = 1$. Then, by Proposition 3.2.1, we have that $\rho_f \colon S_n \times S_n \to \{1, z\}$ defined by $\rho_f(\sigma, \sigma') = f(\sigma) f(\sigma') f(\sigma\sigma')^{-1}$, for all $\sigma, \sigma' \in S_n$, is a 2-cocycle. Thus, there exists a set map $a \colon S_n \times S_n \to \{0, 1\}$ such that $\rho_f(\sigma, \sigma') = z^{a(\sigma, \sigma')}$, for all $\sigma, \sigma' \in S_n$. Define $\rho \colon S_n \times S_n \to \mathbb{C}$ by setting $\rho(\sigma, \sigma') = (-1)^{a(\sigma, \sigma')}$, for all $\sigma, \sigma' \in S_n$. Since ρ_f is a 2-cocycle, so is ρ. We will prove that ρ is a non-trivial one.

First, notice that, since $f(1) = 1$, we have, for all $\sigma \in S_n$:

$$1 = f(1) = f(\sigma\sigma^{-1}) = z^{a(\sigma, \sigma^{-1})} f(\sigma) f(\sigma^{-1}).$$

Thus we have

$$f(\sigma)^{-1} = z^{a(\sigma, \sigma^{-1})} f(\sigma^{-1}), \qquad \text{for all } \sigma \in S_n. \tag{5.4}$$

Let $s_1, s_3 \in S_n$ be the images of $t_1, t_3 \in \tilde{S_n}$ under ϕ. Then, by the presentation of $\tilde{S_n}$ and (5.4), we have:

$$z = [t_1, t_3] = f(s_1) f(s_3) f(s_1)^{-1} f(s_3)^{-1} = z^N f([s_1, s_3]) = z^N,$$

where $N = a(s_1, s_1^{-1}) + a(s_3, s_3^{-1}) + a(s_1, s_3) + a(s_1 s_3, s_1^{-1}) + a(s_1 s_3 s_1^{-1}, s_3^{-1})$. Therefore $N = 1$. But, by definition of ρ, we have:

$$-1 = \rho(s_1, s_1^{-1}) \rho(s_3, s_3^{-1}) \rho(s_1, s_3) \rho(s_1 s_3, s_1^{-1}) \rho(s_1 s_3 s_1^{-1}, s_3^{-1}). \tag{5.5}$$

Suppose that there exists a set map $\delta \colon S_n \to \mathbb{C}$, such that ρ is its coboundary. Then, by (5.5), we have:

$$-1 = \delta(s_1)^2 \delta(s_3)^2 \delta(s_1^{-1})^2 \delta(s_3^{-1})^2. \tag{5.6}$$

Since $\delta(1) = 1$, it follows from (5.6) that:

$$-1 = \rho(s_1, s_1^{-1})^2 \rho(s_3, s_3^{-1})^2.$$

This is a contradiction, since ρ takes values only ± 1. Therefore $H^2(S_n, \mathbb{C}) \cong \mathbb{Z}_2$ and $(\tilde{S_n}, \tau)$, $(\hat{S_n}, \tau')$ are representation groups of S_n.

To conclude the theorem, we have only to prove that $(\tilde{S}_n, \tau)$ and $(\hat{S}_n, \tau')$ are non-isomorphic if, and only if, $n \neq 6$. Such a proof can be found in [HH92, Theorem 2.12]. ☐

Remark 5.1.5. Let $n \geqslant 4$. Since $\tilde{S}_n$ is a representation group for S_n and $H^2(S_n, \mathbb{C}) \cong \mathbb{Z}_2$, by Proposition 2.3.2 and Theorem 4.2.7, we conclude that the projective representations of S_n are naturally partitioned into two sets, those projectively equivalent to linear representations of S_n and those corresponding to linear representations of $\tilde{S}_n$ with z acting by $-\operatorname{Id}$. We call those of the second type *spin representations of* S_n.

Let $n \geqslant 3$. Since all permutation of S_n are products of transpositions we have that $[\sigma, \sigma']$ is an even permutation, for all $\sigma, \sigma' \in S_n$. Therefore $(S_n)'$ is a subgroup of A_n, the alternating group of degree n. Furthermore, let $(a\ b\ c) \in S_n$ be a 3-cycle. A simple calculation gives us that $(a\ b\ c) = [(a\ b), (a\ c)]$. Therefore, since A_n is generated by the 3-cycles (for $n \geqslant 3$), we conclude that A_n is a subgroup of $(S_n)'$. Hence $(S_n)' = A_n$ and $\dfrac{S_n}{(S_n)'} \cong \mathbb{Z}_2$.

Therefore, by Theorem 4.2.11, we have that there exist no more than 2 representation groups for S_n, when $n \geqslant 4$. By Theorem 5.1.4, we conclude that $\tilde{S}_n$ and $\hat{S}_n$ are the only representation groups for S_n, for $n \geqslant 4$ and $n \neq 6$.

Let $\alpha \in Z^2(S_n, \mathbb{C})$ be a non-trivial 2-cocycle and consider its twisted group algebra $\mathbb{C}_\alpha S_n$, given in Definition 2.2.3. We will denoted $\mathbb{C}_\alpha S_n$ by $\mathcal{T}_n$. Therefore, by Remark 5.1.5 and Theorem 2.2.4, we conclude that the study of projective representation of S_n is equivalent to representation theory of $\mathbb{C}S_n$ and $\mathcal{T}_n$.

Remark 5.1.6. It is not hard to show that $\mathcal{T}_n$ and $\mathbb{C}\tilde{S}_n/\langle z-1\rangle$ are isomorphic algebras. Therefore, by the given presentation of $\tilde{S}_n$, $\mathcal{T}_n$ can be defined as the algebra generated by $t_1, \dots, t_n$ subject to the relations:

$$t_i^2 = 1, \quad (t_j t_{j+1})^3 = 1, \quad [t_i, t_j] = -1,$$

where $1 \leqslant i, j \leqslant n$, and $|i-j| \geqslant 2$. See [Kle05, Section 13.1] for more details.

5.2. **Digression on superalgebras.**

Definition 5.2.1 (Vector superspace)**.** A *vector superspace* over $\mathbb{K}$, is a $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. If $m = \dim V_{\bar{0}}$ and $n = \dim V_{\bar{1}}$, then we write $\operatorname{sdim} V = (m, n)$. The elements of $V_{\bar{0}}$ are called *even* and elements of $V_{\bar{1}}$ are called *odd*. A vector v is called *homogeneous* if is either even or odd and we denote its *degree* by $|v| \in \mathbb{Z}_2$.

A *subsuperspace* of V is a superspace $W \subseteq V$ with grading $W = (W \cap V_{\bar{0}}) \oplus (W \cap V_{\bar{1}})$. We say that such a W is *homogeneous*.

For any super vector space V, we define the *parity reversed space* ΠV to be the super vector space with the even and odd subspaces interchanged.

Let V be a superspace. Defining the linear map $\delta_V \colon V \to V$ by setting $\delta_V(v) = (-1)^{|v|}v$, for all homogenious $v \in V$, we can notice that a superspace W is a subsuperspace of V if, and only if, it is a subspace of V stable under δ_V.

Let W be another superspace. We can view the direct sum $V \oplus W$ and the tensor product $V \otimes W$ as superspace in the following way:

$$(V \oplus W)_i = V_i \oplus W_i, \quad (i \in \mathbb{Z}_2) \tag{5.7}$$

$$(V \otimes W)_{\bar{0}} = (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}) \text{ and} \tag{5.8}$$

$$(V \oplus W)_{\bar{1}} = (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}}). \tag{5.9}$$

We can see $\mathrm{Hom}_{\mathbb{K}}(V, W)$ as a superspace by defining $\mathrm{Hom}_{\mathbb{K}}(V, W)_i$ to be the linear maps $f\colon V \to W$ such that $f(V_j) \subseteq W_{i+j}$, for all $i, j \in \mathbb{Z}_2$. Elements of $\mathrm{Hom}_{\mathbb{K}}(V, W)_{\bar{0}}$ and $\mathrm{Hom}_{\mathbb{K}}(V, W)_{\bar{1}}$ are called *even* and *odd* linear maps, respectively. The parity of a homogeneous linear map f will be denoted by $|f|$. The dual superspace $\mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ is denoted by V^* and called the *dual superspace*.

Definition 5.2.2 (Superalgebra)**.** A *superalgebra* $\mathcal{A}$ is a vector superspace $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ with a bilinear multiplication such that $A_i A_j \subseteq A_{i+j}$, for all $i, j \in \mathbb{Z}_2$.

A *superideal* of A is a homogeneous ideal $\mathcal{I}$ of $\mathcal{A}$. A superalgebra is called *simple* when it has no non-trivial superideals.

A *superalgebra homomorphism* is an even linear map that is also an algebra homomorphism in the usual sense.

Let $\mathcal{A}$ and $\mathcal{B}$ be two superalgebras. We can view the tensor product of superspaces $\mathcal{A} \otimes \mathcal{B}$ as a superalgebra, by defining the multiplication:

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|}(aa') \otimes (bb'),$$

for all homogeneous $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$.

An superalgebra $\mathcal{A}$ viewed only as an algebra is denoted by $|\mathcal{A}|$.

Now we are able to show three important examples.

Example 5.2.3 (Superalgebra $\mathcal{M}_{m,n}$)**.** Let V be a superspace with $\operatorname{sdim} V = (m, n)$. And define $\mathcal{M}(V)$ to be $\mathrm{Hom}_{\mathbb{K}}(V, V)$. By grading $\mathcal{M}(V)$ as the direct sum of even and odd linear maps, and defining the multiplication to be the composition of maps, we can easily see that $\mathcal{M}(V)$ is a superalgebra with $\operatorname{sdim} \mathcal{M}(V) = (m^2 + n^2, 2mn)$.

Also, let W be another superspace. It can be proved (see [Kle05, Example 12.1.1]) that:

$$\mathcal{M}(V) \otimes \mathcal{M}(W) \cong \mathcal{M}(V \otimes W). \tag{5.10}$$

Since, up to isomorphism, the algebra $\mathcal{M}(V)$ does not depend on the supervector space V, but only its superdimention $(m, n) \in V$, we can identify $\mathcal{M}(V)$ with the matrix superalgebra $\mathcal{M}_{m,n}$. By (5.8), (5.9) and (5.10), we have

$$\mathcal{M}_{n,m} \otimes \mathcal{M}_{k,l} \cong \mathcal{M}_{mk+nl,ml+nk}. \tag{5.11}$$

Example 5.2.4 (Superalgebra $\mathcal{Q}_n$)**.** Let V be a vector superspace with $\operatorname{sdim} V = (n, n)$ and let J be an involution in $\mathrm{Hom}_{\mathbb{K}}(V, V)$ of degree $\bar{1}$. We define the superalgebra $\mathcal{Q}(V, J)$ to be:

$$\mathcal{Q}(V, J) = \{f \in \mathrm{Hom}_{\mathbb{K}}(V, V) \mid fJ = (-1)^{\bar{f}} Jf\}$$

It is possible to show that the superalgebra $\mathcal{Q}(V, J)$ can be identified with the superalgebra $\mathcal{Q}_n$ of all matrices of the form

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix},$$

where A and B are arbitrary $n \times n$ matrices.

By the definition of $\mathcal{Q}(V, J)$, (5.10) and (5.11), it is possible to show the following equivalences of superalgebras:

$$\mathcal{M}(V) \otimes \mathcal{Q}(W, J) \cong \mathcal{Q}(V \otimes W, \mathrm{Id}_V \otimes J) \text{ and}$$
$$\mathcal{M}_{m,n} \otimes \mathcal{Q}_k \cong \mathcal{Q}_{(m+n)k}.$$

For more details see [Kle05, Exemple 12.1.2].

Example 5.2.5 (Clifford superalgebra $\mathcal{C}l_n$)**.** Define the *Clifford superalgebra* $\mathcal{C}l_n$ to be the superalgebra given by odd generators $c_1, \ldots, c_n$, subject to the relations

$$c_i^2 = 1 \quad (1 \leqslant i \leqslant n), \tag{5.12}$$
$$c_i c_j = -c_j c_i \quad (1 \leqslant i \neq j \leqslant n). \tag{5.13}$$

Now we can define supermudules.

Definition 5.2.6 (Supermodule)**.** Let $\mathcal{A}$ be a superalgebra. A *(left) $\mathcal{A}$-supermodule* is a vector superspace wich is a left $\mathcal{A}$-module in the usual sense, such that $\mathcal{A}_i V_j \subseteq V_{i+j}$, for all $i, j \in \mathbb{Z}_2$. Right supermodules are defined similarly.

A *subsupermodule* of an $\mathcal{A}$-supermodule is a subsuperspace which is $\mathcal{A}$-stable. A non-zero $\mathcal{A}$-supermodule is *irreducible* (or *simple*) if has no non-zero proper $\mathcal{A}$-subsupermodules.

We call an $\mathcal{A}$-supermodule M *completely reducible* if any subsupermodule of M is a direct summand of M.

A *homomorphism* $f\colon V \to W$ of $\mathcal{A}$-supermodules V and W is a linear map such that

$$f(av) = (-1)^{|f||a|} a f(v) \qquad (v \in V, \text{ homogeneous} a \in \mathcal{A}).$$

Notice that an $\mathcal{A}$-supermodule M can be considered as a usual $|\mathcal{A}|$-module denoted by $|M|$. There exists an isomorphism of vector spaces between $\mathrm{Hom}_{\mathcal{A}}(V, W)$ and $\mathrm{Hom}_{|\mathcal{A}|}(|V|, |W|)$ (see [Kle05, Lemma 12.1.5]). Let V be an irreducible $\mathcal{A}$-supermodule. It might happen that $|V|$ is a reducible $|\mathcal{A}|$-module. In this case we say that V is a supermodule of *type* $\mathtt{Q}$. Otherwise we say that V is of *type* $\mathtt{M}$. By [Kle05, Lemma 12.2.1] an $\mathcal{A}$-supermodule V of type $\mathtt{Q}$ is a direct sum of two non-isomorphic irreducible $|\mathcal{A}|$-modules.

The category of finite-dimensional $\mathcal{A}$-supermodules will be denoted by $\mathcal{A}\text{-}\mathrm{smod}$. We have the *(left) parity change functor*

$$\Pi\colon \mathcal{A}\text{-}\mathrm{smod} \to \mathcal{A}\text{-}\mathrm{smod},$$

where, for an object V, ΠV is view as an $\mathcal{A}$-supermodule under the new action defined by $a \cdot v = (-1)^{|a|} av$, for all $v \in \Pi V$ and homogeneous $a \in \mathcal{A}$, where the juxtaposition denote the original action of $\mathcal{A}$ on V.

When studying the representation theory of algebras, we come across two important results: Schur's lemma and Wedderburn's Theorem. There are also analogous versions of these results in the theory of supermodules, but taking into account the two types of irreducible supermodules. For more detail statements see [Kle05, Lemma 12.2.2] and [Kle05, Theorem 12.2.9].

Let $\mathcal{A}$ be a superalgebra. The result shown in [Kle05, Corollary 12.2.10] allows us to construct a complete set of pairwise non-isomorphic irreducible $|\mathcal{A}|$-modules from a complete set of pairwise non-isomorphic irreducible $\mathcal{A}$-supermodules. That way, we do not lose information in studying supermodule theory instead module theory, providing we keep track of types of irreducible supermodules.

5.3. **Sergeev and Hecke-Clifford superalgebras.** Back to our main discussion of this section, in this subsection we define two important superalgebras and explain how their study is equivalent to the study of projective representation of S_n.

As shown in Subsection 5.1, studying $\mathcal{T}_n$-modules is equivalent to studying spin representations of S_n. Furthermore, we have a superalgebra structure on $\mathcal{T}_n$, defining the $\mathbb{Z}_2$-grading:

$$(\mathcal{T}_n)_{\bar{0}} = \operatorname{span}\{g \mid g \in A_n\}, \quad (\mathcal{T}_n)_{\bar{1}} = \operatorname{span}\{g \mid g \in S_n \backslash A_n\},$$

where A_n is the alternating group of degree n.

Definition 5.3.1 (Sergeev superalgebra)**.** We define the *Sergeev superalgebra* $\mathcal{Y}_n$ to be the tensor product of superalgebras

$$\mathcal{Y}_n = \mathcal{T}_n \otimes \mathcal{C}l_n.$$

Now, notice that there is a natural action of the group S_n on the generators $c_1, \ldots, c_n$ of the Clifford superalgebra $\mathcal{C}l_n$ by defining $\sigma \cdot c_i = c_{\sigma(i)}$. We can extend this action and define a new algebra structure on the space $\mathbb{K}S_n \otimes \mathcal{C}l_n$. Precisely:

Definition 5.3.2 (Heck-Clifford algebra)**.** Let $c_1, \ldots, c_n$ be the generators of the Clifford superalgebra. Identify $1 \otimes c_i$ with c_i, for all $i \in \{1, \ldots, n\}$, and $\sigma \otimes 1$ with σ, for all $\sigma \in S_n$. Then we define the *Heck-Clifford superalgebra* $\mathcal{H}_n$ to be the smash product $\mathbb{K}S_n \rtimes \mathcal{C}l_n$, where

$$\sigma c_i = c_{\sigma(i)} \sigma,$$

for all $i \in \{1, \ldots, n\}$ and $\sigma \in S_n$, and extending linearly.

The algebra $\mathcal{H}_n$ is naturally a superalgebra by defining the $\mathbb{Z}_2$-grading:

$$(\mathcal{H}_n)_{\bar{0}} = \operatorname{span}\{\sigma \mid \sigma \in S_n\}, \quad (\mathcal{H}_n)_{\bar{1}} = \operatorname{span}\{c_i \mid i = 1, \ldots, n\}.$$

It can be proved that Hecke-Clifford and Sergeev superalgebras are isomorphic, by the isomorphism $\varphi\colon \mathcal{Y}_n \to \mathcal{H}_n$, defined by:

$$\begin{aligned} \varphi(1 \otimes c_i) &= c_i, && (1 \leqslant i \leqslant n), \\ \varphi(t_i) &= \frac{1}{\sqrt{-2}} s_i (c_i - c_{i+1}), && (1 \leqslant i \leqslant n-1), \end{aligned}$$

where the s_i are the usual generators of S_n and the t_i the generators of $\tilde{S}_n$, given in Remark 5.1.6.

It can be shown that $\mathcal{C}l_n$ is a simple superalgebra with a unique, up to isomorphism, supermodule U_n of dimension $2^{n/2}$ and type $\mathtt{M}$, if n is even, and of dimension $2^{(n+1)/2}$ and type $\mathtt{Q}$, if n is odd (see [Kle05, Exemple 12.1.3]). Then, define the functors:

$$\begin{aligned} \mathfrak{F}_n &\colon \mathcal{T}_n\text{-}\operatorname{smod} \to \mathcal{H}_n\text{-}\operatorname{smod}, \quad V \mapsto V \otimes U_n, \\ \mathfrak{G}_n &\colon \mathcal{H}_n\text{-}\operatorname{smod} \to \mathcal{T}_n\text{-}\operatorname{smod}, \quad V \mapsto \operatorname{Hom}_{\mathcal{C}l_n}(U_n, V). \end{aligned}$$

These functors define a Morita super-equivalence between the superalgebras $\mathcal{H}_n$ and $\mathcal{T}_n$ in the sense of:

Lemma 5.3.3 ([Kle05, Proposition 13.2.2])**.**

(a) If n is even, then $\mathfrak{F}_n$ and $\mathfrak{G}_n$ are equivalences of categories with

$$\mathfrak{F}_n \circ \mathfrak{G}_n \cong \mathrm{Id}, \quad \mathfrak{G}_n \circ \mathfrak{F}_n \cong \mathrm{Id}\,.$$

(b) If n is odd then $\mathfrak{F}_n$ and $\mathfrak{G}_n$ satisfy:

$$\mathfrak{F}_n \circ \mathfrak{G}_n \cong \mathrm{Id} \oplus \Pi, \quad \mathfrak{G}_n \circ \mathfrak{F}_n \cong \mathrm{Id} \oplus \Pi.$$

In this way, Hecke-Clifford and Sergeev superalgebras give us two new approaches to studying spin representations of S_n.

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