SCHUR-WEYL DUALITY

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ABSTRACT. We give an exposition of Schur-Weyl duality accessible to undergraduate students.

CONTENTS

1. Introduction

The symmetric group, S_k, is one of the most fundamental structures within mathematics. It is the most notable of the permutation groups, which were first defined in [Jor70] and have since been studied extensively. Another fundamental structure is the Lie group $GL(V)$, which is the group of invertible transformations from a vector space V to itself. As these objects are of great interest to the field of mathematics, many tools have been developed in order to better understand them. One such tool is the field of representation theory. A group representation allows group elements to be viewed as invertible matrices, and the extensive theory of linear algebra can then be applied to study the group structure. Since S_k and $GL(V)$ are fundamental groups within mathematics, much is known about representations of these objects over $\mathbb{C}$.

In [Sch27], Issai Schur showed that S_k and $GL(V)$ generate each other's commutants. Today, we know this relationship as Schur-Weyl Duality, which provides a strong relationship between the representation theory of these groups.

In this paper we consider $\mathfrak{gl}_n(V)$, the set of $n \times n$ matrices, and present an existing proof of Schur-Weyl Duality for this set. In fact, $\mathfrak{gl}_n(V)$ is the Lie algebra of $GL(V)$ and their actions are related by the map

$$\exp\colon \mathfrak{gl}_n(V) \to GL(V) \quad X \mapsto e^X.$$

From Schur-Weyl duality for $\mathfrak{gl}_n(V)$, it is easy to show Schur-Weyl Duality for $GL(V)$. The goal of this paper is to present these results in such a way that they become accessible to undergraduate students.

Prerequisites. This document should be accessible to students with a basic understanding of representation theory and upper year linear algebra.

Acknowledgements. This document was written as my undergraduate honour's project at the

University of Ottawa. I would like to thank Professor Alistair Savage for his guidance and support throughout the project.

2. Background

Throughout the paper, $\mathrm{Hom}(V)$ is used to mean $\mathrm{Hom}_{\mathbb{C}}(V)$ and $\mathrm{End}(V)$ for $\mathrm{End}_{\mathbb{C}}(V)$ for a complex vector space V. We will need some elementary results in this paper which are stated below.

Lemma 2.1 (Schur's Lemma). *Let R be a ring and let M, N be simple R-modules. If $f \in \mathrm{Hom}_R(M, N)$, then f is either identically 0 or an isomorphism.*

For a proof see [FH04, (1.7)].

Theorem 2.2 (Artin-Wedderburn Theorem). *If R is a semisimple ring then it is isomorphic as a ring to a finite direct product of simple rings.*

For a proof see [EH18, Chapter 5]. The following discussion is needed for the proof of the Double Commutant Theorem stated below. Let R be a ring and let

$$V = \bigoplus_{j=1}^{n} V_j \quad \text{and} \quad W = \bigoplus_{i=1}^{m} W_i$$

be two R-modules. Let $\phi \colon V \to W$ be an R-module homomorphism. Then we have the components

$$\phi_{i,j} \colon V_j \hookrightarrow V \to W \twoheadrightarrow W_i.$$

Thus we can describe ϕ by a matrix $[\phi_{i,j}]$, where each $\phi_{i,j} \colon V_j \to W_i$ is a module homomorphism. So

$$\phi(v_1 + \cdots + v_n) = [\phi_{i,j}] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad v_j \in V_j, 1 \leqslant j \leqslant n.$$

So we have an abelian group isomorphism from $\mathrm{Hom}_R(V, W)$ to $[\mathrm{Hom}_R(V_j, W_i)]$, which is the collection of $m \times n$ matrices whose (ij)th entry is an R-module homomorphism $V_j \to W_i$. Furthermore, if $V = W$ and R is a $\mathbb{C}$-algebra, then this is an isomorphism of $\mathbb{C}$-algebras.

Furthermore, for A, a semisimple subalgebra of $\mathrm{End}(V)$, $B = \mathrm{End}_A(V)$ acts on $\mathrm{Hom}_A(U, V)$ where U is a submodule of A by $(b \cdot f)(u) = bf(u)$ for $b \in B$, $f \in \mathrm{Hom}_A(U, V)$ and $u \in U$.

Theorem 2.3 (Double Commutant Theorem). *Let V be a finite-dimensional vector space, A be a semisimple subalgebra of $\mathrm{End}(V)$ and $B = \mathrm{End}_A(V)$. Then*

(a) *B is semisimple*
(b) $A = \mathrm{End}_B(V)$
(c) *As an $A \otimes B$-module we have the decomposition*

$$V \simeq \bigoplus_{i=1}^{N} U_i \otimes W_i$$

where U_i are pairwise non-isomorphic simple modules of A and $W_i := \mathrm{Hom}_A(U_i, V)$ are simple modules of B.

Proof. Since A is semisimple, we have the following A-module decomposition

$$V \simeq \bigoplus_{i=1}^{N} U_i \otimes W_i \tag{2.1}$$

where $W_i = \mathrm{Hom}_A(U_i, V)$ is the multiplicity space of U_i and $a \in A$ acts on $u \otimes f \in U_i \otimes W_i$ by $a(u \otimes f) = au \otimes f$. Furthermore we have

$$\begin{aligned} B &= \mathrm{End}_A(V) \\ &= \mathrm{Hom}_A\left(\bigoplus_{i=1}^{N} U_i \otimes W_i, \bigoplus_{j=1}^{N} U_j \otimes W_j\right) \\ &\simeq \left[\mathrm{Hom}_A(U_i \otimes W_i, U_j \otimes W_j)\right]_{i,j=1,\dots,N}. \end{aligned}$$

Let $n_i = \dim W_i$ for $1 \leqslant i \leqslant N$. Since W_i is the multiplicity space of U_i we have that $U_i \otimes W_i \simeq U_i^{\oplus n_i}$ as A-modules. So,

$$\begin{aligned} \mathrm{Hom}_A\left(U_i \otimes W_i, U_j \otimes W_j\right) &\simeq \mathrm{Hom}_A(U_i^{n_i}, U_j^{n_j}) \\ &\simeq \left[\mathrm{Hom}_A(U_i, U_j)\right]_{k=1,\dots,n_i, \ell=1,\dots,n_j} \\ &\simeq \begin{cases} 0 & \text{if } i \neq j, \\ M_{n_i}(\mathbb{C}) & \text{if } i = j. \end{cases} \qquad \text{by Lemma 2.1} \end{aligned}$$

Thus we have that

$$\begin{aligned} B &\simeq \begin{bmatrix} M_{n_1}(\mathbb{C}) & & 0 \\ & \ddots & \\ 0 & & M_{n_N}(\mathbb{C}) \end{bmatrix} \\ &\simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}) \\ &\simeq \bigoplus \mathrm{End}(W_i). \end{aligned}$$

Therefore, since matrix algebras are simple, we can conclude B is semisimple and so (a) is verified. By Theorem 2.2, we also have

$$A \simeq \bigoplus_i \mathrm{End}(U_i)$$

as algebras. So, by a similar argument as above we have

$$\mathrm{End}_B(V) \simeq A.$$

So we have verified (b). We can see that $U_i \simeq \mathrm{Hom}_B(W_i, V)$ with the isomorphism $u \mapsto ev_u$ where $ev_u \colon \mathrm{Hom}_A(U_i, V) \to V$ is defined by $ev_u(f) = f(u)$. Then, since B is semisimple by (a), we can write (2.1) as

$$V \simeq \bigoplus_i W_i \otimes \mathrm{Hom}_B(W_i, V).$$

So, the decomposition is also a B-module isomorphism, hence it is an $A \otimes B$-module isomorphism, and (c) follows. □

For a finite dimensional vector space U, S_k acts naturally on $U^{\otimes k}$ by permuting the terms

$$\sigma(u_1 \otimes \cdots \otimes u_k) = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(k)}, \ \sigma \in S_k, \ u_1, \ldots, u_k \in U.$$

So, *the* S_k *invariants of* $U^{\otimes k}$ is an algebra consisting of elements of $U^{\otimes k}$ that are invariant under the above action and is denoted $(U^{\otimes k})^{S_k}$. Furthermore, the k*th symmetric power*, $S^k U$ is the quotient space $U^{\otimes k}/\langle \sigma u - u \colon \sigma \in S_k, u \in U^{\otimes k}\rangle$. *The algebra of symmetric polynomials in* k *variables*, Sym_k is defined as $\mathbb{C}[X_1, \ldots, X_k]^{S_k}$. For $r \geqslant 0$, *the* r*th elementary symmetric polynomial* e_r is given by

$$e_r(X_1, \ldots, X_n) = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n} X_{j_1} \cdots X_{j_r}$$

so that in particular $e_0 = 1$ and $e_n = X_1 X_2 \cdots X_n$.

Theorem 2.4. *Any* $f \in \mathbb{C}[X_1, \ldots, X_n]^{S_k}$ *can be uniquely expressed as a polynomial in the elementary symmetric polynomials.*

A proof of this statement can be found in [Mac79, (2.4)]. Another class of elementary polynomial we will need is the power sums p_r. For $r \geqslant 1$, the rth power sum is

$$p_r(X_1, \ldots, X_n) = \sum_{j=1}^{n} X_i^r.$$

It is clear that the power sums are symmetric and so $p_r \in \mathbb{C}[X_1, \ldots, X_n]^{S_k}$. Therefore, by Theorem 2.4, the p_r can be expressed as a polynomial in the e_r. We will use this fact in the proof of Schur-Weyl duality for $\mathfrak{gl}_n(V)$.

For a finite-dimensional $\mathbb{C}$-algebra A, we have the following inclusion

$$\iota \colon (A^{\otimes k})^{S_k} \hookrightarrow A^{\otimes k}$$

and projection onto the quotient

$$P \colon A^{\otimes k} \twoheadrightarrow S^k A.$$

Lemma 2.5. *Let* A *be a finite dimensional* $\mathbb{C}$*-algebra. The map*

$$P \circ \iota \colon (A^{\otimes k})^{S_k} \to S^k A, \quad a^{\otimes k} \mapsto a^{\otimes k}$$

is an isomorphism of algebras. Note that here $a^{\otimes k} \in S^k A$ *is actually a representative of an equivalence class.*

Proof. Let $\Phi = P \circ \iota$. We verify Φ is an isomorphism by constructing an inverse. Define $\Psi \colon A^{\otimes k} \to (A^{\otimes k})^{S_k}$ by

$$\Psi(w) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma w \quad w \in A^{\otimes k}.$$

Indeed $\Psi(w) \in (A^{\otimes k})^{S_k}$ since for $\tau \in S_k$

$$\tau \Psi(w) = \frac{1}{k!} \sum_{\sigma \in S_k} \tau \sigma w = \frac{1}{k!} \sum_{\mu \in S_k} \mu w$$

because any permutation $\mu \in S_k$ can be written $\mu = \tau\sigma$ for $\sigma = \tau^{-1}\mu$. By the same argument we have that

$$\Psi(\tau w - w) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma(\tau w - w) = 0.$$

Thus $U := \langle \sigma w - w \colon \sigma \in S_k, w \in A^{\otimes k} \rangle \subseteq \ker \Psi$. Recall that $A^{\otimes k}/U = S^k A$. We have an induced map

$$\widetilde{\Psi} \colon S^k A \to (A^{\otimes k})^{S_k}$$

which we claim is the inverse of Φ. For $x \in (A^{\otimes k})^{S_k}$ we have that

$$\begin{aligned}
\widetilde{\Psi}\Phi(x) &= \widetilde{\Psi}(x) \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \sigma x \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} x \\
&= \frac{1}{k!} k! x \\
&= x.
\end{aligned}$$

For $w \in A^{\otimes k}$ we have

$$\begin{aligned}
\Phi\widetilde{\Psi}(w + U) &= \Phi \frac{1}{k!} \sum_{\sigma \in S_k} \sigma w + U \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \sigma w + U \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} (\sigma w) - w + w + U \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} (\sigma w) - \frac{1}{k!} \sum_{\sigma \in S_k} (w) + w + U \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} (\sigma w - w) + w + U \\
&= w + U
\end{aligned}$$

since

$$\frac{1}{k!} \sum_{\sigma \in S_k} (\sigma w - w) \in U.$$

So $\Phi\widetilde{\Psi} = \mathrm{id}$ and we are done. □

3. Schur-Weyl Duality for $\mathfrak{gl}_n(V)$

The set of $n \times n$ matrices with entries in $\mathbb{C}$, $\mathfrak{gl}_n(\mathbb{C})$, is a Lie algebra with Lie bracket $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{gl}_n(\mathbb{C})$. In fact, for $V := \mathbb{C}^n$, $\mathfrak{gl}_n(V)$ acts on the space $V^{\otimes k}$ as follows

$$X(v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k} v_1 \otimes \cdots \otimes X v_i \otimes \cdots \otimes v_k, \ X \in \mathfrak{gl}_n(V), \ v_1, \ldots, v_k \in V.$$

The group algebra of the symmetric group, $\mathbb{C}S_k$, also acts naturally on $V^{\otimes k}$ by permuting the terms,

$$\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}, \ \sigma \in S_k, \ v_1, \ldots, v_k \in V$$

and extending by linearity. These two actions on $V^{\otimes k}$ commute since

$$X\sigma(v_1\otimes\cdots\otimes v_k)=X(v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(k)})=\sum_{\sigma^{-1}(i)=1}^{k} v_{\sigma^{-1}(1)}\otimes\cdots\otimes Xv_{\sigma^{-1}(i)}\otimes\cdots\otimes v_{\sigma^{-1}(k)}$$

$$=\sigma\left(\sum_{i=1}^{k} v_1\otimes\cdots\otimes Xv_i\otimes\cdots\otimes v_k\right)=\sigma X(v_1\otimes\cdots\otimes v_k).$$

Theorem 3.1 (Schur-Weyl Duality). *We have the following decomposition of* $V^{\otimes k}$ *as a representation of* $S_k\times\mathfrak{gl}_n(V)$

$$V^{\otimes k}\simeq\bigoplus_{\lambda\vdash k}V_\lambda\otimes L_\lambda$$

where V_λ *are irreducible representations of* S_k *and* $L_\lambda\simeq\mathrm{Hom}_{\mathbb{C}S_k}(V_\lambda,V)$ *are irreducible representations of* $\mathfrak{gl}_n(V)$ *or are 0.*

The following lemmas are needed in the proof of Schur-Weyl duality which will be proved at the end of this section.

Lemma 3.2. *Let* V *be a finite-dimensional vector space and let* W *be some finite-dimensional vector space containing* V*. Fix* $v_0,\dots v_d\in W$*. Suppose for any* $t\in\mathbb{C}$*,* V *contains*

$$f(t)=\sum_{i=0}^{d}v_it^i.$$

Then V *contains* $v_0,\dots,v_d$*.*

Proof. Let $\{u_1,\dots,u_n\}$ be a basis of V. Let $\{u_1,\dots,u_n,u_{n+1},\dots,u_m\}$ be a basis of W. We proceed by induction on d. If $d=0$ the result is clear. Now suppose the result holds for polynomials of degree $d-1$. Setting $t=0$ we can conclude $v_0\in V$. It then follows that for $t\neq 0$

$$\frac{f(t)-f(0)}{t}=\sum_{i=1}^{d}v_it^{i-1}\in V.$$

Since this is a polynomial of degree $d-1$ if we can show it is in V of t we can conclude $v_1,\dots,v_d\in V$ by our induction hypothesis. Since each $v_i\in W$ we can write

$$v_i=\sum_{j=1}^{m}a_{i,j}u_j.$$

So for all $t\neq 0$ we have

$$\sum_{i=1}^{d}v_it^{i-1}=\sum_{i=1}^{d}\sum_{j=1}^{m}a_{i,j}u_jt^{i-1}$$
$$=\sum_{j=1}^{m}\left(\sum_{i=1}^{d}a_{i,j}t^{i-1}\right)u_j\in V.$$

Then for all $n+1\leqslant j\leqslant m$,

$$\sum_{i=1}^{d}a_{i,j}t^{i-1}=0,\quad t\neq 0.$$

However, this holds for infinitely many $t \in \mathbb{C}$, so it has infinitely many zeros and must be the zero polynomial by the fundamental theorem of algebra. Therefore

$$\sum_{i=1}^{d} v_i t^{i-1} \in V$$

for all values of t and we are done. □

Lemma 3.3. *Suppose U is a finite-dimensional vector space. The symmetric power $S^\ell U$ is spanned by elements of the form $u^{\otimes \ell}$ for $u \in U$. In fact, it is enough to let u range over a Zariski-dense subset of U.*

Proof. We will prove the first assertion and and refer the reader to [Lor18, (3.37)] for a proof of the second assertion.

Let $u_1, \dots u_k$ be a basis for U. Let W be the subspace of $S^\ell U$ spanned by the $u^{\otimes \ell}$. It suffices to show that for all $\ell \geqslant 1$, $u_1^{m_1} \cdots u_k^{m_k} \in W$ for all $m_1, \dots, m_k$ such that $\sum_{i=1}^{k} m_i = \ell$. If $k = 1$ the result is clear. Suppose the result holds for $k-1$. Let $t_1, \dots, t_k \in \mathbb{C}$. We know

$$(t_1 u_1 + \cdots + t_{k-1} u_{k-1} + t_k u_k)^\ell \in W$$

by definition. By the binomial theorem we have

$$(t_1 u_1 + \cdots + t_{k-1} u_{k-1} + t_k u_k)^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} (t_1 u_1 + \cdots t_{k-1} u_{k-1})^{\ell - j} u_k^j t_k^j.$$

Then, applying Lemma 3.2 and taking

$$v_j = \binom{\ell}{j} (t_1 u_1 + \cdots + t_{k-1} u_{k-1})^{\ell - j} u_k^j$$

we can conclude $v_j \in W$ for all $0 \leqslant j \leqslant \ell$. Since W is a vector space we can conclude $(t_1 u_1 + \cdots + t_{k-1} u_{k-1})^{\ell-j} u_k^j \in W$ for all $0 \leqslant j \leqslant \ell$. Let $U' = \operatorname{Span}\{u_1, \dots, u_{k-1}\}$. Since $\dim U' = k-1$, we can apply our induction hypothesis and so we have that $(S^{\ell-j} U') u_k^j \in W$ and we are done. □

Lemma 3.4. *The algebra of symmetric polynomials in k variables, Sym_k, is isomorphic as an algebra to the S_k invariants of $\mathbb{C}[X]^{\otimes k}$.*

Proof. Define $\Psi \colon \mathbb{C}[X]^k \to \mathbb{C}[X_1, \dots, X_k]$ by $\Psi(f_1(X), \dots, f_k(X)) = f_1(X_1) \cdots f_k(X_k)$. It is clear Ψ is multilinear. Thus, by the universal property of the tensor product, Ψ induces a map

$$\Phi \colon \mathbb{C}[X]^{\otimes k} \to \mathbb{C}[X_1, \dots, X_k] \tag{3.1}$$

defined by $\Phi(f_1(X) \otimes \cdots \otimes f_k(X)) = f_1(X_1) \cdots f_k(X_k)$. It is clear Φ is an algebra homomorphism. It is also straightforward to see that any element of $\mathbb{C}[X]^{\otimes k}$ can be written in the form

$$\sum_{m_j \in \mathbb{Z}_{\geqslant 0}} a_{m_j} X^{m_1} \otimes \cdots \otimes X^{m_k}$$

for $a_{m_j} \in \mathbb{C}, m_j \in \mathbb{Z}_{\geqslant 0}$ using the properties of the tensor product. It is clear Φ is surjective. To see injectivity note that if

$$\Phi\left(\sum_{m_j \in \mathbb{Z}_{\geqslant 0}} a_{m_j} X^{m_1} \otimes \cdots \otimes X^{m_k}\right) = \Phi\left(\sum_{m_j \in \mathbb{Z}_{\geqslant 0}} b_{m_j} X^{m_1} \otimes \cdots \otimes X^{m_k}\right)$$

then

$$\sum_{m_j \in \mathbb{Z}_{\geqslant 0}} a_{m_j} \Phi\left(X^{m_1} \otimes \cdots \otimes X^{m_k}\right) = \sum_{m_j \in \mathbb{Z}_{\geqslant 0}} b_{m_j} \Phi\left(X^{m_1} \otimes \cdots \otimes X^{m_k}\right)$$

and

$$\sum_{m_j \in \mathbb{Z}_{\geqslant 0}} (a_{m_j} - b_{m_j}) X_1^{m_1} \cdots X_k^{m_k} = 0$$

so $a_{m_j} = b_{m_j}$ for all $m_j \in \mathbb{Z}_{\geqslant 0}$. Thus Φ is an isomorphism so $\mathbb{C}[X]^{\otimes k} \simeq \mathbb{C}[X_1, \dots, X_k]$. Finally, it is clear Φ commutes with the S_k action so we have $(\mathbb{C}[X]^{\otimes k})^{S_k} \simeq \mathrm{Sym}_k$ as desired. □

Lemma 3.5. *The algebra* $(\mathbb{C}[X]^{\otimes k})^{S_k}$ *is generated as an algebra by the elements*

$$\Delta_k(X^a) = X^a \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} + \mathrm{id} \otimes X^a \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} + \cdots\cdots + \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes X^a \quad a \in \mathbb{N}$$

Proof. By Lemma 3.3 we know $S^k(\mathbb{C}[X])$ is spanned by elements of the form $X^{\otimes k}$. By Lemma 2.5 we know $(\mathbb{C}[X]^{\otimes k})^{S_k}$ is isomorphic to $S^k(\mathbb{C}[X])$. So, it suffices to verify that any $X^{\otimes k}$ can be expressed as a polynomial in the $\Delta_k(X^a)$. By Lemma 3.4 we know $(\mathbb{C}[X]^{\otimes k})^{S_k}$ is also isomorphic to Sym_k. The image of $X^{\otimes k}$ under (3.1) is $X_1 \cdots X_k$, which is the kth elementary symmetric polynomial, $e_k(X_1, \dots, X_k)$. Furthermore, the image of $\Delta_k(X^a)$ under Φ is $X_1^a + X_2^a + \cdots + X_k^a$ which is the ath power sum, p_a. From Theorem 2.4 we know that elementary symmetric polynomials can be written as a polynomial in the power sums. So $(\mathbb{C}[X]^{\otimes k})^{S_k}$ is generated as an algebra by the $\Delta_k(X^a)$. □

Lemma 3.6. *For any finite dimensional* $\mathbb{C}$*-vector space* V, $\mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k})$ *is generated as an algebra by the* $\Delta_k(X)$ *for* $X \in \mathrm{End}(V)$.

Proof. Let $A = \mathrm{End}(V)$. By [Lor18, (4.7.2)] we have that $\mathrm{End}(V^{\otimes k}) \simeq A^{\otimes k}$. Since this action commutes with the S_k action we can take the S_k invariants to get $\mathrm{End}_{S_k}(V^{\otimes k}) = \mathrm{End}(V^{\otimes k})^{S_k} \simeq (A^{\otimes k})^{S_k}$. Let B be the subalgebra of $(A^{\otimes k})^{S_k}$ generated by $\Delta_k(a)$ for $a \in A$. By Lemma 2.5 we have that $(A^{\otimes k})^{S_k} \simeq S^k A$. Lemma 3.3 implies that $(A^{\otimes k})^{S_k}$ is spanned by elements of the form $a^{\otimes k}$ for $a \in A$. Thus it suffices for show that $a^{\otimes k} \in B$ for all $a \in A$. Fix $a \in A$ and consider the algebra homomorphism

$$\theta \colon \mathbb{C}[X] \to A, \quad X \mapsto a.$$

Then we have $\theta^{\otimes k} \colon \mathbb{C}[X]^{\otimes k} \to A^{\otimes k}$ which can be restricted to the S_k invariant spaces,

$$\theta^{\otimes k} \colon (\mathbb{C}[X]^{\otimes k})^{S_k} \to (A^{\otimes k})^{S_k}.$$

By Lemma 3.5, $X^{\otimes k}$ can be written as a polynomial in $\Delta_k(X^r)$, $r \in \mathbb{N}$. Then, applying $\theta^{\otimes k}$ we have that $a^{\otimes k}$ can be written as a polynomial in the $\Delta_k(a^r)$, $r \in \mathbb{N}$. So $a^{\otimes k} \in B$ as desired. □

Proposition 3.7 (Schur-Weyl Duality for $\mathfrak{gl}_n(V)$)**.** *The images of* $\mathbb{C}S_k$ *and* $\mathfrak{gl}_n(V)$ *in* $\mathrm{End}(V)$ *generate each other's commutants.*

Proof. First we will verify the image of $\mathfrak{gl}_n(V)$ in $\mathrm{End}(V^{\otimes k})$ is $\mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k})$. Recalling the action of $\mathfrak{gl}_n(V)$ on $V^{\otimes k}$, we can see the image of $X \in \mathfrak{gl}_n(V)$ in $\mathrm{End}(V^{\otimes k})$ is $\Delta_k(X)$. By Lemma 3.6, the $\Delta_k(X)$ generate $\mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k})$ so we are done.

Now, let A be the image of $\mathbb{C}S_k$ in $\mathrm{End}(V^{\otimes k})$. By the First Isomorphism Theorem and the fact that $\mathbb{C}S_k$ is semisimple we can conclude A is semisimple since quotients of semisimple rings are semisimple. Then, by Theorem 2.3, $A = \mathrm{End}_{\mathfrak{gl}_n(V)}(V^{\otimes k})$. □

4. Schur-Weyl Duality for GL(V)

The group of $n \times n$ invertible matrices, $GL(V)$, acts on $V^{\otimes k}$ as follows

$$X(v_1 \otimes \cdots \otimes v_k) = Xv_1 \otimes \cdots \otimes Xv_k, \quad X \in GL(V), v_1, \dots, v_k \in V.$$

This action commutes with the action of S_k on $V^{\otimes k}$ defined in Section 3 since

$$\begin{aligned} X\sigma(v_1 \otimes \cdots \otimes v_k) &= X(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}) \\ &= Xv_{\sigma^{-1}(1)} \otimes \cdots \otimes Xv_{\sigma^{-1}(k)} \\ &= \sigma\,(Xv_1 \otimes \cdots \otimes Xv_k) \\ &= \sigma X(v_1 \otimes \cdots \otimes v_k). \end{aligned}$$

Thus it can be shown that there is a similar Schur-Weyl duality statement for $\mathrm{GL}(V)$.

Theorem 4.1 (Schur-Weyl Duality for $\mathrm{GL}(V)$)**.** *The images of* $\mathrm{GL}(V)$ *and* S_k *in* $\mathrm{End}(V)$ *generate each other's commutants.*

Before we prove Schur-Weyl duality for $\mathrm{GL}(V)$, recall that while $\mathfrak{gl}_n(V)$ is the space of $n \times n$ matrices, $\mathrm{GL}(V)$ is the group of $n \times n$ invertible matrices and so we have that $\mathrm{GL}(V) \subseteq \mathfrak{gl}_n(V)$. Thus, to verify Schur-Weyl duality for $\mathrm{GL}(V)$ we need only to verify that the algebra generated by $\mathrm{GL}(V)$ in $\mathrm{End}(V)$ is the algebra generated by $\mathfrak{gl}_n(V)$.

First recall a *topology* on a set X is a collection, T, of subsets of X such that

(a) $X \in T$ and $\emptyset \in T$,
(b) finite intersections of elements of T are in T, and
(c) any union of elements of T is in T.

The elements of T are said to be open and a subset of X is closed if its complement is in T. We may now define the *Zariski topology* on affine space $\mathbb{A}^n(k) = \{(a_1, \ldots, a_n) : a_1, \ldots, a_n \in k\}$. For an ideal I of $k[X_1, \ldots, X_n]$, the zero set

$$V(I) = \{x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in S\}$$

is called an affine variety. In the Zariski topology, we define such varieties to be the closed sets and the complements to be the open sets. It is known that in the Zariski topology, open sets are dense (for example, see [Lor18, (C.3)]).

Proof of Theorem 4.1. We need to verify the image of $\mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes})$ is $\mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k})$. Recalling the action of $\mathrm{GL}(V)$, we can see the image of $X \in \mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes k})$ is $X^{\otimes k}$. By Lemma 2.5 $S^k(\mathrm{End}(V)) \simeq (\mathrm{End}(V))^{S_k} = \mathrm{End}_{S_k}(V)$ and by Lemma 3.3, $S^k(\mathrm{End}(V))$ is spanned by elements of the form $X^{\otimes k}$ for X in a Zariski-dense subset of $\mathrm{End}(V)$. By the above discussion, $V(\{\det\})$ is closed. Then, since $\mathrm{GL}(V) = \{X \in M_n(V) : \det(X) \neq 0\}$, we have that $\mathrm{GL}(V) = \mathbb{A}^{n^2} \setminus V(\{\det\})$ is open and therefore Zariski-dense. So the image of $\mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes k})$ is $\mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k})$. The rest of the proof follows from Proposition 3.7. □

References

[EH18] K. Erdmann and T. Holm. *Algebras and Representation Theory*. Springer, 2018.
[FH04] W. Fulton and J. Harris. *Representation Theory: A First Course*. Springer, 2004.
[Jor70] C. Jordan. *Traité des substitutions et des équations algébriques*. Gauthier-Villars, 1870.
[Lor18] Martin Lorenz. *A Tour of Representation Theory*. American Mathematical Society, 2018.
[Mac79] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford University Press, 1979.
[Sch27] I. Schur. *´Uber die rationalen Darstellungen der allgemeinen linearen Gruppe*. Springer, 1927.

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